Moment approach to the drift kinetic equation for toroidal plasmas

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NIMROD Team Meeting

November 6, 2010, Chicago, IL

Overview

- General moment equations
  - Linearized collision operators Ji Held, PoP 2006 2008
  - Nonlinear collision terms Ji Held PoP 2009
- Analytic solution for a uniform plasma Ji Held, Phys. Rev. E 2010
- Parallel integral closures for general collisionality
  - Heat flow Ji Held Sovinec, PoP 2009
  - Viscosity Ji Held, J Fusion Energy 2009
- High collisionality closures for electron-ion plasmas
  - Electrons: require more moments
  - Ions: keep ion-electron operator and time derivative terms
- General collisionality closures in a strong magnetic field NIMROD Team Meeting 2009 APS

Outline

- General moment equations
- Closures
- Drift kinetic equation
- Parallel moment equations
- Solving parallel moment equations for toroidal plasmas

General moment equations

- Landau (Fokker-Planck) kinetic equation

$$\frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \nabla f_a + \frac{q_a}{m_a}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_a}{\partial \mathbf{v}} = \sum_b C(f_a, f_b)$$

- Moment expansion

$$f_a(t, \mathbf{x}, \mathbf{v}) = f_a^{\mathrm{M}} \sum_{lk} \mathsf{m}_a^{lk}(t, \mathbf{x}) \cdot \hat{\mathsf{p}}_a^{lk}$$

$$\mathsf{n}_a^{lk} \equiv n_a \mathsf{m}_a^{lk}(t, \mathbf{x}) = \int d\mathbf{v} \hat{\mathsf{p}}_a^{lk} f_a(t, \mathbf{x}, \mathbf{v})$$

where $\hat{\mathsf{p}}^{lk}$'s are orthonormal polynomials of $\mathbf{c} = (\mathbf{v} - \mathbf{V})/v_T$

- General moment equations: $\int d\mathbf{v} \hat{\mathsf{p}}^{jp}$(kinetic eq.)$\Rightarrow$

$$d_a \mathsf{n}_a^{jp} + \Omega_a \mathbf{b} \check{\times} \mathsf{n}_a^{jp} + \{\hat{\Xi}^j(d_a \ln T) + \hat{U}_{\mathrm{c}}^j \nabla \cdot \mathbf{V} + \hat{U}_{\mathrm{l}}^j(\underline{\nabla} \mathbf{V}) \cdot + \hat{U}_{\mathrm{r}}^j(\nabla \underline{\mathbf{V}}) \cdot\}_{pk} \mathsf{n}_a^{jk}$$
$$+ \{v_T \hat{\Psi}^{j\pm} \nabla + v_T \hat{\Phi}^{j\pm} \nabla \ln T + v_T^{-1} \hat{\Theta}^{j\pm} \mathbf{a}_a\}_{pk}(\cdot) \mathsf{n}_a^{j\pm1,k}$$
$$+ \hat{U}_{pk}^{j\pm} \nabla \mathbf{V}(:) \mathsf{n}_a^{j\pm2} = (\hat{C}_{aa}^{jpk} + \hat{A}_{ab}^{jpk}) \mathsf{n}_a^{jk} + \hat{B}_{ab}^{jpk} \mathsf{n}_b^{jk} + C_{ab}^{(2)jp}$$

where $d_a \equiv \partial_t + \mathbf{V}_a \cdot \nabla$ and $\mathbf{a}_a \equiv \dfrac{q_a}{m_a}(\mathbf{E} + \mathbf{V}_a \times \mathbf{B}) - d_a \mathbf{V}_a$

Moment equations for closures

- Maxwellian moment $(n_a, \mathbf{V}_a, T_a)$ equations

$$(0,0): \quad d_a n_a + n_a \nabla \cdot \mathbf{V}_a = 0$$

$$(0,1): \quad \tfrac{3}{2} n_a d_a T_a + n_a T_a \nabla \cdot \mathbf{V}_a + \nabla \cdot \mathbf{h}_a + \nabla \mathbf{V}_a : \boldsymbol{\pi}_a = Q_a$$

$$(1,0): \quad \underbrace{m_a n_a d_a \mathbf{V}_a - n_a q_a (\mathbf{E} + \mathbf{V}_a \times \mathbf{B})}_{-m_a n_a \mathbf{a}_a} + \nabla p_a + \nabla \cdot \boldsymbol{\pi}_a = \mathbf{R}_a$$

