Simulation of secondary islands with a Lorentz ion/fluid electron hybrid model

Jianhua Cheng, Scott Parker, Yang Chen, Dmitri Uzdensky

University of Colorado-Boulder

Outline

- Model equations
- Second order semi-implicit method
- Multiple islands in large Δ' tearing mode
  - Secondary islands coalescence
  - Energy conversion
- Summary

Ion equations of motion and field equations

- Lorentz force ions

$$\frac{d\boldsymbol{v}_i}{dt} = \frac{q_i}{m_i}(\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{B})$$
$$\frac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i$$

- Ampere's equation

$$\bigtriangledown \times \boldsymbol{B} = \mu_0 \left(n_i q_i \boldsymbol{u}_i - n_e e \boldsymbol{u}_e\right)$$

- Faraday's law

$$\bigtriangledown \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

The generalized Ohm's law

- Starting from the electron momentum equation:

$$en_e(\boldsymbol{E} + \boldsymbol{u}_e \times \boldsymbol{B}) = en_e\eta\, \boldsymbol{j} - \nabla \cdot \boldsymbol{\Pi}_e - m_e \frac{\partial(n_e\boldsymbol{u}_e)}{\partial t}$$

where $\boldsymbol{\Pi}_e = \int f_e m_e \boldsymbol{v}\boldsymbol{v} d\boldsymbol{v}$.

- Substitute in Ampere's equation $\boldsymbol{j} = (n_i q_i \boldsymbol{u}_i - n_e e \boldsymbol{u}_e) = \frac{1}{\mu_0}\ \nabla \times \boldsymbol{B}$, the above equation could be rewritten as

$$en_e\boldsymbol{E} = -\boldsymbol{j}_i \times \boldsymbol{B} + \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \frac{en_e}{\mu_0}\eta\,(\nabla \times \boldsymbol{B}) - \nabla \cdot \boldsymbol{\Pi}_e - m_e \frac{\partial(n_e\boldsymbol{u}_e)}{\partial t}$$

where $\boldsymbol{j}_i = n_i q_i \boldsymbol{u}_i$.

Electron inertia

- Taking the time derivative of Ampere's equation

$$\mu_0 \left(q_i \frac{\partial n_i \boldsymbol{u}_i}{\partial t} - e \frac{\partial n_e \boldsymbol{u}_e}{\partial t} \right) = \nabla \times \frac{\partial \boldsymbol{B}}{\partial t} = - \nabla \times \nabla \times \boldsymbol{E}$$

- The first term on the left hand side is obtained from the ion momentum equation, thus the electron inertial term can be written as

$$m_e \, \frac{\partial (n_e \boldsymbol{u}_e)}{\partial t} = \frac{m_e}{m_i} \frac{q_i}{e} \left(q_i n_i (\boldsymbol{E} + \boldsymbol{u}_i \times \boldsymbol{B}) - \nabla \cdot \boldsymbol{\Pi}_i - q_i n_i \frac{\eta}{\mu_0} \nabla \times \boldsymbol{B} \right) + \frac{m_e}{\mu_0 e} \nabla \times \nabla \times \boldsymbol{E}.$$

Generalized Ohm's law

- Using quasi-neutrality $n_i = n_e$, the electron density and flow can be calculated directly from particle ions

$$
\begin{aligned}
& e n_i \left(1 + \frac{m_e}{m_i}\frac{q_i^2}{e^2}\right)\boldsymbol{E} + \frac{m_e}{\mu_0 e}\nabla \times (\nabla \times \boldsymbol{E}) \\
= & -\left(1 + \frac{m_e}{m_i}\frac{q_i}{e}\right)\boldsymbol{j}_i \times \boldsymbol{B} + \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \\
& + \eta \frac{e n_i}{\mu_0}\left(1 + \frac{m_e}{m_i}\frac{q_i^2}{e^2}\right)\nabla \times \boldsymbol{B} - \nabla \cdot \boldsymbol{\Pi}_e + \frac{m_e}{m_i}\frac{q_i}{e}\nabla \cdot \boldsymbol{\Pi}_i,
\end{aligned}
$$

- In general, we need an electron model to calculate Π_e. Here we assume the electrons are isothermal and Π_e reduces to

$$P_e = n_e T_e = n_i T_e$$

Future plans include drift-kinetic and gyro-kinetic electron models.

