Continuum solution to Chapman-Enskog-like drift kinetic equation (CEL-DKE) in NIMROD

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Motivation for continuum solution to closure problem.

- Treat time-dependent problems such as Landau damping or coupling of closures to rapidly evolving instabilities.
- Easily incorporate nonlinearities and particle trapping effects.
- Increase the efficiency of the closure calculation.
- Incorporate accelerations effects.

- Solve lowest-order Chapman-Enskog-like drift kinetic equation:

\[
\frac{\partial F}{\partial t} + \vec{v}_|| \cdot \vec{\nabla} F + \left( \frac{q}{m} \frac{\vec{v}_||}{v} \cdot \vec{E} \frac{\partial}{\partial v} - \frac{\mu}{B} \frac{\partial B}{\partial t} \frac{\partial}{\partial \mu} \right) F - C( F + f_M ) = \\
-\frac{2}{3} L_1^{1/2} \left( \frac{d \ln T}{dt} + \vec{\nabla} \cdot \vec{V}_1 \right) f_M + L_1^{3/2} \vec{v}_|| \cdot \vec{\nabla} \ln T + \vec{v}_|| \cdot (\vec{\nabla} \cdot \vec{\Pi} - \vec{R}) f_M \\
+ 2 s^2 P_2 \left( \frac{\vec{v}_||}{v} \right) (\vec{b} \cdot \vec{b} - \frac{I}{3}) : \vec{\nabla} \vec{V}_1 f_M
\]
Advantages of continuum approach.

- Using NIMROD's infrastructure we can
  - test different representations of velocity space basis functions including 2D finite elements,
  - handle time dependence in fully implicit manner,
  - allow for multiple parallelization strategies, and
  - test accuracy against existing, steady-state integral closures.

- Successful formulation of mixed finite-element advance for T bodes well for continuum solution:

\[
\frac{3}{2} n \Delta T + \kappa_0 \sqrt{\theta \Delta t} \vec{B} \cdot \vec{\nabla} q_{\parallel} + \ldots = \ldots,
\]

\[
\frac{\kappa_0^2 B^2}{\kappa_{\parallel} - \kappa_{\perp}} q_{\parallel} + \kappa_0 \sqrt{\theta \Delta t} \vec{B} \cdot \vec{\nabla} \Delta T = -\kappa_0 \sqrt{\frac{\Delta t}{\theta}} \vec{B} \cdot \vec{\nabla} T^n
\]

\[
\]
Simplify.

- Consider solving CEL-DKE by expanding $F = \sum F_i(x,t) \phi_i(\nu_\parallel, \nu)$:

$$\frac{\partial F}{\partial t} + \nu L(F) + \nabla \cdot \vec{v} F + \left( \frac{q}{m} \frac{\vec{v} \cdot E}{v} \frac{\partial}{\partial \nu} - \frac{\mu}{B} \frac{\partial B}{\partial t} \frac{\partial}{\partial \mu} \right) F =$$

$$C_{(-L)}(F + f_M) + CEL \text{ drives}$$

- Acceleration and collision terms, $C_{(-L)}(F + f_M)$, couple velocity expansion coefficients in speed variable, $\nu$.

- Preliminary implementation ignores acceleration term and uses a moment approach for $C_{(-L)}$ (J-Y Ji's work and J. James' thesis).
Existing implementation solves for coefficients of F expansion on grid in $s=v/v_T$.

- Expanding $F = \sum F_i(x,v,t) P_i(v_\parallel/v)$ yields:

$$
(I \frac{\partial}{\partial t} - \nu_L L) \tilde{F} + A \nu \hat{b} \cdot \tilde{\nabla} \tilde{F} - B \nu (\hat{b} \cdot \tilde{\nabla} \ln B) \tilde{F} = \text{drives},
$$

where $\nu_L = \sum_b \nu_{ab} \left( (2 - \frac{1}{s_b^2}) E(s_b) + \frac{E'(s_b)}{s_b} \right)$

- Matrices $A$, $B$, and $L$ represent free-streaming, $|B|$ and pitch-angle collisional couplings, respectively.
- With Lorentz pitch-angle scattering operator, speed enters as a parameter only.
- Solve equation on grid in $s$. 

Coupling to the fluid equations: time discretization and parallelization issues (I).

- Nimrod uses staggered advance.

\[ \vec{V}(n, \vec{B}, T) \]

- For parallel heat flow closure, can couple fully implicit solves for T and F.

\[ \vec{V}(n, \vec{B}, (T, F)) \]

- Must solve simultaneously for coefficients, \( F_i \), on speed grid, \( s_i, i=1,...,n \).
- Closure moment \( q_{\parallel} \) couples to T equation.
- Leads to large system of equations with “parallelization” performed inside solver.
Coupling to the fluid equations: time discretization and parallelization issues (I).