- Non-Maxwellian moment equations $(j,p) \neq (0,0), (0,1), (1,0)$

$$\hat{D}_a \mathsf{n}_a + \Omega_a \mathbf{b} \breve{\times} \mathsf{n}_a = (\hat{C}_{aa} + \hat{A}_{ab}) \mathsf{n}_a + \mathsf{G}_a + \hat{B}_{ab} \mathsf{n}_b + \hat{C}_a^{(2)}$$

where $\mathsf{n}_a = (\mathsf{n}_a^{02}, \mathsf{n}_a^{03}, \cdots, \underbrace{\mathsf{n}_a^{11}}_{\mathbf{h}_a}, \mathsf{n}_a^{12}, \cdots, \underbrace{\mathsf{n}_a^{20}}_{\boldsymbol{\pi}_a}, \mathsf{n}_a^{21}, \cdots, \cdots)^{\mathrm{T}}$,

$$\mathsf{G}_a^1 = \begin{pmatrix} \frac{\sqrt{5}}{2} \frac{n_a v_{Ta}}{T_a} \nabla T_a + \delta_{ae} \sqrt{2} a_{\mathrm{ei}}^{110} \frac{n_e}{\tau_{\mathrm{ei}}} \frac{\mathbf{V}_{\mathrm{ei}}}{v_{Te}} \\ \delta_{ae} a_{\mathrm{ei}}^{120} \frac{n_e}{\tau_{\mathrm{ei}}} \frac{\mathbf{V}_{\mathrm{ei}}}{v_{Te}} \\ \delta_{ae} a_{\mathrm{ei}}^{130} \frac{n_e}{\tau_{\mathrm{ei}}} \frac{\mathbf{V}_{\mathrm{ei}}}{v_{Te}} \\ \vdots \end{pmatrix}, \quad \mathsf{G}_a^2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} n_a \mathsf{W}_a \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

$a_{\mathrm{ei}}^{1p0} = -\sqrt{\frac{3(p+\frac{1}{2})!}{(2p+3)p!(\frac{1}{2})!}}$, and $\mathsf{W} = \nabla \mathbf{V} + (\nabla \mathbf{V})^{\mathrm{T}} - \frac{2}{3} \mathsf{I} \nabla \cdot \mathbf{V}$

From kinetic equation to parallel moment equations

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}} = C(f)$$

$$\int d\gamma$$

$$\mathbf{v}_{\rm D}\cdot\nabla\bar{f}_0 + (\frac{q}{m}\mathbf{v}_\parallel\cdot\mathbf{E}_1 + \mathbf{v}_{\rm D}\cdot\mathbf{E}_0)\frac{\partial\bar{f}_0}{\partial\varepsilon} + \mathbf{v}_\parallel\cdot\nabla f_1 = C(\bar{f}_1)$$

$$\mathbf{v}_{\rm D} = \tfrac{1}{\Omega}\mathbf{b}\times[(v_\parallel^2+\tfrac{1}{2}v_\perp^2)\tfrac{\nabla B}{B} + \tfrac{q}{m}\nabla\Phi] + \tfrac{v_\parallel^2}{\Omega}\tfrac{\mu_0\mathbf{J}_\perp}{B}$$

$$\int d\mathbf{v}\mathsf{P}^{lk}$$

$$\Omega \gg \mathbf{v}\cdot\nabla,\ \frac{q}{m}\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{v}},\ C \gg \frac{\partial}{\partial t}$$
$$(n,T) = (n,T)_0(\psi) + (n,T)_1$$
$$\mathbf{V} = \mathbf{V}_1$$