The second-order semi-implicit δf algorithm

- In order to eliminate the fast compressional wave, we have implemented a second-order accurate semi-implicit method.

- For particle ions, the usual δf method is employed. Given a distribution function $f = f_0 + \delta f$, if a weight of $w_j = \frac{\delta f}{f}|_{x=x_j,v=v_j}$ is assigned to each particle, we could then calculate the field quantities by weight δf to the grids. According to Vlasov equation, the particle weight evolves as

$$\frac{d}{dt}w_j = -\frac{d \ln f_0}{dt}|_{x=x_j,v=v_j}$$

Yang Chen, Scott E. Parker, Phys. Plasmas **16**, 052305 (2009)

Second order semi-implicit scheme

- The velocity, length and time are normalized to $c_s^2 = T_e/m_i$, $\rho_s = m_i c_s/eB_0$ and $\Omega_{ci}^{-1} = m_i/eB_0$. $\beta_e = \mu_0 n_0 T_e/B_0^2$ is defined upon the uniform background plasma.
- The equations of motion are

$$\frac{\boldsymbol{x}^{n+1} - \boldsymbol{x}^n}{\Delta t} = (1-\theta)\ \boldsymbol{v}^n + \theta\ \boldsymbol{v}^{n+1},$$
$$\frac{\boldsymbol{v}^{n+1} - \boldsymbol{v}^n}{\Delta t} = (1-\theta)\ \boldsymbol{a}^n + \theta\ \boldsymbol{a}^{n+1},$$
$$\frac{w^{n+1} - w^n}{\Delta t} = -(1-\theta)\ (\boldsymbol{v}^n \cdot \bigtriangledown + \boldsymbol{a}^n \cdot \partial_{\boldsymbol{v}}) \ln f_0(\boldsymbol{x}^n, \boldsymbol{v}^n)$$
$$-\theta\ (\boldsymbol{v}^{n+1} \cdot \bigtriangledown + \boldsymbol{a}^{n+1} \cdot \partial_{\boldsymbol{v}}) \ln f_0(\boldsymbol{x}^{n+1}, \boldsymbol{v}^{n+1}),$$

  where $\boldsymbol{a} = \frac{q_i}{m_i}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$.
- Generalized Ohm's law:

$$(n_{i0} + \delta n_i^{n+1})(1 + \frac{m_e}{m_i} q_i^2)\boldsymbol{E}^{n+1} + \frac{m_e}{m_i}\frac{1}{\beta_e} \bigtriangledown \times (\bigtriangledown \times \boldsymbol{E}^{n+1})$$
$$= -(1 + \frac{m_e}{m_i} q_i)\delta \boldsymbol{j}_i^{n+1} \times (\boldsymbol{B}_0 + \delta \boldsymbol{B}^{n+1}) + \frac{1}{\beta_e}(\bigtriangledown \times \delta \boldsymbol{B}^{n+1}) \times \boldsymbol{B}_0$$
$$+\frac{1}{\beta_e}(\bigtriangledown \times (\boldsymbol{B}_0 + \delta \boldsymbol{B}^{n+1})) \times \delta \boldsymbol{B}^{n+1} + \frac{\eta}{\beta_e}(1 + \frac{m_e}{m_i} q_i^2)(n_{i0} + \delta n_i^{n+1}) \bigtriangledown \times \delta \boldsymbol{B}^{n+1}$$
$$-\nabla \delta n_i^{n+1} + \frac{m_e}{m_i} q_i\ \bigtriangledown \cdot P_i^{n+1},$$

Ion current and nonlinear terms

- The first term on the right hand side of the generalized Ohm's law involves the future ion current density

$$en_i(1+\frac{m_e}{m_i}\frac{q_i^2}{e^2})\boldsymbol{E}^{n+1}+\cdots=-(1+\frac{m_e}{m_i}\frac{q_i}{e})\ \delta\boldsymbol{j}_i^{n+1}\times\boldsymbol{B}^{n+1}+\cdots$$

