Progress on the Implicit Kinetic MHD model

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Model Motivation

- Current Gyrokinetic-Maxwell equations are not fully electromagnetic.
  - The $\boldsymbol{A}_{\parallel} - \phi$ model does not have $\delta \boldsymbol{B}_{\parallel}$.
- Gyrokinetic ordering may not be valid in some problems, such as Tokamak edge ETG or reconnection.
  - $\mathbf{E} \times \mathbf{B}$ flow of thermal speed
  - density or temperature over $\sim 10\rho_i$
- By treating electrons as massless fluid, this simple hybrid model does not require a guide field, and it is capable of capturing MHD physics in a natural way.
- We are using the GEM code as a test bed for the model and algorithm.

The Lorentz ion/Drift kinetic electron model

Lorentz ions:

$$\frac{d\mathbf{v}_i}{dt} = \frac{q}{m_i}(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}), \qquad \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i$$

Drift kinetic electrons: $\varepsilon = \frac{1}{2}m_e v^2$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}_G \equiv v_\parallel \left(\mathbf{b} + \frac{\delta\mathbf{B}_\perp}{B_0}\right) + \mathbf{v}_D + \mathbf{v}_E$$

$$\frac{d\varepsilon}{dt} = -e\mathbf{v}_G \cdot \mathbf{E} + \mu\frac{\partial B}{\partial t}, \qquad \frac{d\mu}{dt} = 0$$

Ampere's equation

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J}_i - en_e(\mathbf{V}_{\mathrm{e}\perp} + u_{\parallel\mathrm{e}}\mathbf{b}))$$

$$\mathbf{V}_{\mathrm{e}\perp} = \frac{1}{B}\mathbf{E} \times \mathbf{b} - \frac{1}{enB}\mathbf{b} \times \nabla P_{\perp\mathrm{e}}$$

$$\mathbf{J}_i = \int f_i \mathbf{v}\, d\mathbf{v}, \quad u_{\parallel\mathrm{e}} = \int f_e\, v_\parallel\, d\mathbf{v}, \quad P_{\perp\mathrm{e}} = \int f_e \frac{1}{2} m_e v^2\, d\mathbf{v}$$

Faraday's equation,

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

Refer to Yang Chen and Scott Parker's poster TP6.00143, Thursday morning.

- Quasi-neutral
  - No displacement current in the Faraday's equation
- No transverse electron inertia (no electron polarization current). Electron FLR and polarization current can be added for reconnection problems.
- The magnetic field perturbation is 3-D, whereas in the $A_\parallel - \phi$ model $\delta\mathbf{B} = \nabla \times (A_\parallel \mathbf{b})$ is 2-D
- Unable to combine $A_\parallel - \phi$ field model with Vlasov ions. With GK ions ϕ is obtained from GK Poisson equation. With Vlasov ions the equation

$$n_i = n_e$$

does not determine ϕ! However, taking time derivative of this equation to the second order results in

$$\frac{n_0 q}{m_i}\nabla_\perp^2\phi - \frac{q}{m_i}\nabla\phi\cdot\nabla n_0 + \frac{e}{m_e}\nabla_\parallel E_\parallel$$
$$= \nabla\cdot\left(\frac{1}{m_e}\nabla\cdot\mathbf{P}_i - \frac{qn_0}{m_i}\mathbf{V}_i\times\mathbf{B}\right) - \frac{1}{m_e}\nabla_\parallel\left(\frac{\delta\mathbf{B}}{B}\cdot\nabla(n_0T_0) + \nabla_\parallel\delta P_{\parallel\mathrm{e}}\right) + \frac{\dot{\mathbf{E}}\times\mathbf{b}}{B}\cdot\nabla n_0$$

We have not been able to produce the Alfven waves solving this equation.

