Spectral-element convergence studies with NIMEQ

Eric C. Howell

NIMROD Meeting October 27, 2012

NIMEQ solves the Grad-Shafranov equation to calculate axis-symmetric static MHD equilibria.

- $\Delta^* \psi = \frac{\partial^2 \psi}{\partial z^2} + R \frac{\partial}{\partial R} \frac{1}{R} \frac{\partial \psi}{\partial R} = -\mu_0 R^2 \frac{dP}{d\psi} - F \frac{dF}{d\psi}$
- $\vec{B} = \nabla \phi \times \nabla \psi + F \nabla \phi$
- $\vec{J} = \Delta^* \psi \nabla \phi + \frac{dF}{d\psi} \nabla \phi \times \nabla \psi$
- User specifies the flux functions $P(\psi)$ and $F(\psi)$
  - A normalized ring flux is used to constrain ψ.
  - $\hat{\psi} = \frac{\psi - \psi_{axis}}{\psi_{sep} - \psi_{axis}}$

To assure regularity near the geometric axis, NIMEQ solves for the field $\frac{\psi}{R^2}$

- After substituting A in for $\frac{\psi}{R^2}$, the Grad-Shafranov operator is converted into a pure divergence.
  - $\Delta^* \psi = \nabla \cdot R^2 \nabla A$
- In axisymmetry, Taylor expanding ψ in R near the magnetic axis yields:
  - $\psi = \psi_0 + \psi_2 R^2 + \ldots$
  - $A = \frac{\psi}{R^2} = \frac{\psi_0}{R^2} + \psi_2 + \ldots$
- The physical fields only depend on derivatives of ψ, pick $\psi_0 = 0$ and then A is well behaved near the geometric axis.

NIMEQ solves the weak form of the G-S equation.

- The solution field is split into a homogeneous and an inhomogeneous part.
  - $A = A_h + A_b$
- The homogeneous part is projected onto NIMROD's spectral element basis $\alpha_i(R, Z)$.
  - $A_h = \sum_i A_i \alpha_i$
- The Grad-Shafranov equation is multiplied by the test functions α_j and then integrated by parts.
  - $M_{ij} A_i = \sum_i A_i \int dV R^2 \nabla \alpha_i \cdot \nabla \alpha_j$
  - $Q_j = \int dV [(FF' - \mu_0 R^2 P')\alpha_j - R^2 \nabla A_b \cdot \nabla \alpha_j]$
  - $M_{ij} A_i = Q_j$
- A modified Picard iteration is used to solve the nonlinear equation.
  - $M_{ij} A_i^n = (1 - \theta) M_{ij} A_i^{n-1} + \theta Q_j^{n-1}$

Direct comparison is made between NIMEQ calculated field ψ_{eq} and analytic solutions ψ_{an}.

- For this talk I will define error as:
  $error = \sqrt{\sum_{nodes}(\psi_{eq} - \psi_{an})^2}$
- h-convergence studies are performed by increasing the number of elements at fixed polynomial degree.
  - Algebraic convergence
  - Straight line on a log-log plot.
- p-convergence studies keep the number of elements fixed while increasing the polynomial degree.
  - Geometric convergence is expected for smooth solutions.
  - Straight line on a log-linear plot.

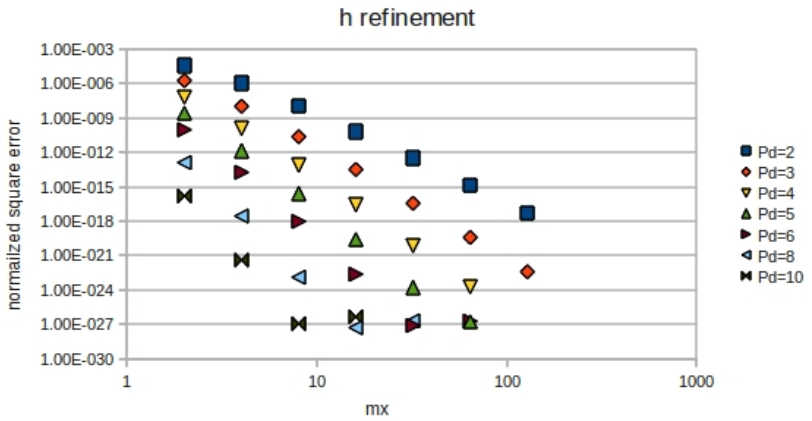
The spectral energy of the highest order term provides an additional measure of the error.