we approximate $\delta\boldsymbol{j}_i^{n+1}$ as follows

$$\begin{aligned}\delta\boldsymbol{j}_i^{n+1} &= q_i\ \sum_j w_j^{n+1}\boldsymbol{v}_j^{n+1}\\ &= \delta\boldsymbol{j}_i^{\star}+q_i\ \theta\ \Delta t\ \sum_j \frac{q_i}{T_i}\boldsymbol{E}^{n+1}(\boldsymbol{x}_j^{n+1})\cdot\boldsymbol{v}_j^{n+1}\boldsymbol{v}_j^{n+1}\\ &\simeq \delta\boldsymbol{j}_i^{\star}+\theta\ \Delta t\ \frac{q_i^2}{m_i}\ \boldsymbol{E}^{n+1}\equiv\boldsymbol{J}_i'.\end{aligned}$$

- For accuracy issues, we iterate on the differences between $\delta\boldsymbol{j}_i^{n+1}$ and $\boldsymbol{J}_i'$.
- For every k_y and k_z mode, the generalized Ohm's law is solved in x direction using finite difference. The equilibrium part is solved by direct matrix inversion. And the nonlinear terms are treated iteratively.

Equilibrium and Boundary conditions

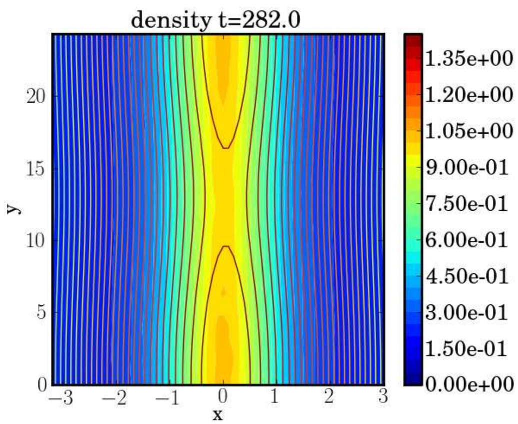
- Zero-order magnetic field $\boldsymbol{B}_0(\boldsymbol{x}) = B_{y0}\tanh(\frac{x}{a})\ \hat{\boldsymbol{y}} + B_G\ \hat{\boldsymbol{z}}$

- Perfect conducting wall boundary condition is employed in x while periodic boundary conditions in y and z direction.

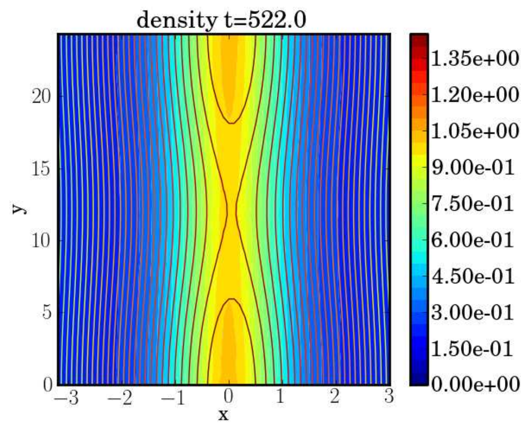
$$\boldsymbol{E}_{y,z}|_{x=\pm l_x/2} = 0$$
$$\delta\boldsymbol{B}_x|_{x=\pm l_x/2} = 0$$

- Particles are reflected at $x = \pm l_x/2$.

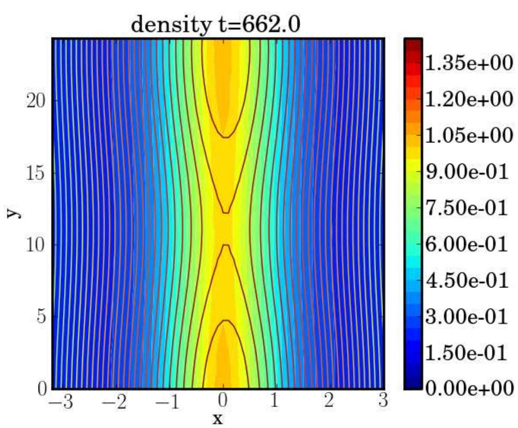
Island evolution I



(a)



(b)



