Nonlinear Modeling of Mode Locked States Induced by Transient Magnetic Perturbations

M. T. Beidler, J. D. Callen, C. C. Hegna, and C. R. Sovinec

Department of Engineering Physics,
University of Wisconsin

Supported by DOE OFES grants DE-FG02-92ER54139, DE-FG02-86ER53218, and the U.S. DOE FES Postdoctoral Research program administered by ORISE and managed by ORAU under DOE contract DE-SC0014664. This research used resources of the National Energy Research Scientific Computing Center, a DOE Office of Science User Facility supported by the Office of Science of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231.
Motivation: ELM Can Precipitate Transition to ELM-Free State

• External 3D fields force magnetic reconnection (FMR), whose islands can lock plasma in place to 3D field structure
  • Fundamental physics controlled by external forcing, flow, resistivity, and viscosity
• Transient MHD events [sawteeth and edge localized modes (ELMs)] are an additional source of 3D fields that may effect FMR processes
  • Figure shows post-ELM evolution to ELM-suppressed state for large background resonant magnetic perturbation (RMP) compared to small RMP (black and grey traces, respectively)
    • Paz Soldan et al., PRL (2015); Nazikian et al., PRL (2015); Callen et al., UW-CPTC Report 16-4
Motivation: ELM and Sawteeth Can Precipitate NTM Growth

- External 3D fields force magnetic reconnection (FMR), whose islands can lock plasma in place to 3D field structure
  - Fundamental physics controlled by external forcing, flow, resistivity, and viscosity
- Transient MHD events [sawteeth and edge localized modes (ELMs)] are an additional source of 3D fields that may effect FMR processes
  - Figure shows m/n=2/1 NTM triggered by ELM after perturbation from previous ELM and sawtooth decayed
    - La Haye, private communication (2016)
Outline

• Model of mode penetration due to transient perturbation
  • Time-asymptotic Quasilinear Force balance
  • Self-consistent evolution of force balance during transient

• Computational results of penetration dynamics
  • Comparison of analytic and computation results
  • Penetration for large magnitude and duration transient
Mode Penetration Determined by Transient-Induced Force Evolution

• Begin in time-asymptotic, metastable state
  • External 3D magnetic field $B_{\text{ext,0}}$ flow-screened at rational surface
  • Electromagnetic (EM) and viscous forces balance at rational surface

• Transient 3D field $B_{\text{ext,T}}$ added to $B_{\text{ext,0}}$
  • EM force increases due to greatly increased current and evolving magnetic field at rational surface
  • Flow at rational surface decreases due to EM force, affecting flow-screening of 3D field; flow profile evolution induces large viscous force

• Transient turns off and system continues to evolve
  • Mutual evolution of forces during transient determine final state
  • If flow is decreased enough, evolution determined by nonlinear modified Rutherford equation (MRE)
Taylor’s Slab Model Paradigm Explored to Simplify Analytics

- Slab geometry with uniform out-of-plane current density
  - Stable equilibrium with $\Delta' a \cong -2k_y a$
  - $a = 1\text{m}, B_{y,0}(x=a) \equiv B_{\text{norm}} = 0.1\text{T}, B_{z,0}(x=0) \equiv B_0 = 100 B_{\text{norm}}$

- Sinusoidal applied normal magnetic field at edge $B_{x,1}(|x|=a) = B_{\text{ext}} \sin(k_y y)$
  - Drives reconnection at $x = 0$
  - $k_y = 2\pi n/L_y \Rightarrow k_y(n=1) = \pi$

- Visco-Resistive (VR) dissipation
  - $S = 1.1 \times 10^7, P_m = 20$
  - Linear layer width: $\delta_{\text{VR}} = S^{-1/3} P_m^{1/6} a = 7.4 \times 10^{-3} a$
  - Viscosity profile $\sim [1 + (\Delta_{\text{mag}}^{1/2} - 1)(x'/a)^{\Delta_{\text{width}}}]^2$

- Constant flow with $V_{y,0}$
  - $V_{\text{norm}} = V_A(B_{\text{norm}}) \Rightarrow t_{\text{norm}} = a / V_{\text{norm}} \equiv \tau_A = 1.45 \mu\text{s}$
  - Flow frequency $\omega = k \cdot V = k_y V_y$
Electromagnetic and Viscous Forces Balance At Rational Surface