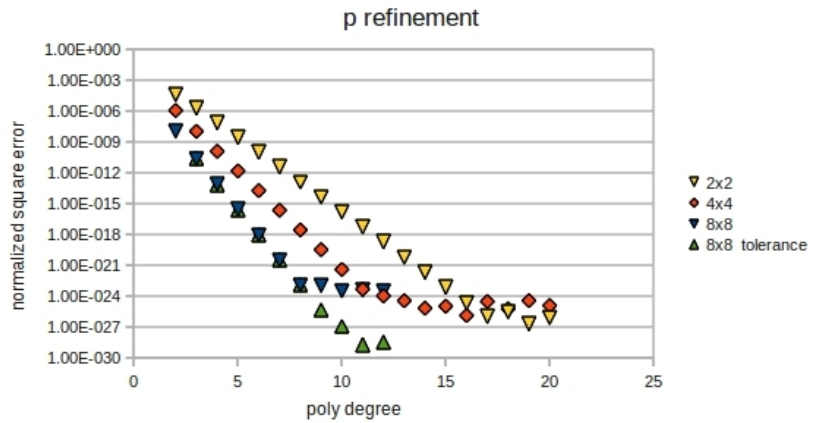
- Rule of thumb: Truncation error is comparable to the magnitude of the highest order term kept in the expansion.
- To calculate the energy NIMEQ solution is projected onto Legendre polynomials within each element.
  - Uses Josh's subroutine.
- Similar to error estimates for a truncated Taylor or Fourier series.

Solev'ev Equilibrium (Simple Tokamak Case)

- $\mu_0 P'(\Psi) = -C$ and $FF' = A$
- $\psi = -\frac{A}{2}Z^2 + \frac{C}{8}R^4 + c_1 + c_2 R^2 + c_3(R^4 - 4R^2 Z^2)$
- Remember that NIMEQ solves for ψ/R^2.
  - The terms $AZ^2/2R^2$ and c_1/R^2 require high order spectral elements to fully resolve.

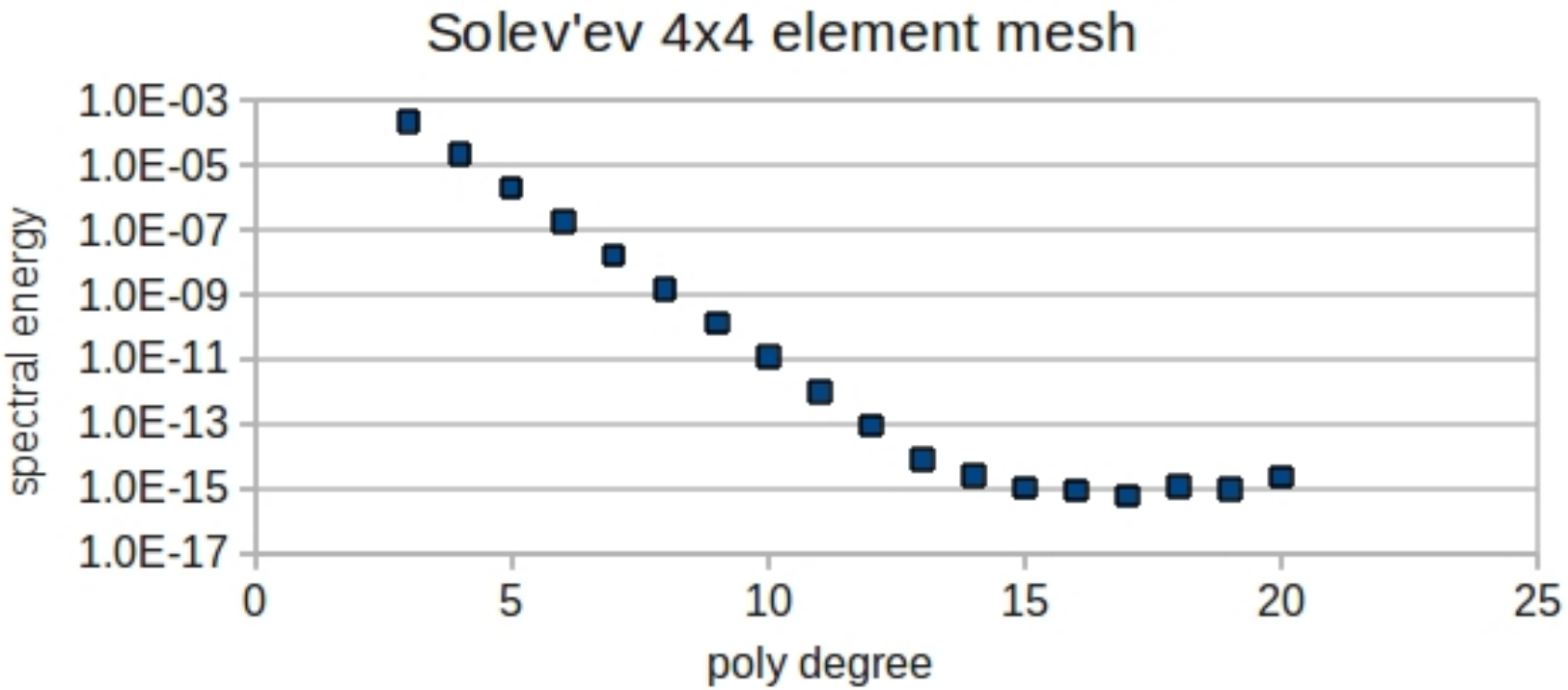
Good h- and p-convergence is observed for the Solev'ev equilibrium