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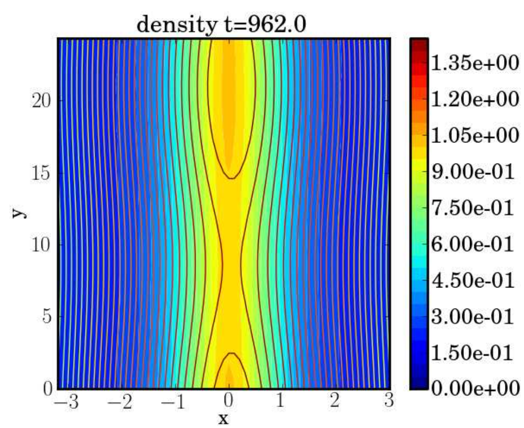
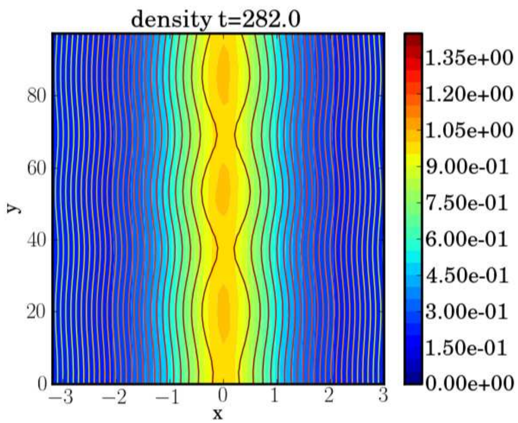


(d)

$128 \times 64 \times 16$ grids, 1048576 particles. $\frac{a}{\rho_i} = 1.0, \beta_e = \frac{\mu_0 n_0 T_e}{B_0^2} = 0.5$,
$\eta \frac{e n_0}{B_0} = 15 \times 10^{-4}$, $\frac{\boldsymbol{B}_G}{B_0} = 0$, $\frac{T_i}{T_e} = 1$, $\frac{l_x}{\rho_i} = 12.8$, $\frac{l_y}{\rho_i} = 25.12$

Island evolution II



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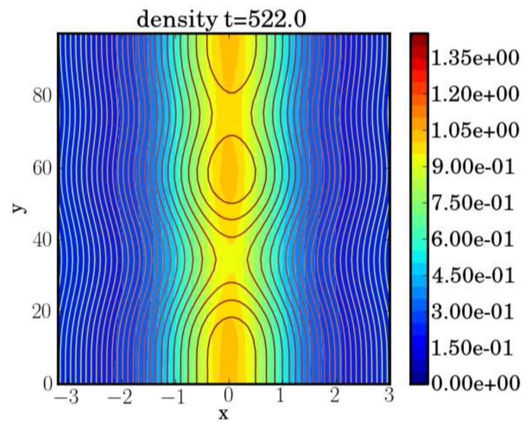

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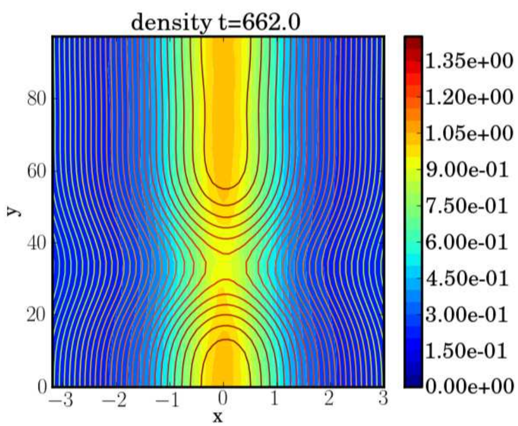