• Quasilinear $n=0$ force EM force per unit length in $z$ at $x=0$

$$\hat{F}_{y,EM} = \int_{-\delta y/2}^{\delta y/2} dx \int_{-L_y/2}^{L_y/2} dy (J \times B) \cdot \hat{y} = -\frac{n\pi}{\mu_0} \text{Im} \{ \psi^*_\text{res} \left[ \partial_x \psi_{\text{res}} \right]_{x=0} \}$$

• Linear, time-asymptotic, VR, field response:
  • $J_{z,1}$ in phase with $B_{\text{ext}}$, $B_{x,1}$ is phase shifted
  • $F_{y,EM}$ localized at $x=0$

$$\hat{F}_{y,EM} = -\frac{\omega_{\text{res}} \tau_{\text{VR}}}{(-a\Delta')^2 + (\omega_{\text{res}} \tau_{\text{VR}})^2} \frac{n\pi (a\Delta_{\text{ext}}')^2}{\mu_0} \frac{B_{\text{ext}}^2}{ak_y^2}$$

• Viscous force per unit length in $z$ at $x=0$

$$\hat{F}_{y,VS} = \int_{-\delta y/2}^{\delta y/2} dx \int_{-L_y/2}^{L_y/2} dy \left[ \nabla \cdot \rho \nu \nabla \nu \right] \cdot \hat{y}$$

$$= \frac{L_y \rho \nu_0}{k_y} \left[ \varrho(x) \partial_x \omega(x, t) \right]_{x=0}$$

• Viscosity profile $\nu(x) = \nu_0 \varrho(x)$ moves no-slip boundary from $a$ to $a_{\nu} \equiv \int_0^a \left[ dx / \varrho(x) \right]$

• For $0 < x < \vert a \vert \nu$, $\partial_{xx} \omega = 0$ yields $\omega(x) = \omega_{\text{res}} + (\omega_0 - \omega_{\text{res}}) (x/\vert a \vert \nu)$

$$\hat{F}_{y,VS} = \frac{4\pi \rho \nu_0}{a_{\nu} k_y^2} \left( \omega_0 - \omega_{\text{res}} \right)$$
EM and Viscous Force Balance Gives Rise to Bifurcation

- Time-asymptotic force balance gives cubic relation for \( \omega_{\text{res}} \)
  \[
  \frac{\omega_0}{\omega_{\text{res}}} - 1 + \omega_0 \omega_{\text{res}} \tau_{\text{VR}}^2 - \omega_{\text{res}}^2 \tau_{\text{VR}}^2 = \frac{a \nu \tau_{\text{VR}}}{4 \alpha \rho \nu_0} \left( \frac{\Delta'_{\text{ext}}}{-\Delta'} \right)^2 \frac{B_{\text{ext}}^2}{\mu_0}
  \]

  Here, \( \tau_{\text{VR}} = 2.104 \tau_A S^{2/3} P_m^{1/6} \) and \( \tau'_{\text{VR}} \equiv \tau_{\text{VR}} / (-a \Delta') \)

  System bifurcates and exhibits hysteresis for \( \omega_0 > 3 \sqrt{3} / \tau'_{\text{VR}} \)

- Metastable: two stable equilibria are flow-screened and mode-penetrated
  - Shaded region is metastable
  - Unstable equilibrium (○) divides phase space into regions that approach one of the two stable equilibria (●)
Effect of Magnetic Transient Depends on EM and Viscous Force Evolution

• Hypothesis: if transient causes flow to decrease below unstable equilibrium, mode will penetrate

• Flow evolution equation with viscous and EM forces:

\[
\frac{L_y \delta_{VR} \rho}{k_y} \partial_t \omega_{res} = \hat{F}_{VS} + \hat{F}_{EM}
\]

\[
= \frac{L_y \rho v_0}{k_y} \left[ \partial_x \omega \right]_{x=0} - \frac{n \pi}{\mu_0} \text{Im} \{ \psi_{res}^{*} \left[ \partial_x \psi_{tot} \right]_{x=0} \}
\]