- Error in ψ is found to scale as $mx^{-(p+2)}$ for h-refinement.
- At moderate resolutions, the accuracy is limited by double precision arithmetic.

Spectral energy calculations of Solev'ev equilibrium show geometric convergence

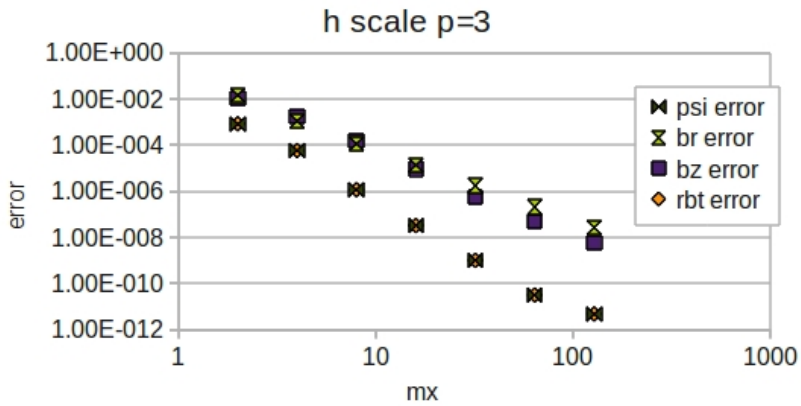


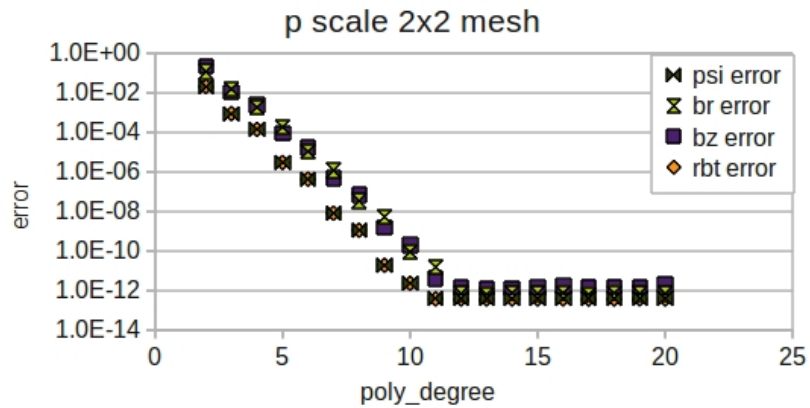
- Qualitatively agrees with p-refinement studies.

Isolated constant λ $0 - \beta$ spheromak

- $F = f_1(1 - \hat{\psi})$
- The solution has the form $\psi \sim -rsin(\pi z/H)J_1(x_{11}r/R)$
- x_{11} is the first zero of J_1
- There are two tricks to creating constant lambda spheromaks in NIMEQ.
  - If you use $\lambda = -F'$ with the physical flux, then the problem reduces to an eigenvalue problem.The nonlinear solver will not converge:
  - The ring flux is calculated from the max value of ψ on a node. If the magnetic axis does not lie on a node, this will introduce a small error, and degrades convergence.
- These equilibria include the geometric axis in their domain and tests regularity.