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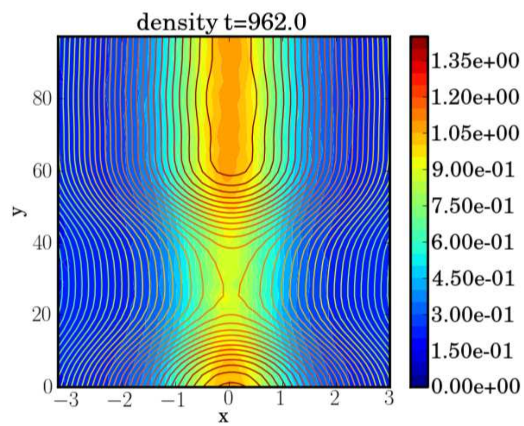
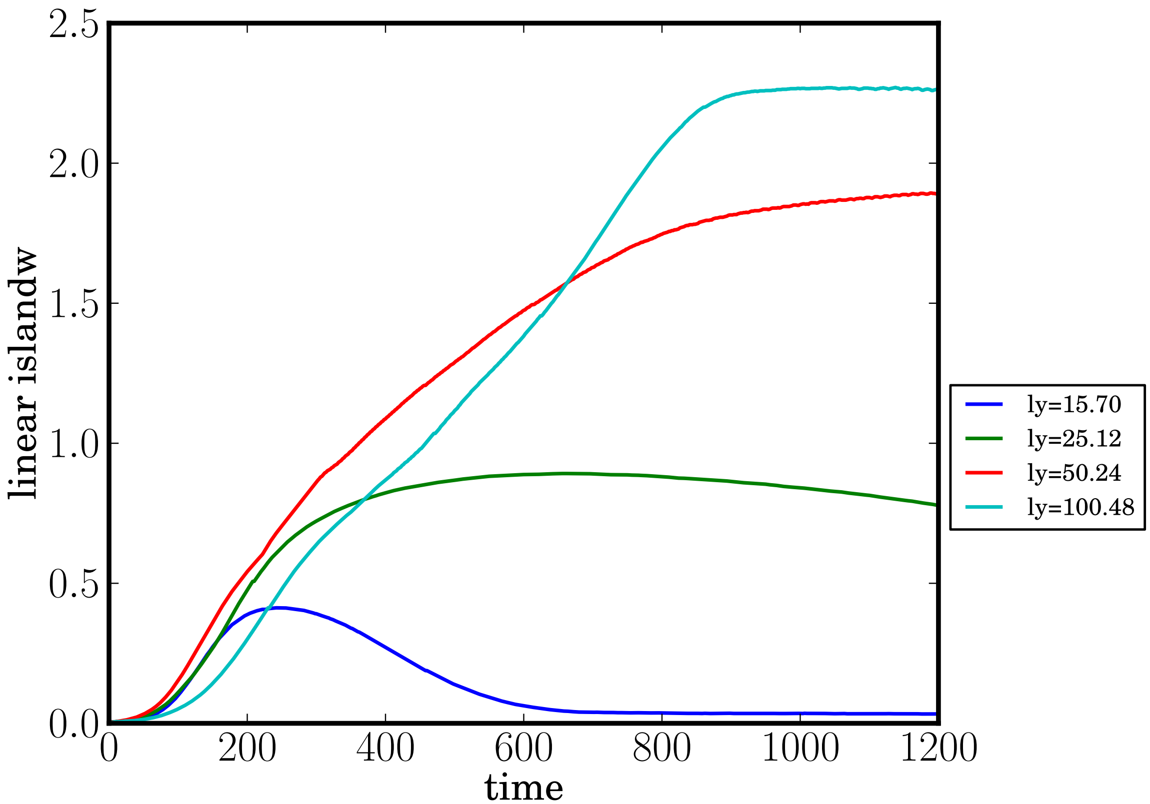


(d)

$128 \times 64 \times 16$ grids, 1048576 particles. $\frac{a}{\rho_i} = 1.0, \beta_e = \frac{\mu_0 n_0 T_e}{B_0^2} = 0.5,$
$\eta \frac{e n_0}{B_0} = 15 \times 10^{-4}, \frac{\boldsymbol{B}_G}{B_0} = 0, \frac{T_i}{T_e} = 1, \frac{l_x}{\rho_i} = 12.8, \frac{l_y}{\rho_i} = 100.48$

Island width with different aspect ratio

- Island growth with various aspect ratio.



$128 \times 64 \times 16$ grids, 1048576 particles. $\frac{a}{\rho_i} = 1.0, \beta_e = \frac{\mu_0 n_0 T_e}{B_0^2} = 0.5, \frac{\boldsymbol{B}_G}{B_0} = 0, \frac{T_i}{T_e} = 1, \frac{l_x}{\rho_i} = 12.8$

