Applying Differential Approximation to the Implicit Leapfrog

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Outline

- Introduction: origins of present two-fluid algorithm
- Results from von Neumann analysis
  - Limiting cases
  - Numerical evaluation
- Differential approximation
  - Advection
  - Implicit Hall
- Conclusions

Introduction: The implicit leapfrog was based on numerical analysis following other attempts.

- NIMROD's first two-fluid algorithm used a semi-implicit Hall advance based on Harned & Mikic (JCP **83**).
  - SI Hall differential operator was symmetric. (4th order derivatives)
  - 1/2-level information came from prediction.
- Analysis and NIMROD results for this method show:
  - Numerical stability for EMHD
  - Numerical instability for HMHD
  - Applying the SI operator only on the corrector step is stable for uniform equilibria but not for nonuniform equilibria.
- Centered implicit Hall in a temporally staggered magnetic-field advance is stable.
  - Implicit Hall operator is 2nd-order and asymmetric.
  - Advection needs to be centered for all fields; no more p/c advection.

Von Neumann Analysis

Modes of the time-step operation assess stability for detailed physics models with a uniform background.

- Analysis of the implicit leapfrog includes Hall, gyroviscosity, electron inertia, separate COM and electron flows, and resistivity.
- Thermal response is adiabatic with electron pressure being a fixed fraction of the total pressure.
- Spatially, the linear response is assumed to vary as exp(*iky*).
  - Spatial discretization effects are not considered.
- λ is the eigenvalue of the time-step operation.
  - Modes satisfy:

$$\begin{pmatrix} \underline{v}^{j+1} \\ \underline{b}^{j+3/2} \\ p^{j+3/2} \end{pmatrix} = \lambda \begin{pmatrix} \underline{v}^{j} \\ \underline{b}^{j+1/2} \\ p^{j+1/2} \end{pmatrix}$$

The algebraic system has 6 components and as many as 6 modes at each set of physical and numerical parameters.

- Time is normalized by $1/\Omega_i$; length is normalized by d_i; $v_A \rightarrow 1$.
- The cosine and sine of the angle between **k** and $\mathbf{B}_0$ are c and s.

$$\left(\frac{1}{\Delta t}+ikV_0\theta_v-\Delta t\underline{\underline{L}}\right)\left(\underline{v}^{j+1}-\underline{v}^{j}\right)=-ikV_0\underline{v}^{j}+ik\begin{pmatrix}c&0&0\\0&-s&-\beta\\0&c&0\end{pmatrix}\begin{pmatrix}b_x\\b_z\\p\end{pmatrix}^{j+1/2}+\frac{1}{2}\underline{\underline{F}}\left(\underline{v}^{j+1}+\underline{v}^{j}\right)$$

$$\underline{\underline{L}}=-C_0k^2\begin{pmatrix}c^2&0&0\\0&s^2+\Gamma\beta&-cs\\0&-cs&c^2\end{pmatrix}\qquad\underline{\underline{F}}=\frac{(1-f_e)\beta k^2}{4}\begin{pmatrix}0&-F_{21}&-F_{31}\\2s\left(1+3c^2\right)&0&0\\-c\left[1+3\left(c^2-s^2\right)\right]&0&0\end{pmatrix}$$

$$\left[\frac{1}{\Delta t}+ikV_{e_0}\theta_b+\left(\frac{m_e}{m_i\Delta t}+D_\eta\theta_\eta\right)k^2\right]\left(\underline{b}^{j+3/2}-\underline{b}^{j+1/2}\right)=-ikV_{e_0}\underline{b}^{j+1/2}$$

$$+ik\begin{pmatrix}c&0&0\\0&-s&c\end{pmatrix}\underline{v}^{j+1}-D_\eta k^2\underline{b}^{j+1/2}+\frac{k^2c}{2}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\left(\underline{b}^{j+3/2}+\underline{b}^{j+1/2}\right)$$

$$\left(\frac{1}{\Delta t}+ikV_0\theta_p\right)\left(p^{j+3/2}-p^{j+1/2}\right)=-ikV_0p^{j+1/2}-ik\Gamma v_y^{j+1}$$

Results of the von Neumann analysis establish the stability of the implicit leapfrog.