\[
\frac{3}{2} n \Delta T + \Delta t \hat{\nabla} \cdot q_\parallel \hat{b} = \ldots, \quad \text{T equation}
\]

\[
((a_{l,l}/s_i) + \theta \Delta t (\nu L (s_i/l s_i) L_{l,l}) \Delta F_l + \theta \Delta t \hat{b} \cdot \hat{\nabla} (a_{l,l+1} \Delta F_{l+1} + a_{l,l-1} \Delta F_{l-1}) + \theta \Delta t (\hat{b} \cdot \hat{\nabla} \ln (B)) [b_{l,l+1} \Delta F_{l+1} + b_{l,l-1} \Delta F_{l-1}] + \delta_{l,l} a_{l,l} L_1^{3/2} f_M \hat{b} \cdot \Delta \ln T = \ldots, \quad \text{coupled equations for } F_l
\]

where \( q_\parallel = -T \int d \hat{v} v_\parallel L_1^{3/2} P_1(v/\nu) F_1 \)
A simple quadrature scheme for $s$ dependence.

- For heat flow, need speed integrals of form:
  \[
  \int_0^\infty ds \ s^3 \left( \frac{5}{2} - s^2 \right)^2 \ e^{-s^2} \ F = \sum_{i=1}^N w_i \ F(s_i^2).
  \]

- Linearize about $f_M$:
  \[
  \hat{F}_l = F_l / (L_1^{3/2} f_M)
  \]

- Simple quadrature scheme with $N=1$ exact for linear functions of $s^2$ when $s_1^2 = 2.25$ and $w_1 = 2.888889$.

- Use as test case for more sophisticated solution methods.
Applied to heat transport problem in cylindrical geometry (Hölzl et al., POP 2008).

2/1 island added into zero pressure, cylindrical eq. Heat source finite for $r < 0.2$ and zero outside.
Can solve for T coupled to F equations.

- T profiles as function of flux show flattening across 2/1 island.
- All cases used 3 Fourier modes and 10 x 10 grid.
- Predicted T from continuum solution spatially accurate for bicubic (pd=2) finite elements.
- Here \( \nu \) chosen to yield heat flow consistent with Braginskii closure.
- Solution not resolved in velocity variables!
Coupling to the fluid equations: time discretization and parallelization issues (II).

- Can stagger $T$ and $F$.

\[ (\vec{V}, F) \quad (n, \vec{B}, T) \]

- Solution for coefficients, $F_i$, performed on groups of processors with different speeds, $s_i, i=1,...,n$.

- Processors communicate to compute $q_{\parallel}$ which is centered in $T$ advance.

- Small systems of equations with parallelization performed outside solver.

- Limits time step.
Coupling to the fluid equations: time discretization and parallelization issues (II).

(1) Solve coupled equations for $F_l$, on separate groups of processors for single $s_i$.

\[
\begin{align*}
\left( (a_{l, l} / s_i) + \theta \Delta t (\nu_L (s_i) / s_i) L_{l, i} \right) \Delta F_l + \\
\theta \Delta t \hat{b} \cdot \vec{\nabla} (a_{l, l+1} \Delta F_{l+1} + a_{l, l-1} \Delta F_{l-1}) + \\
\theta \Delta t (\hat{b} \cdot \vec{\nabla} \ln (B)) \left[ a_{l, l+1} \Delta F_{l+1} + b_{l, l-1} \Delta F_{l-1} \right] + \\
\delta_{ll} a_{l, l} L_1^{3/2} f_M \hat{b} \cdot \Delta \ln T = \ldots,
\end{align*}
\]

(2) Compute parallel heat flow closures.

(3) Advance T.
Generalize velocity space basis to include 2-D finite elements.

- Expand $$F = \sum F_i(x,t) \phi_i(\xi, \mu)$$, where $$\xi$$ and $$\mu$$ are appropriate velocity variables and insert into CEL-DKE:

$$\frac{\partial F}{\partial t} + \mathbf{v}_\parallel \cdot \nabla F + \left( \frac{q}{m} \frac{\mathbf{v}}{v} \cdot \mathbf{E} \right) \frac{\partial}{\partial v} - \frac{\mu}{B} \frac{\partial B}{\partial t} \frac{\partial}{\partial \mu} \right) F - C \left( F + f_M \right) = \text{drives}$$

- Integrate equation using

$$\int d\mathbf{x} \int d\mathbf{v} \phi_i(\xi, \mu) \alpha_j(R, Z) e^{-i n^\prime \phi}$$

- Use NIMROD finite element and Gaussian quadrature machinery to compute velocity integrals.
- Consider $$C^0$$ or possibly discontinuous basis functions.
- Requires new data types and solver development.
Choose basis functions based on problem.

- Legendre functions for pitch-angle with simple grid in speed, $v$, is easiest.
- Allows for packed grids.

![Diagram](image)

- Pack near RF resonance in $v_\perp$
- Pack in speed and angle for slowing down of hot particles
Future work.

- At present, can solve for coefficients of Legendre expansion, $F_l(x,t)$, on grid in $s$.

- Future work includes:
  
  - testing convergence of fully implicit advance as Legendre polynomials and grid points in speed are added.

  - implementing staggered advance and comparing with fully implicit solutions.

  - generalizing continuum solution to allow for 2-D finite-element basis functions for velocity dependence.