Higher-Order Advection-Based Remap of Magnetic Fields: Update

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The discrete de Rham complex illustrates important properties of the $H(\nabla \times)$ and $H(\nabla \cdot)$ conforming finite element spaces.

Continuous and discrete de Rham sequences:

$$
\begin{array}{cccccc}
H(\nabla) & \downarrow \nabla & H(\nabla \times) & \downarrow \nabla \times & H(\nabla \cdot) & \downarrow \nabla \cdot & L^2 \\
\downarrow \Pi \nabla & & \downarrow \Pi \nabla \times & & \downarrow \Pi \nabla \cdot & & \downarrow \Pi^0 \\
V \nabla & \rightarrow & V \nabla \times & \rightarrow & V \nabla \cdot & \rightarrow & Q \\
G & & C & & D & & 
\end{array}
$$

Important properties:

- The interpolation operators ($\Pi$) commute with the differential operators.
- The discrete operators have the appropriate kernels (direct result of commuting diagram property), i.e. $Gx \in \ker(C)$ and $Cx \in \ker(D)$
- $H(\nabla \times)$ conforming elements have continuity of tangential components only
In the BLAST code, the arbitrary Lagrangian-Eulerian method is formulated as two separate phases.

### Hydrodynamics Equations:

<table>
<thead>
<tr>
<th>Conservation Law</th>
<th>Lagrangian Phase</th>
<th>Remap Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mass</strong></td>
<td>( \frac{d\rho}{dt} = -\rho \nabla \cdot \vec{v} )</td>
<td>( \frac{d\rho}{d\tau} = \vec{u} \cdot \nabla \rho )</td>
</tr>
<tr>
<td><strong>Momentum</strong></td>
<td>( \rho \frac{d\vec{v}}{dt} = \nabla \cdot \vec{\pi} )</td>
<td>( \frac{d\rho \vec{v}}{d\tau} = \vec{u} \cdot \nabla (\rho \vec{v}) )</td>
</tr>
<tr>
<td><strong>Energy</strong></td>
<td>( \rho \frac{de}{dt} = \vec{\pi} : \nabla \vec{v} )</td>
<td>( \frac{d\rho e}{d\tau} = \vec{u} \cdot \nabla (\rho e) )</td>
</tr>
</tbody>
</table>
During the Lagrangian phase, magnetic diffusion was implemented such that $\nabla \cdot \vec{B}$ is preserved exactly.

Magnetic diffusion equations (Lagrangian phase):

Charge Conservation: \[ \nabla \cdot \sigma \nabla \phi = 0 \]

Ampère’s Law: \[ \sigma \vec{E}^{\text{ind}} = \nabla \times \frac{1}{\mu} \vec{B} + \sigma \nabla \phi \]

Faraday’s Law: \[ \frac{d\vec{B}}{dt} = -\nabla \times \vec{E}^{\text{ind}} \]

Discrete form using $\tilde{\vec{B}}^n = \vec{B}^n - (1 - \alpha) \Delta t \nabla \times \vec{E}^n$:

\[
\begin{align*}
S_0 \phi^{n+1} &= g_0 \\
\left[ M_1(\sigma) - \alpha \Delta t S_1 (\mu^{-1}) \right] e^{n+1} &= \left( D_1 \left( \mu^{-1} \right) \right)^T b^n + D_0(\sigma) \phi^{n+1} - g_1 \\
b^{n+1} &= \tilde{b}^n - \alpha \Delta t C e^{n+1} \\
\tilde{b}^{n+1} &= b^{n+1} - (1 - \alpha) \Delta t C e^{n+1}
\end{align*}
\]
We would like to design remap using the magnetic advection equation that preserves $\nabla \cdot \vec{B}$ exactly.

Magnetic advection equation (Remap phase):

$$\frac{d\vec{B}}{d\tau} = -\nabla \times (\vec{u} \times \vec{B})$$

The advection equation can be split by introducing an auxiliary variable, the pseudo-electric field.

$$\frac{d\vec{B}}{d\tau} = -\nabla \times \vec{E}
\quad \vec{E} = \vec{u} \times \vec{B}$$

$\vec{B}$ is discretized with a $H(\nabla \cdot)$ finite element representation, while $\vec{E}$ is discretized with a $H(\nabla \times)$ finite element representation.
A straightforward scheme of discretizing the magnetic advection equation produced the desired qualities.

$$\frac{db}{d\tau} = -CM_1^{-1}Ub$$

Here $C$ is the metric-free discrete $\nabla \times$ operator, $M_1$ is the mass matrix of the $H(\nabla \times)$ finite element basis functions, and $U$ corresponds to the bilinear operator $\langle \vec{u} \times \vec{\beta}, \vec{\psi} \rangle$, where $\vec{\psi} \in H(\nabla \times)$ and $\vec{\beta} \in H(\nabla \cdot)$. 
Once properly implemented the scheme showed qualitatively correct behavior.

(a) Lagrangian

(b) Eulerian
An interesting "snag" occurs early in the simulation, but then disappears.

(a) Eulerian

(b) Eulerian (zoomed)