Explicit Hyperviscosity: Formulation and Use

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Motivation

- Computations of strongly nonlinear conditions often lead to noise at the node-spacing scale.
  - Large dimensionless parameters = low physical dissipation
  - Without other FE stabilization methods, coercive Galerkin projection relies on physical dissipation terms.
    - The semi-implicit operator is also coercive but not for $\partial \mathbf{V} / \partial t \rightarrow 0$.
  - Noisy flow leads to noisy *n*, *T*, **B** and can cause aliasing errors.
- The flow stabilization coding targets specific effects and is not always adequate.
  - High-differential-order V-dissipation is more general.
  - Implicit computation would be expensive.

Formulation: Hyper-viscosity adds a term to the **V**-advance.

- The modified center-of-mass flow velocity evolution is:

$$\rho\left(\frac{\partial}{\partial t}\mathbf{V}+\mathbf{V}\cdot\nabla\mathbf{V}\right)=\mathbf{J}\times\mathbf{B}-\nabla p-\nabla\cdot\underline{\Pi}-\nabla\cdot\bar{\nu}_h\nabla\nabla^2\mathbf{V}$$

where the $\bar{\nu}_h$ hyper-viscosity has units of m•L/t.

- Focusing on the hyper-dissipation and ignoring density dependence,

$$\frac{\partial}{\partial t}\mathbf{V}=-\nabla\cdot\nu_h\nabla\nabla^2\mathbf{V}$$

with ν_h being a hyper-diffusivity ~ L^4/t.

Implicit vs. explicit is a tradeoff.

- With C^0 elements, implicit computations would need an auxiliary field.

$$\mathbf{W} = \nabla^2 \left[f\,\mathbf{V}^{n+1} + \left(1 - f\right)\mathbf{V}^n \right]$$

$$\Delta\mathbf{V} = -\Delta t \nabla \cdot \nu_h \nabla \mathbf{W}$$

- Explicit hyper-dissipation is only possible for very small coefficients.
  - Basic von Neumann analysis implies $\Delta t \leq 2 / \nu_h k_{\max}^4$.
  - For node-scale smoothing, this may be sufficient.
  - Some sub-cycling can also help if each pass if quick.

A diagonal mass matrix can make each pass quick.

- With minimal Gauss-Lobatto-Legendre integration and GLL nodal bases, integration points are located at the nodes of the expansion.
  - The resulting mass matrix is diagonal.
  - Integration is only exact for polynomials of degree 2*N_p - 1. [N_p = poly_degree.]
- Explicit computation needs the auxiliary **W**, but steps can be separated. For uniform ν_h ,
  - Step 1: $\mathbf{W} = \nabla^2 \mathbf{V}^n$
  - Step 2: $\Delta \mathbf{V} = -\nu_h \Delta t \nabla^2 \mathbf{W}$
- The differential operation is a fast matrix-vector multiplication, where the matrix is created once.

$$\nabla^2 \mathbf{V} \rightarrow \left(\underline{\underline{M}}^{-1} \underline{\underline{K}} \right) \underline{V}$$

$$M_{ij} = \int dVol\, \mathbf{A}_i^* \cdot \mathbf{A}_j \qquad K_{ij} = -\int dVol \left(\nabla \mathbf{A}_i^* \right)^T : \nabla \mathbf{A}_j$$

The implementation uses splitting and sub-cycling to relax the one-step explicit limit.

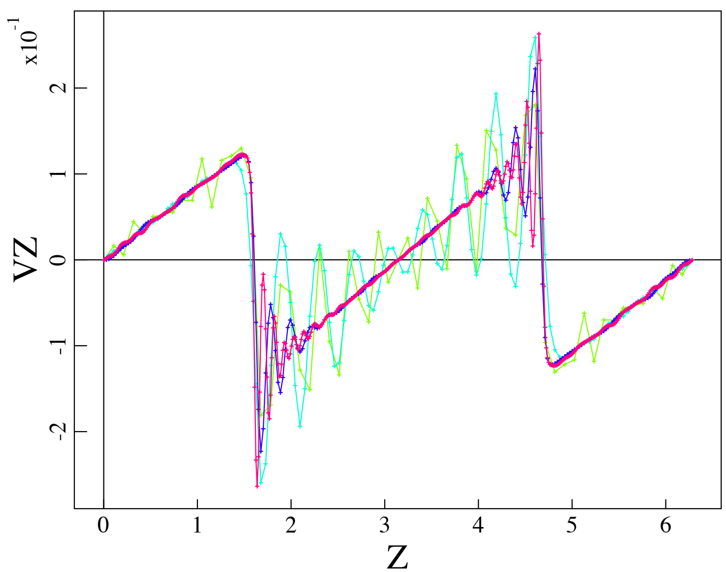
- Strang splitting is used to avoid $O(\Delta t)$ truncation error.
  1. Apply hyper-viscous diffusion of **V** over $\Delta t/2$.
  2. Advance **V** without hyper-viscosity over Δt.
  3. Apply hyper-viscous diffusion over $\Delta t/2$.
- Each hyper-viscous diffusion step is sub-cycled.
- The product $\nu_h \Delta t_{sub}$ is kept less than $2/k_{num}^4$ by using N_{sub} sub-steps, where

$$\left(\nu_h \Delta t_{sub}, N_{sub}\right) = \begin{cases} \left(\nu_h \Delta t / 2N_{sub}, \left\lceil \nu_h k_{num}^4 \Delta t / 4 \right\rceil\right), & \Delta t \le 4N_{s\max} / \nu_h k_{num}^4 \\ \left(2 / N_{s\max} k_{num}^4, N_{s\max}\right), & \Delta t > 4N_{s\max} / \nu_h k_{num}^4 \end{cases}$$