- The time-step matrix is found from products of matrices for explicit terms and inverses of matrices for implicit terms.
- Analytical relations for the eigenvalues can be found for limiting cases.
- Results for general cases are evaluated numerically with LAPACK routines that can handle matrices with geometric multiplicity < algebraic multiplicity.
- Key findings (presented at APS 2005 & Sherwood 2006) are:
  - **The implicit leapfrog is numerically stable with the Hall and GV terms centered in the B- and V- advances, respectively.**
  - **Predictor/corrector advection leads to severe time-step restrictions with the implicit Hall advance.**
  - **Implicit advection is numerically stable if $V \cdot \nabla$ terms are centered in each advance.**
  - **Dissipation is numerically stable for centered or backward differencing.**

An example of a limit that can be found analytically is parallel propagation without flow or dissipation.

- With $\chi \equiv k\Delta t$ (CFL number for Alfvén waves),

$$(\lambda-1)^2\left(1+C_0\chi^2\right)\pm i\left(\lambda^2-1\right)\left(\frac{k\chi}{2}\right)\left[1+C_0\chi^2+\beta_i\right]-(\lambda+1)^2\left(\frac{k^2\chi^2\beta_i}{4}\right)+\lambda\chi^2=0$$

- The form of this dispersion relation is similar to that from plasma theory.

- Coefficients of λ^0 and λ^2 are complex conjugates, and the coefficient of λ^1 is real.

- Setting $C_0 \geq 1/4$ ensures numerical stability without numerical dissipation for all Δt-values.

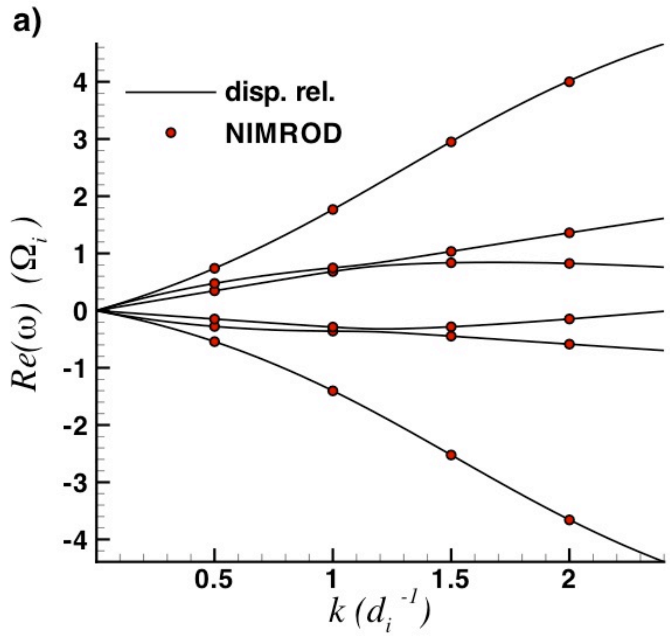
- With $C_0=1/4$,

$$\lambda=\frac{1-\frac{\chi^2}{4}\left(1-\beta_i k^2\right)\pm i\chi\sqrt{1+\frac{k^2}{4}\left(1-\beta_i+\frac{\chi^2}{4}\right)^2+\frac{k^2\chi^2\beta_i}{4}}}{1+\frac{\chi^2}{4}\left(1-\beta_i k^2\right)\pm\frac{i}{2}k\chi\left(1+\beta_i+\frac{\chi^2}{4}\right)}$$

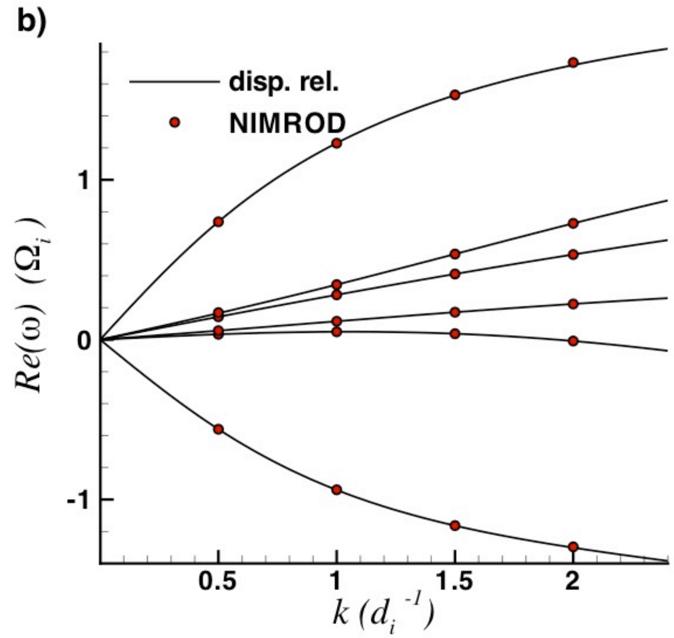
- This matches the expansion of $\exp(-i\omega\Delta t)$ to second order for the theoretical $\omega=\pm k\sqrt{1+k^2(1-\beta_i)^2/4}\pm k^2(1+\beta_i)/2$

Numerical evaluation produces eigenvectors that can be used to initialize NIMROD tests.