$$\int d(v,v_\parallel)P^{lk}$$

See NIMROD Team Meeting, 2009 APS

$$\frac{\partial}{\partial t}\mathsf{M} + \frac{1}{T}\frac{\partial T}{\partial t}\hat{\Xi}\mathsf{M} + v_T\hat{\Psi}\nabla\mathsf{M} + v_T\frac{\nabla T}{T}\hat{\Phi}\mathsf{M} + \frac{q}{mv_T}\hat{\Theta}\mathbf{E}\mathsf{M} + \Omega\mathbf{b}\times\mathsf{M} = \frac{1}{\tau}\hat{C}\mathsf{M}$$

$$\mathbf{b}\cdot$$

$$v_T\bar{\Psi}\partial_\parallel M_\parallel - v_T\Upsilon(\partial_\parallel \ln B)M_\parallel = \frac{1}{\tau}CM_\parallel + \bar{G}$$

Solving parallel moment equations

$$[\Psi]\lambda_{\rm mfp}\frac{d[n]}{d\ell} = [c][n] + [g] - [\Upsilon]\lambda_{\rm mfp}\frac{d\ln B}{d\ell}[n]$$ (⊃ Braginskii, integral closure)

- Fourier expansion (axi-symmetric)

$$[\Psi]\lambda\frac{d[n]}{d\vartheta} = [c][n] + [g] - [\Upsilon]\lambda\frac{d\ln B}{d\vartheta}[n], \quad \lambda = \lambda_{\rm mfp}\frac{d\vartheta}{d\ell} = \frac{\lambda_{\rm mfp}}{qR}$$

$$[n] = [n_{0+}] + \sum_{m=1}^{\infty}([n_{m+}]\cos m\vartheta + [n_{m-}]\sin m\vartheta)$$

$$n_1,\, T_1,\, V_{1\|},\, E_{1\|} = \cdots$$

- Large aspect ratio $B = B_0(1-\epsilon\cos\vartheta) \Rightarrow \dfrac{d\ln B}{d\vartheta} \approx \epsilon\sin\vartheta$

$$\sin\vartheta\sin m\vartheta = \tfrac{1}{2}[\cos(m-1)\vartheta - \cos(m+1)\vartheta]$$
$$\sin\vartheta\cos m\vartheta = \tfrac{1}{2}[\sin(m+1)\vartheta - \sin(m-1)\vartheta]$$

$$g^{01},\, g^{20} \sim (\frac{dp_0}{d\psi},\, n\frac{dT_0}{d\psi})c_\psi\sin\vartheta;\; g^{02},\, g^{21} \sim n\frac{dT_0}{d\psi}c_\psi\sin\vartheta \Rightarrow g^{lk}_{1-} \neq 0$$

where $c_\psi = \epsilon\dfrac{\tau_{aa}}{qB^2}\mathbf{b}\times\nabla\psi\cdot\nabla\vartheta$

Linear system of equations for Fourier moments

$$\begin{bmatrix} C & \frac{\epsilon}{2}\lambda\Upsilon & 0 & 0 & 0 & 0 & 0 \\ \frac{\epsilon}{2}\lambda\Upsilon & C & \lambda\Psi & 0 & -\frac{\epsilon}{2}\lambda\Upsilon & 0 & 0 \\ 0 & -\lambda\Psi & C & \frac{\epsilon}{2}\lambda\Upsilon & 0 & 0 & 0 \\ 0 & 0 & \frac{\epsilon}{2}\lambda\Upsilon & C & 2\lambda\Psi & 0 & -\frac{\epsilon}{2}\lambda\Upsilon \\ 0 & -\frac{\epsilon}{2}\lambda\Upsilon & 0 & -2\lambda\Psi & C & \frac{\epsilon}{2}\lambda\Upsilon & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\epsilon\lambda\Upsilon & C & 3\lambda\Psi \\ 0 & 0 & 0 & -\frac{\epsilon}{2}\lambda\Upsilon & 0 & -3\lambda\Psi & C \end{bmatrix} \begin{bmatrix} n_{0+} \\ n_{1-} \\ n_{1+} \\ n_{2-} \\ n_{2+} \\ n_{3-} \\ n_{3+} \end{bmatrix} = - \begin{bmatrix} g_{0+} \\ g_{1-} \\ g_{1+} \\ g_{2-} \\ g_{2+} \\ g_{3-} \\ g_{3+} \end{bmatrix}$$