• Transient magnetic perturbation causes forces to evolve
  • Directly increases EM force local to the rational surface
  • Local change in flow profile increases viscous force
  • Attempts to limit flow changes
EM Force Evolution Depends on Evolution of Penetrated Field

- Evaluate $\hat{F}_{y,EM} = -\frac{n \pi}{\mu_0} \text{Im} \left\{ \psi^*_\text{res} \left[ \partial_x \psi_{\text{tot}} \right]_{x=0} \right\}$ with evolving $\psi_{\text{res}}$
  
  - Induction equation integrated across rational surface yields evolution equation for $\psi_{\text{res}}$:
    $$\tau_{VR} \left[ \partial_t + i \omega_{\text{res}} \right] \psi_{\text{res}} = a \left[ \partial_x \psi_{\text{tot}} \right]_{x=0} \equiv a \Delta' \psi_{\text{res}} + a \Delta'_{\text{ext}} \psi_{\text{ext}}$$

  - RHS is $\propto$ current; $\Delta'_{\text{ext}}$ responds on Alfvénic timescale

- Analytically treat magnetic transient as composed of step functions:
  $$\psi_{\text{ext}}(t) = \psi_{\text{ext},0} + \psi_{\text{ext},T} \left[ H(t) - H(t - \Delta t_T) \right]$$

  - Next step: re-derive with Gaussian transient

- Solve induction equation by Laplace transform and evaluate EM force in each phase
  
  - Phase 2:
    $$\hat{F}_{EM} = -\frac{n \pi}{\mu_0} \frac{a (\Delta'_{\text{ext}})^2}{k_y^2 \tau_{VR}} (B_{\text{ext},0} + B_{\text{ext},T}) \left( \frac{B_{\text{ext},0}}{\omega_{\text{res}}} + \frac{\omega_{\text{res}} t^2}{2} B_{\text{ext},T} \right)$$

  - Phase 3:
    $$\hat{F}_{EM} = -\frac{n \pi}{\mu_0} \frac{a (\Delta'_{\text{ext}})^2}{k_y^2 \tau_{VR}} B_{\text{ext},0} \left[ \frac{B_{\text{ext},0}}{\omega_{\text{res}}} + B_{\text{ext},T} \omega_{\text{res}} \Delta t_T \left( t - \frac{\Delta t_T}{2} \right) \right]$$

  - $J_{z,1}(t)$ in phase and responds with $\psi_{\text{ext}}(t)$; $B_{x,1}$ is phase shifted and evolves on timescale $\geq \tau_A S^{2/3}$
Viscous Force Evolution Depends on Evolution of Flow Profile

• Evaluate \( \hat{F}_{y,V_S} = \frac{L_y \rho \nu_0}{k_y} \left[ \partial_x \omega(x, t) \right]_{x=0} \) with evolving \( \omega(x) \)

• EM force is localized at \( x=0 \); flow profile is solution of \( \partial_t \omega(x, t) = \nu_0 \partial_{xx} \omega(x, t) \) in \( 0 < |x| < a_\nu \)

\[
\omega(x, t) = \omega_{\text{res}}(t) + [\omega_0 - \omega_{\text{res}}(t)] \left( \frac{x}{a_\nu} \right) - \sum_{n=1}^{\infty} \sin \left( \frac{n \pi x}{a_\nu} \right) e^{-\left(\frac{n \pi}{a_\nu}\right)^2 \frac{t}{\tau_\nu}}
\]

\[
\times \left\{ \frac{2}{a_\nu} \int_0^{a_\nu} dx \left[ \omega(x) - \left\{ \omega_{\text{res}}(0) + [\omega_0 - \omega_{\text{res}}(0)] \left( \frac{x}{a_\nu} \right) \right\} \right] \right\}
\times \sin \left( \frac{n \pi x}{a_\nu} \right) - \frac{2}{n \pi} \int_0^t d\tau \partial_\tau \omega_{\text{res}}(\tau) e^{\left(\frac{n \pi}{a_\nu}\right)^2 \frac{\tau}{\tau_\nu}}
\]

• Time-asymptotic; Initial flow profile; Evolving \( \omega_{\text{res}}(t) \)