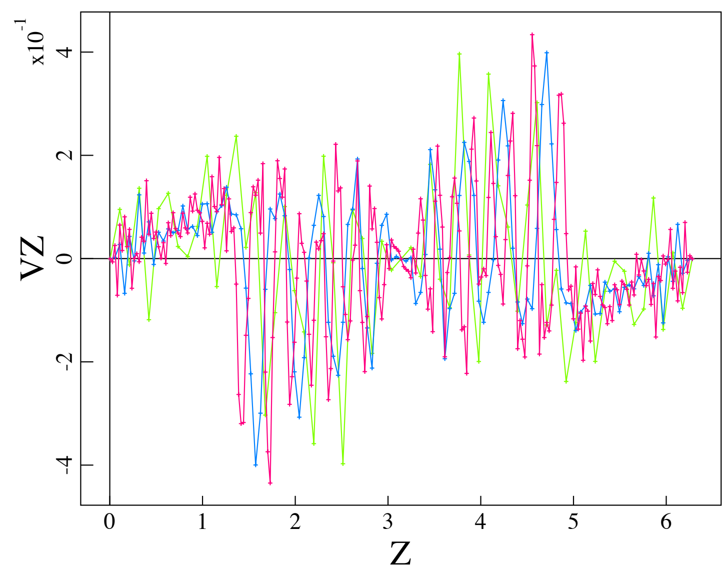
using $\lceil r \rceil$ to represent the smallest integer $\ge r$.

Applications: A 1D nonlinear magneto-acoustic wave demonstrates smoothing of **V**.

- Wave is in a periodic box and starts from a sinusoid in **V** with amplitude of $\sqrt{3}\, c_{ma}/10$.
- Computation loses resolution over time (ran ~ 2.25 linear wave transits).
- Hyper-viscous diffusion is the only dissipation.
- $\max\left(\nu_h\right) = 8\times 10^{-6}$, $N_{s\max} = 20$



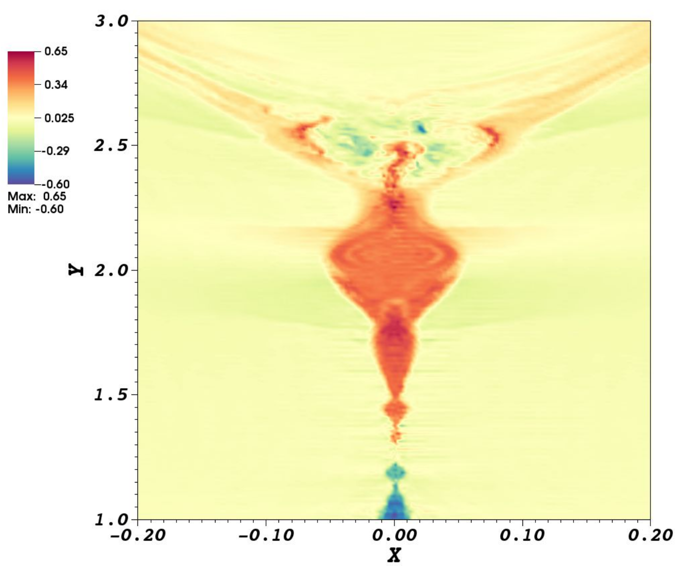
Results at *t* = 10 for 20, 40, 80, 160 cubics.



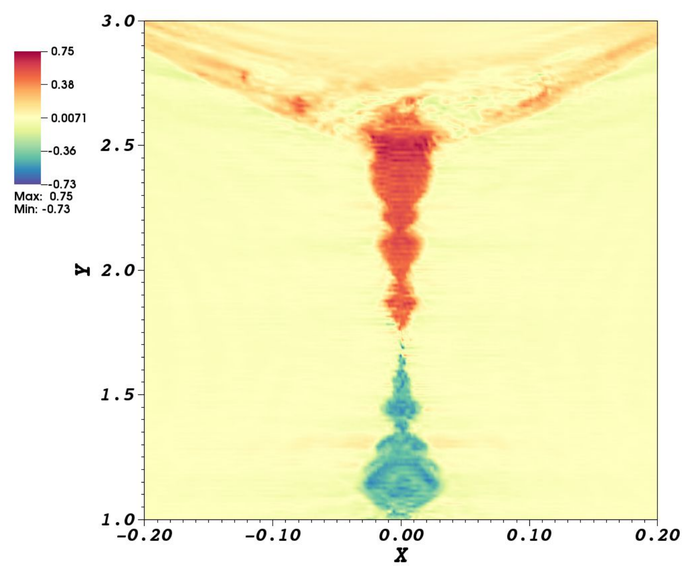
With ν_h = 0 the nonlinear wave is lost.

Slab-geometry plasmoid application prompted this numerical development.

- This is one of the cases that Jake Maddox has considered; S=10^6, Pm=1, χ_{iso}=10^{-6}, no particle diffusivity.
- Mesh is packed along original current layer.
- New case has $\max\left(\nu_h\right) = 1\times10^{-11}$, $N_{s\max} = 20$.
- Comparing with previous result, the new case may have slightly less noise.



Contours of V_z at step 16,000 with hyper-viscosity.



Contours of V_z at step 16,000 without hyper-viscosity.

Conclusions and Discussion

- Explicit hyper-viscosity can provide node-scale smoothing without prohibitive computational cost.
  - Test case run times are about the same as cases without hyper-viscosity.
  - Explicit hyper-dissipation could also be used for n and T.
- Completing this project amounts to the following:
  - Testing the effect of Strang-split computation on temporal convergence.
  - Adding heating from the hyper-viscosity to conserve energy.
  - Checking polar mesh cases.