Energy conservation

- Take the second moment of the Vlasov equation, we have

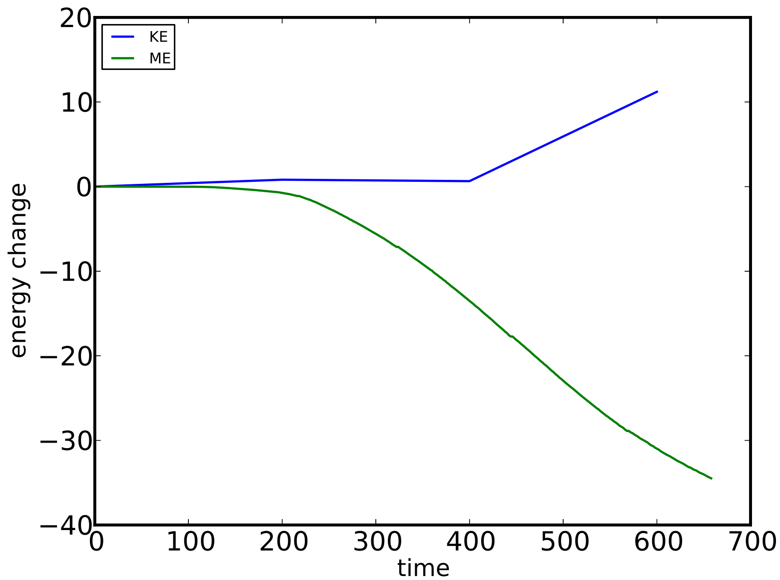
$$\frac{\partial}{\partial t}(KE) = \int \boldsymbol{E} \cdot \boldsymbol{j}_i \; d^3\boldsymbol{x}.$$

- With the generalized Ohm's law, the rhs is rewritten as

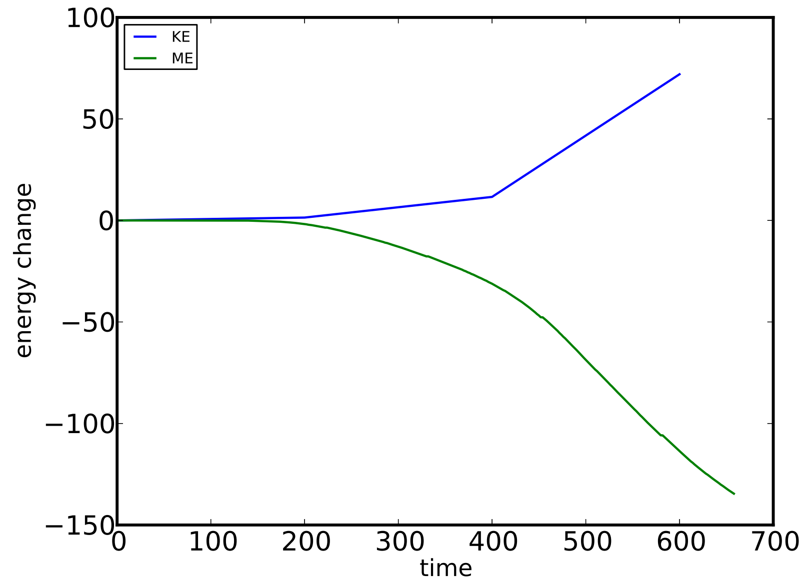
$$\int \boldsymbol{E} \cdot \boldsymbol{j}_i \; d^3\boldsymbol{x} = -\frac{1}{2\mu_0}\frac{\partial}{\partial t}\int \boldsymbol{B}^2 \; d^3\boldsymbol{x} - \int \frac{1}{\boldsymbol{B}^2}\left[(\frac{1}{\mu_0}\nabla \times \boldsymbol{B} - \boldsymbol{j}_i)\cdot \boldsymbol{B}\right]\boldsymbol{B}\cdot\boldsymbol{E} \; d^3\boldsymbol{x}$$
$$-\int \frac{\eta e n_0}{\mu_0}\frac{1}{\boldsymbol{B}^2}\nabla \times \boldsymbol{B} \times \boldsymbol{B}\cdot\boldsymbol{E} \; d^3\boldsymbol{x} + \int \frac{1}{\boldsymbol{B}^2}\nabla \cdot \boldsymbol{\Pi}_e \times \boldsymbol{B}\cdot\boldsymbol{E} \; d^3\boldsymbol{x}$$
$$+m_e \int \frac{1}{\boldsymbol{B}^2}\frac{\partial n_e \boldsymbol{u}_e}{\partial t} \times \boldsymbol{B}\cdot\boldsymbol{E} \; d^3\boldsymbol{x},$$

