A 2D FE + 1D Fourier Treatment of the Fokker-Planck Equation

Andy Spencer, Eric Held, & Jeong-Young Ji

Department of Physics
Utah State University

NIMROD APS meeting, November 12, 2011

Outline of Presentation

1. Test Particle Operator
   - Introduction
   - Implementing the Test Particle Operator
2. Field Operator
   - The Chapman-Enskog Like Approach
   - Spitzer Conductivity

The Fokker-Planck Equation

$$\frac{df_a}{dt} = -\sum_b \frac{\Gamma_{ab}}{2}\nabla_{\mathbf{v}} \cdot \left[\frac{m_a + m_b}{m_b}\left(\nabla_{\mathbf{v}} \cdot \mathsf{D}_b\right) f_a - \nabla_{\mathbf{v}} \cdot \left(\mathsf{D}_b f_a\right)\right]$$

where

$$\mathsf{D}_b\left(\mathbf{v}, t\right) \equiv \int d\mathbf{v}' f_b\left(\mathbf{v}', t\right) \mathsf{U}, \qquad \mathsf{U} \equiv \frac{u^2 \mathsf{I} - \mathbf{u}\mathbf{u}}{u^3}$$

and

$$\Gamma_{ab} \equiv \frac{q_a^2 q_b^2 \ln \Lambda_{ab}}{4\pi\epsilon_0^2 m_a^2}, \qquad \ln \Lambda_{ab} = \ln\left(\frac{r_{\max}}{r_{\min}}\right)$$

Using NIMROD

```
gridshape = 'rect'
geom = 'tor'
```

$$\mathbf{v} \rightarrow \left(v_{\parallel}, v_{\perp}, \gamma\right),$$

$$\mathbf{c}_a \equiv \mathbf{v}/v_{Ta},$$

$$F_a\left(\mathbf{c}_a, t^k\right) =$$

$$\sum_{n=-N}^{N}\sum_{j=0}^{J} F^k_{a,j,n}\alpha_j\left(c_{a\perp}, c_{a\parallel}\right)\mathrm{e}^{in\gamma}$$

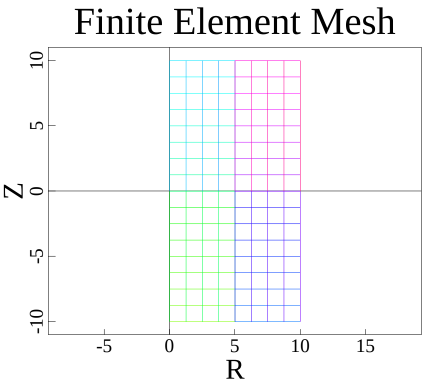

Figure 1: Example velocity grid.

UTAH STATE

δf-linearization about a Maxwellian

$$f_a(\mathbf{v},t) \equiv f_a^{\mathrm{M}}(\mathbf{v}) + F_a(\mathbf{v},t)$$

$$\frac{\partial F_a}{\partial t} = \sum_b \frac{\Gamma_{ab}}{2} \nabla_{\mathbf{v}} \cdot \left[\left(\frac{2}{v_{Tb}} \frac{m_a}{m_b} \mathbf{z}_b F_a + \nabla_{\mathbf{v}} F_a \right) \cdot \mathsf{D}_b^{\mathrm{M}} \right]$$

δf-linearization about a Maxwellian

$$f_a(\mathbf{v},t) \equiv f_a^{\mathrm{M}}(\mathbf{v}) + F_a(\mathbf{v},t)$$

$$\frac{\partial F_a}{\partial t} = \sum_b \frac{\Gamma_{ab}}{2} \nabla_{\mathbf{v}} \cdot \left[\left(\frac{2}{v_{Tb}} \frac{m_a}{m_b} \mathbf{z}_b F_a + \nabla_{\mathbf{v}} F_a \right) \cdot \mathsf{D}_b^{\mathrm{M}} \right]$$

where

$$\mathsf{D}_b^{\mathrm{M}} = \frac{n_b}{v_{Tb}} \left[\frac{3G(z_b) - E(z_b)}{z_b^3} \mathbf{z}_b \mathbf{z}_b + \frac{E(z_b) - G(z_b)}{z_b} \mathsf{I} \right],$$

$E(z_b) = \frac{2}{\sqrt{\pi}} \int_0^{z_b} dx e^{-x^2}$ and $G(z_b) = \frac{E(z_b)}{2z_b^2} - \frac{e^{-z_b^2}}{\sqrt{\pi} z_b}$