• Component due to initial profile measures difference from time-asymptotic profile; vanishes for present case

• Evaluating jump in gradient of flow across rational surface and truncating sum, viscous force yields:

\[
\hat{F}_{VS} = \frac{2 a_\nu L_y \rho}{k_y} \left\{ \frac{\omega_0 - \omega_{\text{res}}(t)}{\tau_\nu} - \frac{2 \omega_{\text{res}}(t)}{\pi} \sqrt{\frac{t}{\tau_\nu}} \right\}
\]
Approximate Solution to Flow Evolution Gives Threshold for Mode Penetration

- Flow evolution equation with viscous and EM forces:
  \[ \frac{L_y \delta_{\text{VR}} \rho}{k_y} \dot{\omega}_{\text{res}} = \dot{F}_{\text{VS}}(\dot{\omega}_{\text{res}}, \omega_{\text{res}}, t) + \dot{F}_{\text{EM}}(\omega_{\text{res}}, \omega_{\text{res}}^{-1}, t) \]

- After cancelling time-asymptotic EM and viscous force balance
  - **Phase 2**: for \(0 < t < \Delta t_T\) : \(\dot{\omega}_{\text{res}} = \dot{A}_{\text{net}}(\dot{\omega}_{\text{res}}, \omega_{\text{res}}, 1, \omega_{\text{res}}^{-1}, t)\)
  - **Phase 3**: for \(\Delta t_T < t\) : \(\dot{\omega}_{\text{res}} = \dot{A}_{\text{net}}(\dot{\omega}_{\text{res}}, \omega_{\text{res}}, 1, t)\)

- General nonlinear ODE with non-constant coefficients is challenging to solve analytically
  - Solution for equation of form \(\dot{\omega}_{\text{res}} + f(t)\omega_{\text{res}} = g(t)\) yields
    \[ \omega_{\text{res}}(t) = \omega_{\text{res}}(t_0) e^{[\mu(t_0) - \mu(t)]} + \int_{t_0}^{t} d\tau g(\tau) e^{[\mu(\tau) - \mu(t)]}, \text{ with } \mu(t) \equiv \int dt f(t) \]

- Assume \(\omega_{\text{res}}^{-1}\) constant to solve phase 2; phase 3 solved exactly
  - Relative magnitude of \(g(t)\) and \(f(t)\) determines final state
Outline

• Model of mode penetration due to transient perturbation
  • Time-asymptotic Quasilinear Force balance
  • Self-consistent evolution of force balance during transient

• Computational results of penetration dynamics
  • Comparison of analytic and computation results
  • Penetration for large magnitude and duration transient
NIMROD Code Employed to Solve Visco-Resistive MHD Equations

- NIMROD capable of solving extended-MHD equations
  \[ \rho \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = \mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbf{\Pi}_i , \]
  \[ \mathbf{\Pi}_i \equiv -\rho \nu \left[ \nabla \mathbf{V} + \nabla \mathbf{V}^T - \frac{2}{3} \nabla \cdot \mathbf{V} \right] , \]
- Time discretization uses finite difference
  \[ \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} , \quad \mu_0 \mathbf{J} = \nabla \times \mathbf{B} , \]
  \[ \mathbf{E} = -\mathbf{V} \times \mathbf{B} + \eta \mathbf{J} \]
- Semi-implicit leapfrog time evolution
- Evolve perturbation fields about a fixed equilibrium
- Spatial discretization uses 2D, $C^0$, spectral elements
  - Third dimension uses finite Fourier series; ignorable here
Large Transient Induces Overdriven Mode Penetration

- **System parameters**
  - \( S = 1.1 \times 10^7 \)
  - \( P_m = 20 \)
  - \( V_0 = 500 \text{ m/s} \)
  - \( B_{\text{ext},0} = 3 \times 10^{-4} \text{ T} \)

- **Transient parameters**
  - 1.75 ms approximately square pulse
  - \( B_{\text{ext},T} = 9 B_{\text{ext},0} \)
  - Overdrives final time-asymptotic state
Transient Precipitates Transition From Flow-Screened to Mode-Penetrated Island