Magnetic energy and Ion kinetic energy

- The kinetic energy and magnetic energy change in the simulation



(a)



(b)

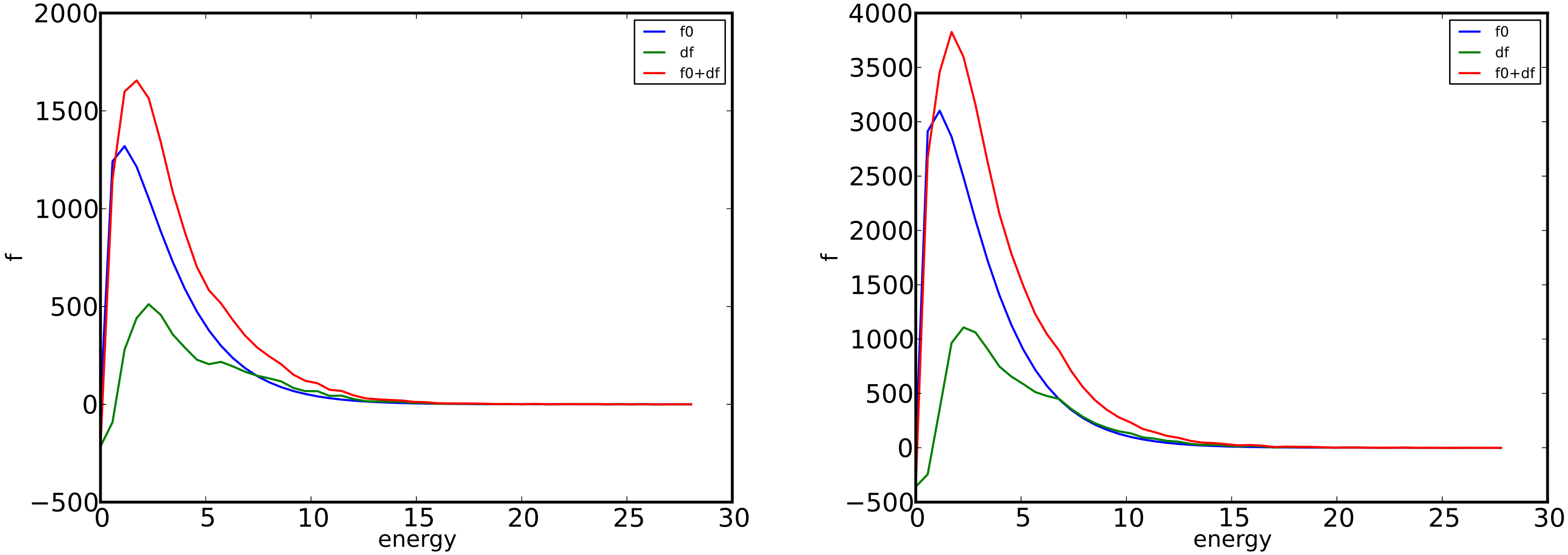
(a) $\frac{l_y}{\rho_i} = 50.24$. (b) $\frac{l_y}{\rho_i} = 100.48$

$128 \times 64 \times 16$ grids, 1048576 particles. $\frac{a}{\rho_i} = 1.0, \beta_e = \frac{\mu_0 n_0 T_e}{B_0^2} = 0.5,$

$\eta \frac{e n_0}{B_0} = 15 \times 10^{-4}, \frac{\boldsymbol{B}_G}{B_0} = 0, \frac{T_i}{T_e} = 1, \frac{l_x}{\rho_i} = 12.8$

Ion heating in the island region

- The distribution function clearly shows that ions are heated.