Frieman-Chen Electron GK Equation in $\mathbf{E}_1$ and $\mathbf{B}_1$

- Gyrokinetic equations are usually derived in terms of $\mathbf{A}$ and ϕ, to make explicit the ordering

$$\frac{\partial \mathbf{A}}{\partial t} \sim \epsilon_\delta \nabla_\perp \phi$$

- The Frieman-Chen gyrokinetic equation, assuming isotropy ($\partial F_0/\partial \mu = 0$),

$$\hat{L}_g \delta H_0 \equiv \left(\frac{\partial}{\partial t} + v_\parallel \mathbf{b} \cdot \nabla + \mathbf{v}_D \cdot \nabla \right) \delta H_0 = -\frac{q}{m} (S_L + \langle R_{\rm NL} \rangle),$$

  where δH_0 is related to the perturbed distribution δF through $\delta F = \frac{q}{m} \phi \frac{\partial F_0}{\partial \epsilon} + \delta H_0$

$$S_L = \frac{\partial}{\partial t} \langle \phi - \mathbf{v} \cdot \mathbf{A} \rangle \frac{\partial F_0}{\partial \epsilon} - \nabla \langle \phi - \mathbf{v} \cdot \mathbf{A} \rangle \times \frac{\mathbf{b}}{\Omega} \cdot \nabla F_0,$$

$$\langle R_{\rm NL} \rangle = -\nabla \langle \phi - \mathbf{v} \cdot \mathbf{A} \rangle \times \frac{\mathbf{b}}{\Omega} \cdot \nabla \delta H_0.$$

- Define $\delta f = \frac{q}{m} \langle \phi \rangle \frac{\partial F_0}{\partial \epsilon} + \delta H_0$. The gyrokinetic equation for δf is, written in terms of $\mathbf{E}_1$ and $\mathbf{B}_1$

$$\frac{D}{Dt} \delta f = -\left(\frac{1}{B_0} \langle \mathbf{E}_1 \rangle \times \mathbf{b} + v_\parallel \frac{\langle \mathbf{B}_{1\perp} \rangle}{B_0} \right) \cdot \nabla F_0 + \frac{1}{m} \dot{\epsilon} \frac{\partial F_0}{\partial \epsilon}$$

$$\frac{D}{Dt} = \hat{L}_g + \left(\frac{1}{B_0} \langle \mathbf{E}_1 \rangle \times \mathbf{b} + v_\parallel \frac{\langle \mathbf{B}_{1\perp} \rangle}{B_0} \right) \cdot \nabla, \qquad \dot{\epsilon} = q \left(v_\parallel \mathbf{b} + \mathbf{v}_D + v_\parallel \frac{\langle \mathbf{B}_{1\perp} \rangle}{B_0} \right) \cdot \langle \mathbf{E}_1 \rangle + q \langle \mathbf{v}_\perp \cdot \mathbf{E}_{1\perp} \rangle$$

- The perturbed electron diamagnetic flow comes from δf,

$$n_0 \mathbf{V}_D(\mathbf{x}) = \int (v_\parallel \mathbf{b} + \mathbf{v}_\perp(\mathbf{R}', \epsilon, \mu, \alpha)) \delta f(\mathbf{R}', \epsilon, \mu) \delta(\mathbf{x} - \mathbf{R}' - \boldsymbol{\rho}) \, J d\mathbf{R}' \, d\epsilon \, d\mu \, d\gamma$$

$n_0 \mathbf{V}_D$ is computed by depositing the particle current along the gyro-ring. In the drift-kinetic limit $\mathbf{V}_D$ reduces to the electron diamagnetic flow.

- The electron $\mathbf{E} \times \mathbf{B}$ flow comes from the first term in δF,

$$n_0 \mathbf{V}_E(\mathbf{x}) = \frac{\mathbf{q}}{\mathbf{m}} \int \mathbf{v} \left(\phi(\mathbf{x}) - \langle \phi \rangle (\mathbf{x} - \boldsymbol{\rho}, \epsilon, \mu) \right) \frac{\partial \mathbf{F_0}}{\partial \epsilon} \mathbf{J} \, \mathbf{d}\epsilon \, \mathbf{d}\mu \, \mathbf{d}\gamma$$

in eikonal form,

$$n_0 \mathbf{V}_E = n_0 \frac{h}{B_0} \delta \mathbf{E}_k \times \mathbf{b}$$

with $b = k_\perp^2 v_T^2 / \Omega^2$ and

$$h(b) = -\frac{1}{b^2} \int_0^\infty e^{-x^2/2b} J_0(b) J_0'(b) x^2 \, dx$$

In the limit of small $k\rho \ll 1$ the factor $h(b)$ become unity, so that $n_0 \mathbf{V}_E$ become the total guiding center $\mathbf{E} \times \mathbf{b}$ flow.