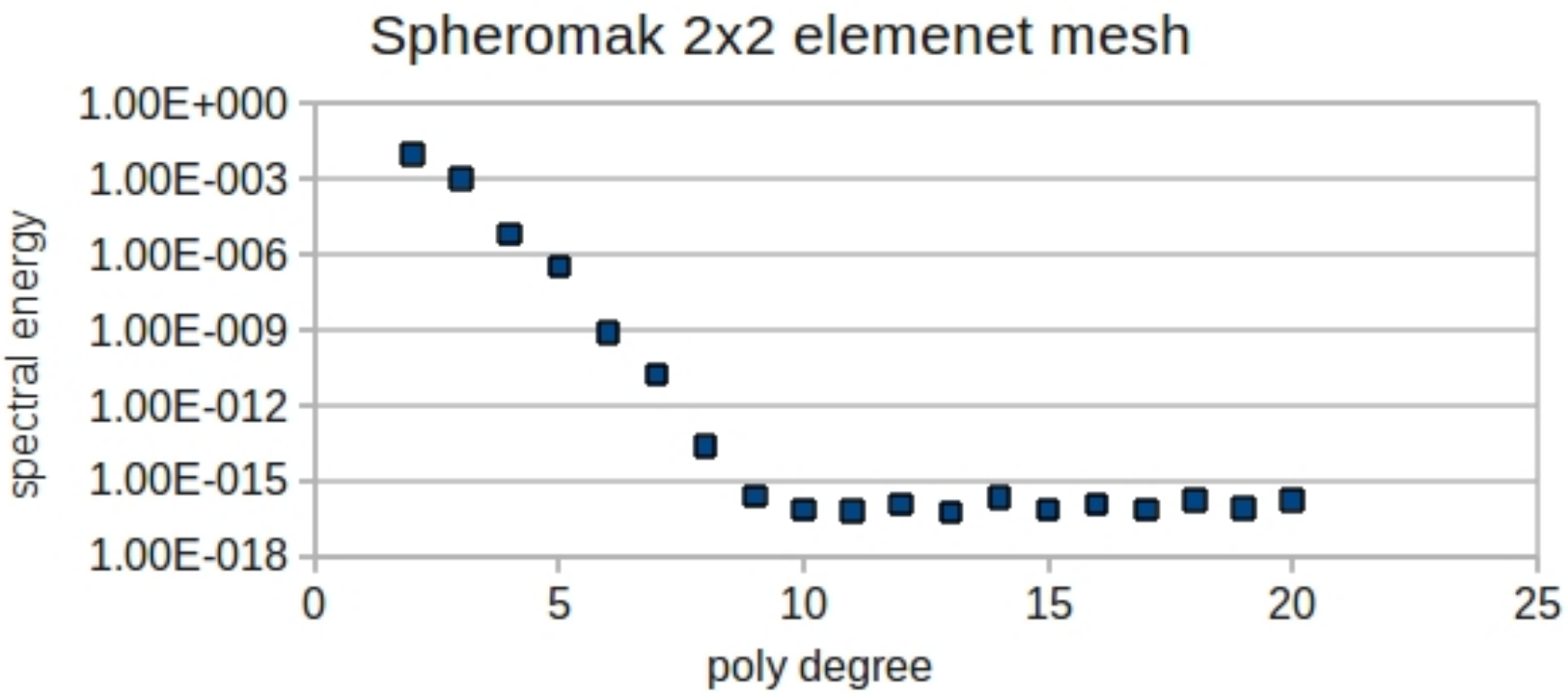
h- and p-convergence test using spheromak equilibria show excellent convergence.





- Error in ψ and RB_ϕ are found to scale as $mx^{-(p+2)}$ for h-refinement.
- Error in B_z and B_r are found to scale as $mx^{-(p)}$ for h-refinement.
- NIMEQ calculates RB_ϕ directly from ψ using F.
- B_r and B_z are calculated from $\nabla\phi \times \nabla\psi$ using a mass matrix inversion.