- A bash script converts eigenmode information from DISPERSION text output into NIMROD input and runs multiple tests.
- Eigenvectors reflect the algorithm's temporal staggering, which is critical for launching a single sine-wave at finite Δt-values.



Case with nearly parallel propagation has β=0.15, V_0=0.2, and Δt=0.5.



Case with nearly perp. propagation has β=0.6, V_0=0.2, and Δt=1.5.

- The agreement (including GV effects) contributes to code verification.
- Other evaluations compare the accuracy of impl. leapfrog and C-N.

Differential Approximation

Differential approximation (Shokin, Springer-Verlag '83) helps explain the cause of numerical instabilities with poor selection of parameters.

- It also provides insight and increases confidence that the von Neumann findings are representative.
- Here, we consider
  - The compatibility of implicit advection with the staggered leapfrog
  - The influence of the semi-implicit operator with centered advection
  - The compatibility of the semi-implicit velocity advance with implicit Hall in the magnetic-field advance.
- In general, differential approximation provides only necessary conditions for numerical stability.

Our approach follows Caramana (JCP 96, ‘91), but we address the temporal staggering directly.

- For perpendicular propagation at low-*k* and $\beta \rightarrow 0$, the algorithm is

$$\left(1-\Delta t^2 C_0 \frac{\partial^2}{\partial y^2}\right)\frac{v^{j+1}-v^j}{\Delta t} = -V_0 \frac{\partial}{\partial y}\left[\theta_v v^{j+1} + (1-\theta_v)v^j\right] - \frac{\partial}{\partial y} b^{j+1/2}$$

$$\frac{b^{j+3/2}-b^{j+1/2}}{\Delta t} = -V_0 \frac{\partial}{\partial y}\left[\theta_b b^{j+3/2} + (1-\theta_b)b^{j+1/2}\right] - \frac{\partial}{\partial y} v^{j+1} \quad ,$$

where *v* is the *y*-component, and *b* is parallel to $\mathbf{B}_0$.

- We expand *v* about half-integer levels and *b* about integer levels and drop terms that would require additional initial conditions (Caramana).

$$\left(1-\Delta t^2 C_0 \frac{\partial^2}{\partial y^2}\right)\left.\frac{\partial v}{\partial t}\right|_{t_{1/2}} = -V_0 \frac{\partial}{\partial y}\left[v + \Delta t\left(\theta_v - \frac{1}{2}\right)\frac{\partial v}{\partial t}\right]_{t_{1/2}} - \frac{\partial}{\partial y}\left(b - \frac{\Delta t}{2}\frac{\partial b}{\partial t}\right)_{t_1}$$

$$\left.\frac{\partial b}{\partial t}\right|_{t_1} = -V_0 \frac{\partial}{\partial y}\left[b + \Delta t\left(\theta_b - \frac{1}{2}\right)\frac{\partial b}{\partial t}\right]_{t_1} - \frac{\partial}{\partial y}\left(v + \frac{\Delta t}{2}\frac{\partial v}{\partial t}\right)_{t_{1/2}} \quad ,$$

- Last terms on the right include terms that account for synchronization.

Manipulation of the differential approximation shows the need for centered advection.

- Change to sum and difference variables, and substitute

$$v = (Z_+ + Z_-)/2 \qquad b = (Z_+ - Z_-)/2$$

- To order Δt^1, the system can be written as

$$\frac{\partial}{\partial t} Z_\pm = -V_0 \frac{\partial}{\partial y} Z_\pm \mp \frac{\partial}{\partial y} Z_\pm$$
$$+ \frac{\Delta t}{2}\left(V_0^2 \pm V_0\right)(\theta_v + \theta_b - 1)\frac{\partial^2}{\partial y^2} Z_\pm - \frac{\Delta t}{2}\left[V_0^2\left(\theta_b - \theta_v\right) \mp V_0\left(\theta_b - \theta_v + 1\right) + 1\right]\frac{\partial^2}{\partial y^2} Z_\mp \; .$$

- Analytically (order Δt^0), the two waves propagate independently and in opposite directions in a frame moving with the flow.
- Considering the 3rd term on the rhs, for V_0<1, one of the two numerical responses dissipates and the other is ill-posed when $\theta_v + \theta_b \neq 1$.
  - This explains why backward (& forward) differencing is not stable.
  - Stability with backward differencing for V_0>1 has been confirmed with von Neumann computations.
- More truncation errors are eliminated with $\theta_v = \theta_b$. Other contributions to the last term are from synchronization (not $O(\Delta t)$ errors).