Lorentz ion and fluid electron model

- Lorentz force ions:

$$\frac{d\boldsymbol{v}_i}{dt} = \frac{q}{m_i}(\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{B}), \qquad \frac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i$$

- Isothermal fluid electrons as a simple test:

$$\delta p_e = \gamma \delta n_e T_e = \gamma \delta n_i T_e.$$

Eventually we will add gyrokinetic electrons.

- Ampere's law:

$$\nabla \times \delta \boldsymbol{B} = \mu_0 e \left(n\boldsymbol{u_i} - n\boldsymbol{u_e}\right)$$

- Faraday's law

$$\nabla \times \boldsymbol{E} = -\frac{\partial\, \delta \boldsymbol{B}}{\partial t}.$$

Ohm's law

- Starting from the electron momentum equation:

$$\boldsymbol{E} = -\boldsymbol{u_i} \times \boldsymbol{B_0} + \frac{1}{\mu_0 en}(\bigtriangledown \times \delta \boldsymbol{B}) \times \boldsymbol{B_0} + \frac{\eta}{\mu_0} \bigtriangledown \times \delta \boldsymbol{B} - \frac{\nabla p_e}{en} - \frac{m_e}{en}\frac{\partial (n\boldsymbol{u_e})}{\partial t}.$$

- With Ampere's law and ion momentum equation

$$\bigtriangledown \times \delta \boldsymbol{B} = \mu_0 e \left(n\boldsymbol{u_i} - n\boldsymbol{u_e}\right)$$

$$\frac{\partial (n\boldsymbol{u_i})}{\partial t} = \frac{en}{m_i}(\boldsymbol{E} + \boldsymbol{u_i} \times \boldsymbol{B_0}) - \frac{1}{m_i} \bigtriangledown p_i.$$

- And neglect terms with m_e/M_i, we obtain Ohm's law

$$\boldsymbol{E} + \frac{c^2}{w_{pe}^2} \bigtriangledown \times (\bigtriangledown \times \boldsymbol{E}) = -\frac{\boldsymbol{J_i}}{en} \times \boldsymbol{B_0} + \frac{1}{\mu_0 en}(\bigtriangledown \times \delta \boldsymbol{B}) \times \boldsymbol{B_0}$$
$$+ \frac{\eta}{\mu_0} \bigtriangledown \times \delta \boldsymbol{B} - \frac{\gamma T_e \nabla n_i}{en}.$$

Implicit δf algorithm

- δf method for ions:

$$\frac{d}{dt} f_{i1} = -\frac{q}{m_i}(\boldsymbol{E} + \boldsymbol{v} \times \delta \boldsymbol{B}_1) \cdot \frac{\partial}{\partial \boldsymbol{v}} f_{i0}.$$

$$\frac{d}{dt} \omega_i = -\frac{q}{T_i} \boldsymbol{E} \cdot \boldsymbol{v}.$$

  where the second equation comes from Maxwellian distribution.

- For ρ_i scale instabilities $k_\perp \rho_i \sim 1, \beta \sim 0.01$, the compressional wave frequency $\frac{\omega}{\Omega_i} \geq 10$, therefore $\Omega_i \Delta t \ll 0.01$ is needed. But in certain cases (e.g. NSTX), $\Omega_i \Delta t \sim 0.1$, which makes implicit method indispensable.