Spectral energies of spheromak equilibrium on a 2x2 mesh

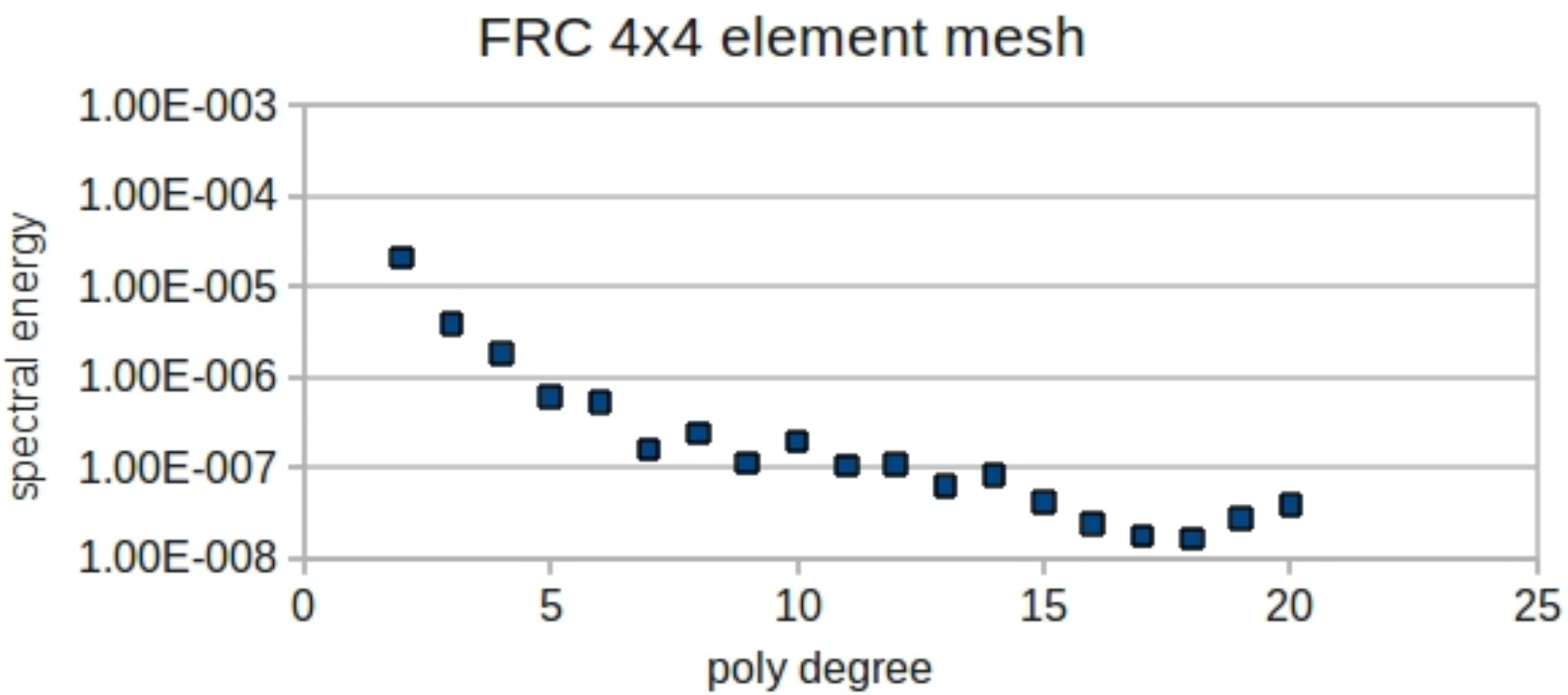


- Good spectral convergence is observed up to machine error.

FRC equilibrium

- Uniform background vertical magnetic field
- $F = 0$
- $\mu_0 P = p_{open} + p_1(1 - Y) + 4p_2 y(y - 1)$
- $\mu_0 P = p_{open}$ on the open field

The spectral energies of the FRC equilibrium on a 4x4 mesh



- The FRC equilibrium has a poorer convergence due to the discontinuity in $P'(\psi)$.

Conclusions

- NIMEQ correctly solves the Grad-Shafranov equation to high precision.
- Algebraic convergence is observed for h-refinement.
- Geometric convergence is observed for p-refinement if the solution is smooth.
- NIMEQ is accurate to Machine level error for modest meshes for smooth equilibria.
  - 4x4 polynomial degree 10 for ideal spheromak
- The rule of thumb spectral error test qualitatively agrees with direct calculations of error.