- Metastable state at $B_{\text{ext},0}$ allows for two stable equilibria
- Magnetic island just before (top), and long after (bottom) transient perturbation
- Flow decrease triggered by transient allows magnetic island to open dramatically
- Nonlinear MRE governs evolution to mode penetrated equilibrium
Analytical and Computational Flow Evolution Compare Favorably

- Transient parameters
  - 1 ms approximately square pulse with a finite activation time
  - $B_{\text{ext,T}} = 7.75 B_{\text{ext,0}}$

- Calculate $g(t)$ and $f(t)$ for analytical solution in phase 2
  - Calculate integral with $g(t)$ numerically, updating $\omega_{\text{res}}$ as time advances

- Evolution of forces (bottom figure) consistent with flow evolution
  - Time-dependent viscous force (green trace) slows flow evolution due to EM force (red trace)
  - Small contribution from time-asymptotic viscous force (blue trace)
Magnitude of Transient Accelerates Flow Response

- **System parameters**
  - \( S = 1.1 \times 10^7 \)
  - \( P_m = 20 \)
  - \( V_0 = 500 \text{ m/s} \)
  - \( B_{\text{ext,0}} = 3 \times 10^{-4} \text{ T} \)

- **Pulse parameters**
  - 1 ms approximately square pulse
  - \( B_{\text{ext,T}} = 7.75 B_{\text{ext,0}} \) mode penetrates, locks
  - \( B_{\text{ext,T}} = 6.5 B_{\text{ext,0}} \) transitions back to high slip state
Mode Penetration Threshold Is Also Evident in Flow-Field Phase Space
Computed Field Response is Similar to Experimental Observations

- $B_{\text{ext}}/B_0$
  - $B_{\text{ext},T} = 7.75 B_{\text{ext},0}$
  - $B_{\text{ext},T} = 6.5 B_{\text{ext},0}$

- $\omega_{\text{res}} \tau_A$

- $B_{\text{res}}/B_0$

- $w_{\text{island}}/2\delta_{VR}$
Longer Transient Produces Increased Flow Response

- System parameters
  - \( S = 1.1 \times 10^7 \)
  - \( P_m = 20 \)
  - \( V_0 = 500 \text{ m/s} \)
  - \( B_{\text{ext},0} = 3 \times 10^{-4} \text{ T} \)

- Transient parameters
  - \( B_{\text{ext},T} = 9 \, B_{\text{ext},0} \) magnitude
  - \( \Delta t_T = 1 \, \text{ms} \) mode penetrates
  - \( \Delta t_T = 0.5 \, \text{ms} \) transitions back to high slip state
Metastable State Determines Magnitude of Transient Needed for Mode Penetration

- System parameters
  - $P_m = 20$ (blue traces) mode penetrates
    - $B_{\text{ext},0} = 3 \times 10^{-4}$ T
  - $P_m = 2$ (red traces) transitions back to high slip state
    - $B_{\text{ext},0} = 9 \times 10^{-5}$ T

- Transient parameters
  - $B_{\text{ext},T} = 9 B_{\text{ext},0}$ Gaussian transient
    - 1 ms full width at half max

- Mode penetration sensitive to metastable state
  - Equivalent transient develops forces smaller by an order of magnitude for low viscosity case
Relevance of Results to Present Experimental Investigations

- Present study relevant to transient MHD, RMP physics
- Analogous seeding of NTMs by ELMs and Sawteeth
  - DIII-D ROF 618: “Better Understanding of NTM Seeding”
  - Need highly resolved data of field and flow response at rational surface preceding and following MHD transients
    - $\Delta x \sim$ cm and $\Delta t \sim$ ms during transient
- Augment modeling with more complete physics in cylindrical and toroidal geometries
  - Improve analytic framework and extended-MHD codes
  - Nonlinear model that improves on linear response calculations
Conclusions

• Model of mode penetration due to transient perturbation is developed from:
  • Quasilinear EM and viscous forces in slab geometry
  • Self-consistent evolution of force balance during transient

• Computational results of mode penetration dynamics
  • Favorable comparison between analytic and computation results
  • Penetration for large magnitude and duration transient, sensitive to metastable state

Take-away: MHD transients can precipitate the transition of metastable plasmas into low-slip, locked mode states