Discretized form

- Faraday's law

$$\frac{\delta \boldsymbol{B}^{n+1} - \delta \boldsymbol{B}^{n}}{\Delta t} = - \bigtriangledown \times \boldsymbol{E}^{n+1}.$$

- Ohm's law

$$\boldsymbol{E}^{\boldsymbol{n+1}} + \frac{1}{\beta_e}(\frac{m_e}{m_i} + \eta \Delta t)(\bigtriangledown \times (\bigtriangledown \times \boldsymbol{E}^{n+1})) + \frac{\Delta t}{\beta_e}(\bigtriangledown \times (\bigtriangledown \times \boldsymbol{E}^{n+1})) \times \boldsymbol{B_0}$$
$$= -\boldsymbol{J}_i^{n+1} \times \boldsymbol{B_0} + \frac{1}{\beta_e}(\bigtriangledown \times \delta \boldsymbol{B}^{n}) \times \boldsymbol{B_0} + \frac{1}{\beta_e}\, \eta(\bigtriangledown \times \delta \boldsymbol{B}^{n}) - \gamma \nabla n_i.$$

- Dependence of $\boldsymbol{J}_i^{n+1}$ on $\boldsymbol{E}_1^{n+1}$

$$\boldsymbol{J}_i^{n+1} = \boldsymbol{J}_i^{n} + \Delta t \frac{V}{N} \sum_j \frac{1}{\Delta V} \frac{q}{T_i} \boldsymbol{v}_j \boldsymbol{E}^{n+1}(\boldsymbol{x}_j^{n+1}) \cdot \boldsymbol{v}_j S(\boldsymbol{x} - \boldsymbol{x}_j^{n+1})$$
$$\simeq \boldsymbol{J}_i^{n} + q n_{i0} \Delta t \boldsymbol{E}^{n+1}(\boldsymbol{x}) \equiv \boldsymbol{J}'_i.$$

Iterate on the differences between $\boldsymbol{J}_i^{n+1}$ and $\boldsymbol{J}'_i$, and 5 iterations are accurate enough.

Field solver

- Zero-order B field

$$\boldsymbol{B}_0 = \boldsymbol{e}_y \boldsymbol{B}_{0y} + \boldsymbol{e}_z \boldsymbol{B}_{0z}.$$

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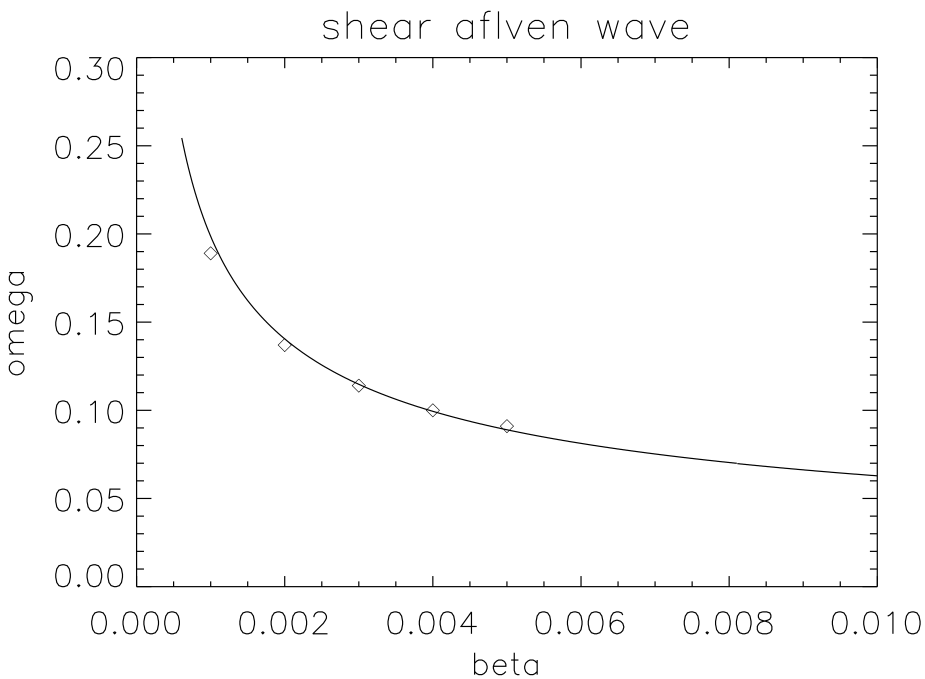
  - If $\boldsymbol{B}_{0y}$ and the guide field $\boldsymbol{B}_{0z}$ are constants, we could solve the Ohm's law by performing Fourier transform.