$$\begin{bmatrix} n_{0+} \\ n_{1-} \\ n_{1+} \\ n_{2-} \\ n_{2+} \\ n_{3-} \\ n_{3+} \end{bmatrix}_a^{jp} = Q_{ua}^{jp} \begin{bmatrix} V_{0+}^{\mathrm{ei}} \\ V_{1-}^{\mathrm{ei}} \\ V_{1+}^{\mathrm{ei}} \\ V_{2-}^{\mathrm{ei}} \\ V_{2+}^{\mathrm{ei}} \\ V_{3-}^{\mathrm{ei}} \\ V_{3+}^{\mathrm{ei}} \end{bmatrix} + Q_{T1a}^{jp} \begin{bmatrix} T_{0+}^{(1)} \\ T_{1-}^{(1)} \\ T_{1+}^{(1)} \\ T_{2-}^{(1)} \\ T_{2+}^{(1)} \\ T_{3-}^{(1)} \\ T_{3+}^{(1)} \end{bmatrix}_a + Q_{Va}^{jp} \begin{bmatrix} V_{0+} \\ V_{1-} \\ V_{1+} \\ V_{2-} \\ V_{2+} \\ V_{3-} \\ V_{3+} \end{bmatrix}_a$$

$$+ \begin{bmatrix} Q_{0+} \\ Q_{1-} \\ Q_{1+} \\ Q_{2-} \\ Q_{2+} \\ Q_{3-} \\ Q_{3+} \end{bmatrix}_{p0}^{jp} c_\psi \frac{dp_a^{(0)}}{d\psi} + \begin{bmatrix} Q_{0+} \\ Q_{1-} \\ Q_{1+} \\ Q_{2-} \\ Q_{2+} \\ Q_{3-} \\ Q_{3+} \end{bmatrix}_{T0}^{jp} c_\psi n_a^{(0)} \frac{dT_a^{(0)}}{d\psi}$$

Solution (heat flux with $\lambda = 1$, $\epsilon = 0.1$, $L = 16$, $K = 16$, $F = 5$)

$$
\begin{bmatrix} n_{0+} \\ n_{1-} \\ n_{1+} \\ n_{2-} \\ n_{2+} \\ n_{3-} \\ n_{3+} \end{bmatrix}^{11}_{\mathrm{e}}
=
\begin{bmatrix}
-0.44 & 0 & -7.6^{-3} & 0 & -1.2^{-3} & 0 & -8.7^{-5} \\
0 & -0.24 & 0 & -6.0^{-3} & 0 & -2.9^{-4} & 0 \\
3.9^{-3} & 0 & -0.24 & 0 & 6.0^{-3} & 0 & 3.1^{-4} \\
0 & 1.3^{-3} & 0 & -0.18 & 0 & -4.5^{-3} & 0 \\
-1.5^{-4} & 0 & 1.3^{-3} & 0 & -0.18 & 0 & 4.5^{-3} \\
0 & 1.8^{-5} & 0 & 6.3^{-4} & 0 & -0.16 & 0 \\
2.7^{-6} & 0 & -2.1^{-5} & 0 & 6.4^{-4} & 0 & -0.16
\end{bmatrix}
\begin{bmatrix} n_{0+} \\ n_{1-} \\ n_{1+} \\ n_{2-} \\ n_{2+} \\ n_{3-} \\ n_{3+} \end{bmatrix}^{10}_{\mathrm{ei}}
\quad \left(\propto \mathbf{b} \cdot \mathbf{V}_{\mathrm{ei}} \right)
$$