(a) (b)

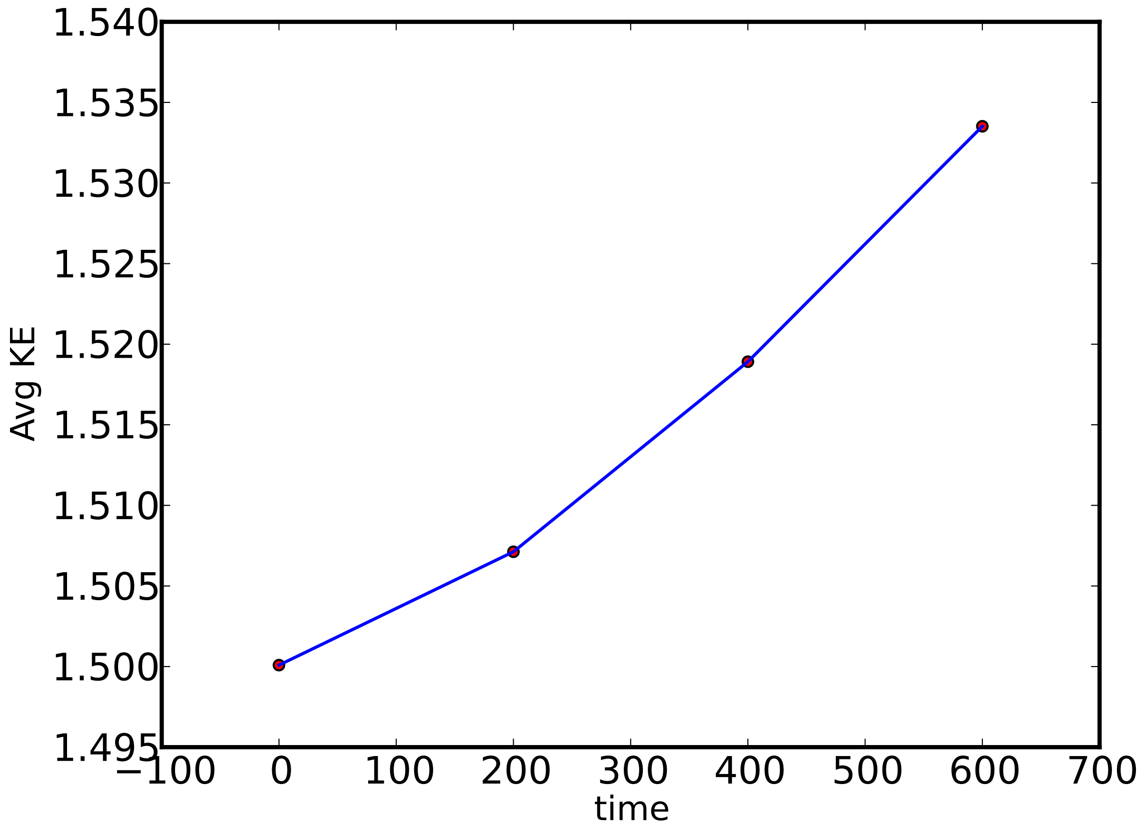
(a) $t = 400\Omega_i^-1$ (b) $t = 600\Omega_i^-1$

$128 \times 64 \times 16$ grids, 1048576 particles.

$\frac{a}{\rho_i} = 1.0, \beta_e = \frac{\mu_0 n_0 T_e}{B_0^2} = 0.5, \frac{\boldsymbol{B}_G}{B_0} = 0, \frac{T_i}{T_e} = 1, \frac{l_x}{\rho_i} = 12.8, \eta = 0.0015, \frac{l_y}{\rho_i} = 100.48$

Average kinetic energy of ions in the island region

- Looking into the island region, the average kinetic energy of ions increases about 13%.



$128 \times 64 \times 16$ grids, 1048576 particles.

$\frac{a}{\rho_i} = 1.0, \beta_e = \frac{\mu_0 n_0 T_e}{B_0^2} = 0.5, \frac{\boldsymbol{B}_G}{B_0} = 0, \frac{T_i}{T_e} = 1, \frac{l_x}{\rho_i} = 12.8, \eta = 0.0015, \frac{l_y}{\rho_i} = 100.48$

Summary

1. We have implemented a second-order accurate implicit algorithm with Lorentz force ions and isothermal fluid electrons which is
   - Quasi-neutral and fully electromagnetic.
   - Suitable for MHD scale plasmas.

2. Secondary islands in the strong tearing mode.
   - For large Δ' tearing mode, we have observed multiple islands forming and coalescence.
   - Using particle ion diagnostics, we have shown that ions are heated in the island region and compared with the magnetic energy change.

3. Future work
   - Further detailed diagnostics about the energy conversion in the process.
   - Use tracer particles to study the particle behaviors (e. g. dissipation) around the X and O points of the islands.