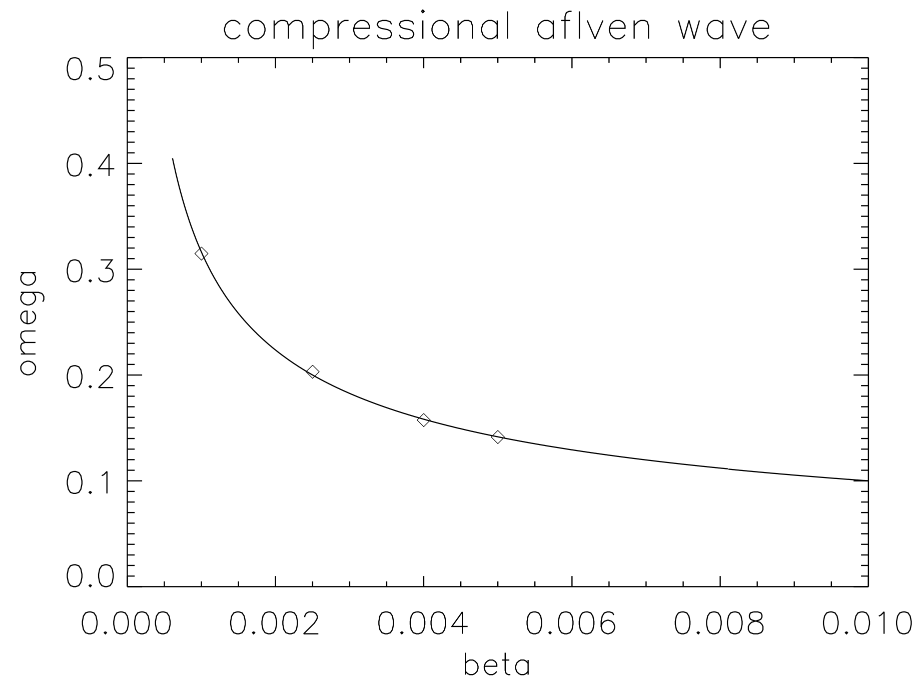
  - If $\boldsymbol{B}_{0y}$ is x dependent, as in the Harris sheet equilibrium, we could solve $\boldsymbol{E}^{n+1}(x_i, k_y, k_z)$ by writting Ohm's law in matrix form. By projecting Ohm's law in x, y,and z, we have 3 matrix equations

$$\begin{pmatrix} M_x \; M_y \; M_z \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{E}}_x \\ \tilde{\boldsymbol{E}}_y \\ \tilde{\boldsymbol{E}}_z \end{pmatrix} = \tilde{N}$$

M_x, M_y, M_z are $l_x \times l_x$ matrices.