$$
+
\begin{bmatrix}
0 & 1.6^{-2} & 0 & 3.3^{-3} & 0 & 2.7^{-4} & 0 \\
0 & 0 & -0.32 & 0 & -1.4^{-2} & 0 & 9.6^{-4} \\
0 & 0.32 & 0 & -1.4^{-2} & 0 & -1.0^{-3} & 0 \\
0 & 0 & 3.7^{-3} & 0 & -0.37 & 0 & -1.3^{-2} \\
0 & -3.5^{-3} & 0 & 0.37 & 0 & -1.3^{-2} & 0 \\
0 & 0 & -4.8^{-5} & 0 & 2.8^{-3} & 0 & -0.41 \\
0 & 5.9^{-5} & 0 & -2.8^{-3} & 0 & 0.41 & 0
\end{bmatrix}
\begin{bmatrix} n_{0+} \\ n_{1-} \\ n_{1+} \\ n_{2-} \\ n_{2+} \\ n_{3-} \\ n_{3+} \end{bmatrix}^{01}_{\mathrm{e}}
\quad \left(\propto \mathbf{b} \cdot \nabla T \right)
$$

$$
+
\begin{bmatrix}
7.4^{-4} & 0 & 9.3^{-5} & 0 & -7.3^{-4} & 0 & -9.3^{-5} \\
0 & 1.3^{-3} & 0 & 5.5^{-3} & 0 & -4.5^{-4} & 0 \\
-5.4^{-3} & 0 & -4.6^{-4} & 0 & 5.3^{-3} & 0 & 4.6^{-4} \\
0 & -3.8^{-3} & 0 & 3.7^{-4} & 0 & 3.8^{-3} & 0 \\
-8.2^{-5} & 0 & -3.8^{-3} & 0 & -2.0^{-4} & 0 & 3.8^{-3} \\
0 & -2.2^{-5} & 0 & -2.9^{-3} & 0 & 2.2^{-4} & 0 \\
-8.6^{-7} & 0 & -2.4^{-5} & 0 & -2.9^{-3} & 0 & -1.8^{-4}
\end{bmatrix}
\begin{bmatrix} n_{0+} \\ n_{1-} \\ n_{1+} \\ n_{2-} \\ n_{2+} \\ n_{3-} \\ n_{3+} \end{bmatrix}^{10}_{\mathrm{e}}
\quad \left(\propto \mathbf{bb} : \overline{\nabla \mathbf{V}} \right)
$$

$$
+
\begin{bmatrix} -1.0^{-2} \\ 0 \\ -7.6^{-2} \\ 0 \\ -1.2^{-3} \\ 0 \\ -1.2^{-5} \end{bmatrix}_{\mathrm{e}}
c_\psi \frac{dp_{\mathrm{e}}^{(0)}}{d\psi}_{\mathrm{e}}
+
\begin{bmatrix} -0.12 \\ 0 \\ -1.5 \\ 0 \\ 2.3^{-2} \\ 0 \\ -5.5^{-4} \end{bmatrix}_{\mathrm{e}}
c_\psi n_{\mathrm{e}0} \frac{dT_{\mathrm{e}}^{(0)}}{d\psi}
$$

Future work

- Transport relations: express also $(n_1, T_1, \mathbf{V}_1)$ in terms of $\left(\frac{dp_0}{d\psi}, \frac{dT_0}{d\psi}\right)$
    - Flux surface average: bootstrap current
    - Compare with other theories and computations

- Closures and transports for general magnetic geometry

$$B = B_0 + B_{mn+}\cos(m\vartheta - n\varphi) + B_{mn-}\sin(m\vartheta - n\varphi)$$

$$[n] = [n_0] + [n_{mn\pm\pm}]\begin{pmatrix} \cos(m\vartheta) \\ \sin(m\vartheta) \end{pmatrix}\begin{pmatrix} \cos(n\varphi) \\ \sin(n\varphi) \end{pmatrix}$$

    - For $L = 40,\ K = 40,\ F_\vartheta = 10,\ F_\varphi = 9$

$$(1600 \times 19)^2 \times 4\,\text{bytes} \approx 3.7\,\text{GB}$$

    - $\Rightarrow$ Use a sparse matrix solver!