Differential approximation also shows the compatibility of centered advection and the semi-implicit advance.

- The equations can be manipulated into a second-order wave equation.

$$\left[1-\Delta t^2\left(C_0-\frac{1}{4}\right)\frac{\partial^2}{\partial y^2}\right]\frac{\partial^2 v}{\partial t^2}+V_0\left(2-\Delta t^2 C_0\frac{\partial^2}{\partial y^2}\right)\frac{\partial}{\partial y}\frac{\partial v}{\partial t}+\left(V_0^2-1\right)\frac{\partial^2}{\partial y^2}v=0$$

- For $C_0 \geq 1/4$, the operator acting on the highest temporal derivative is a positive differential operator in the sense that

$$\int A\left[1-\Delta t^2\left(C_0-\frac{1}{4}\right)\frac{\partial^2}{\partial y^2}\right]A\,dy=\int\left[A^2+\Delta t\left(C_0-\frac{1}{4}\right)\left(\frac{\partial A}{\partial y}\right)^2\right]dy\geq 0$$

with homogeneous boundary conditions.

- This is related to the effective *k*-dependent inertia described by Schnack (JCP **70**, '87) and by Caramana.

- For infinite or periodic systems, where Fourier rep. is appropriate, solutions of the characteristic equation are real for $C_0 \geq 1/4$ for all Δt-values.

$$\omega_k=\frac{kV_0\left(1+\frac{C_0}{2}k^2\Delta t^2\right)\pm k\sqrt{1+k^2\Delta t^2\left(C_0+\frac{V_0^2}{4}-\frac{1}{4}\right)+\frac{C_0^2}{4}V_0^2k^4\Delta t^4}}{1+\left(C_0-\frac{1}{4}\right)k^2\Delta t^2}$$

Compatibility of centered Hall and the semi-implicit advance follows from similar reasoning.

- Here, we consider parallel propagation at arbitrary k with V_0=0 and $\beta \rightarrow 0$.

$$\left(\frac{1}{\Delta t} - \Delta t C_0 \frac{\partial^2}{\partial y^2}\right)\left(\underline{v}^{j+1} - \underline{v}^{j}\right) = \frac{\partial}{\partial y}\underline{b}^{j+1/2}$$

$$\frac{1}{\Delta t}\left(1 - \frac{m_e}{m_i}\frac{\partial^2}{\partial y^2}\right)\left(\underline{b}^{j+3/2} - \underline{b}^{j+1/2}\right) = \frac{\partial}{\partial y}\underline{v}^{j+1} + \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial^2}{\partial y^2}\left(\underline{b}^{j+3/2} + \underline{b}^{j+1/2}\right)$$

- The $\underline{v}$ and $\underline{b}$ vectors have x- and z-components.
- After expanding we arrive at

$$\left[\left(1 - \Delta t^2 C_0 \frac{\partial^2}{\partial y^2}\right)\left(1 - \frac{m_e}{m_i}\frac{\partial^2}{\partial y^2}\right) + \frac{\Delta t^2}{4}\frac{\partial^2}{\partial y^2}\right]\frac{\partial^2}{\partial t^2}\underline{b} + \left(1 - \Delta t^2 C_0 \frac{\partial^2}{\partial y^2}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial t}\underline{b} - \frac{\partial^2}{\partial y^2}\underline{b} = 0$$

- With $C_0 \geq 1/4$ and additional homogenous bcs, the spatial operator acting on the highest temporal derivative is a positive differential operator.
- Electron inertia contributes to this property but is not essential.
- With Fourier expansion, all solutions of the characteristic equation for ω_k^2 are real and positive when $C_0 \geq 1/4$, i.e. stable propagation without numerical dissipation.

Conclusions

- Differential approximation supports the key findings of von Neumann analysis for the implicit leapfrog.
    - In particular, the ill-posed equation for one of the two responses explains the numerical instability that results without centered implicit advection.
- Expanding about separate time-levels is useful but requires terms that account for synchronization.
- NIMROD's two-fluid plane-wave responses at large Δt have been verified by initializing with eigenmode information from von Neumann analysis.