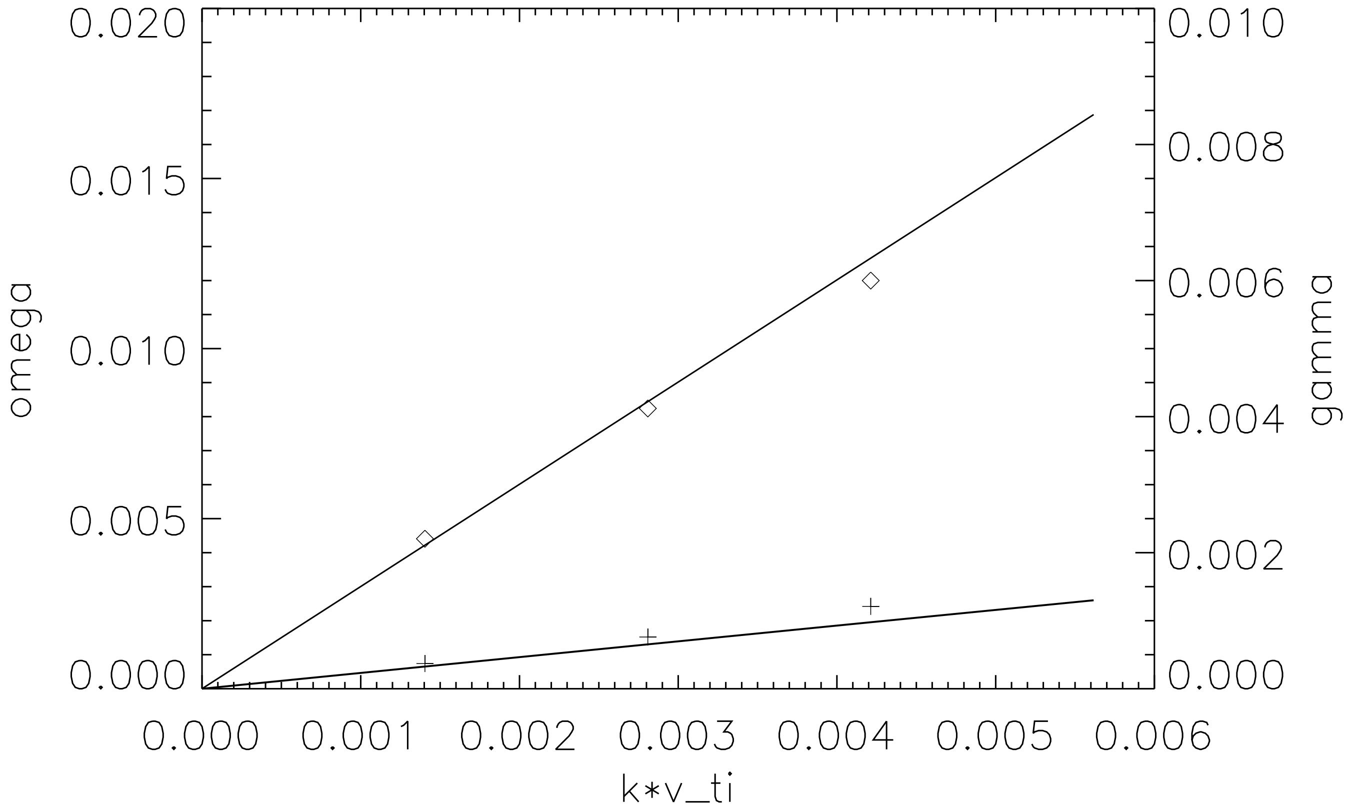
3-D Shearless Slab Alfven waves





$2 \times 32 \times 32$ grids, 131072 particles.
For shear Alfven wave, $k_\perp = 0$, $k_\| \rho_i = 0.00628$, initialize with $\delta \boldsymbol{B}_\perp$.
For compressional Alfven wave, $k_\| = 0$, $k_\perp \rho_i = 0.01$, initialize with $\delta \boldsymbol{B}_\|$. These simulations are done in a tilted B_0 field.

Ion acoustic wave



$2 \times 32 \times 32$ grids, 131072 particles. $k_\perp = 0$.

∇T_e Driven Kinetic Alfven Instability

$$\frac{\partial f_1}{\partial t} + v_\parallel \nabla_\parallel f_1 = \kappa(E_y + v_\parallel B_x) f_0 + (-E_\parallel v_\parallel + \mu \frac{\partial B_\parallel}{\partial t}) f_0$$

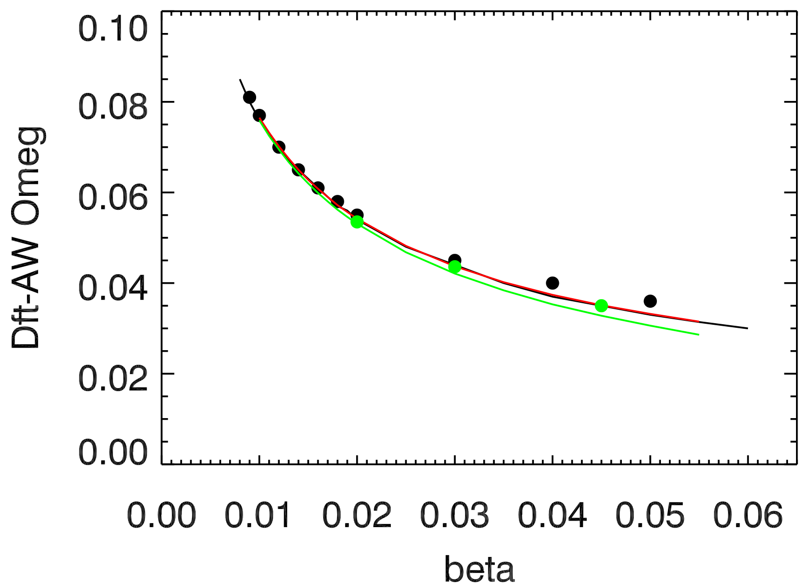
$$E_\parallel + \frac{m_e}{m_i}\frac{1}{\beta}\mathbf{b}\cdot\nabla\times\nabla\times\mathbf{E} - \kappa_T B_x = -\nabla_\parallel \delta P_{\parallel \mathrm{e}}$$

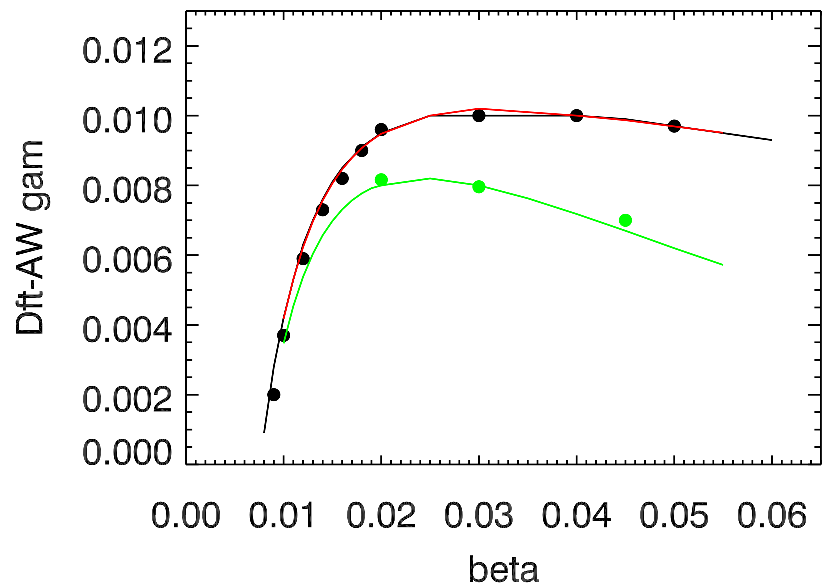
$$\mathbf{E}_\perp + \frac{1}{\beta}\mathbf{b}\times(\nabla\times\mathbf{B}_1) = -\mathbf{J}_i\times\mathbf{B} - \nabla_\perp \delta P_\perp$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = -\nabla\times\mathbf{E}$$

$\kappa = \kappa_T(mv^2/2 - 3/2),\ \kappa_T = -\frac{1}{T}\frac{\partial T_e}{\partial x}$

∇T_e Driven Kinetic Alfven Wave





- $k_x = 0$, $k_y\rho_s = 3.5$, $k_\parallel\rho_s = 0.00284$. $\delta p_{\perp e} = 0$ for simulation data points and dispersion replation (solid black line).
- Green line from dispersion relation with $\delta p_{\perp e}$, which has strong stabilizing effect at high beta.
- Red line from $A_\parallel/\phi$ gyrokinetic dispersion relation. Good agreement with $\delta p_{\perp e} = 0$ Vlasov ion DR, because in GK model only parallel Ampere's equation is used.

Summary

1. We proposed an implicit algorithm with Lorentz force ions and isothermal fluid electrons which is
   - Quasi-neutral and fully electromagnetic.
   - Suitable for MHD scale plasmas.
   - Independent on the existence of a guide field.
2. Fully implicit scheme allows bigger time step, $\Omega_i \Delta t \gtrsim 0.1$.
   - Treat Faraday's law and $\boldsymbol{E} \cdot \boldsymbol{v}$ in the ion weight equation implicitly.
3. Demonstrated 3-D slab simulation for compressional and shear Alfven waves, and ion acoustic wave.
4. Work is underway to apply this model in Harris sheet equilibrium with or without a guide field.