Temporal Discretization and Weak Formulation

$$\Delta F_a - \theta \Delta t \sum_b C\left[\Delta F_a, f_b^{\mathrm{M}}\right] \quad = \quad \Delta t \sum_b C\left[F_a\left(\mathbf{v}, t^k\right), f_b^{\mathrm{M}}\right]$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\int d\mathbf{v}\, \frac{1}{2\pi} \alpha_j \mathrm{e}^{-in'\gamma} \times \qquad\qquad \int d\mathbf{v}\, \frac{1}{2\pi} \alpha_j \mathrm{e}^{-in'\gamma} \times$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

`F_coll_dot`   `F_coll_rhs`

Why We Need the Field Terms

$$\mathbf{R}_{ab} \equiv \int d\mathbf{v} m_a \mathbf{z}_a C\left[f_a, f_b\right]$$

$$Q_{ab} \equiv \int d\mathbf{v} \frac{1}{2} m_a z_a^2 C\left[f_a, f_b\right]$$

Conservation of total momentum and energy

$$\mathbf{R}_{ab}^{\text{test}} + \mathbf{R}_{ba}^{\text{field}} = 0$$

$$Q_{ab}^{\text{test}} + Q_{ba}^{\text{field}} = \mathbf{R}_{ab}^{\text{test}} \cdot \left(\mathbf{V}_b - \mathbf{V}_a\right)$$

Combining Kinetics with Fluid Equations

Now let the Maxwellian be time dependent

$$\begin{aligned} f_a(\mathbf{v},t) &= f_a^{\mathrm{M}}(\mathbf{v},t) + F_a(\mathbf{v},t) \\ \frac{df_a}{dt} &= \sum_b \left\{ C\left[f_a^{\mathrm{M}}, f_b^{\mathrm{M}}\right] + C\left[f_a^{\mathrm{M}}, F_b\right] + C\left[F_a, f_b^{\mathrm{M}}\right] \right\} \end{aligned}$$

By using the fluid equations we can replace:

$$\frac{df_a^{\mathrm{M}}}{dt} = \frac{2 f_a^{\mathrm{M}}}{n_a m_a v_{Ta}} \sum_b \left[\mathbf{z}_a \cdot \mathbf{R}_{ab} + \frac{Q_{ab}}{v_{Ta}} \left(\frac{2}{3} z_a^2 - 1 \right) \right]$$

and we also need

$$\frac{\partial F_a}{\partial t} \to \frac{\partial F_a}{\partial t} + \frac{\partial \mathbf{c}_a}{\partial t} \cdot \frac{\partial F_a}{\partial \mathbf{c}_a}$$

The Field Terms

$$C\left[f_a^{\mathrm{M}}, F_b\right] = -\Gamma_{ab}\nabla_{\mathbf{v}}\cdot\left[f_a^{\mathrm{M}}\left(\frac{m_a}{m_b}\nabla_{\mathbf{v}}h_b + \frac{\mathbf{z}_a}{v_{Ta}}\cdot\nabla_{\mathbf{v}}\nabla_{\mathbf{v}}g_b\right)\right]$$

$$\begin{aligned}
g_b &= \int d\mathbf{v}' F_b\left(\mathbf{v}', t\right) u \\
h_b &= \int d\mathbf{v}' F_b\left(\mathbf{v}', t\right) u^{-1} \\
\nabla_{\mathbf{v}} g_b &= \int d\mathbf{v}' F_b\left(\mathbf{v}', t\right) \hat{\mathbf{u}} \\
\nabla_{\mathbf{v}} h_b &= -\int d\mathbf{v}' F_b\left(\mathbf{v}', t\right) u^{-2}\hat{\mathbf{u}}
\end{aligned}$$

Temporal Discretization

Fluid and Kinetic variables staggered in time

Closes Fluid Equations

$$f_a^{\mathrm{M},k-1}, F_a^{k-1/2} \longrightarrow \mathbf{R}_{ab}^{k-1/2}, Q_{ab}^{k-1/2} \longrightarrow f_a^{\mathrm{M},k}$$

Kinetic Equation Parameters

$$F_a^{k-1/2}, f_a^{\mathrm{M},k} \longrightarrow n_a^k, T_a^k, \mathbf{V}_a^k \longrightarrow F_a^{k+1/2}$$

Spitzer Conductivity

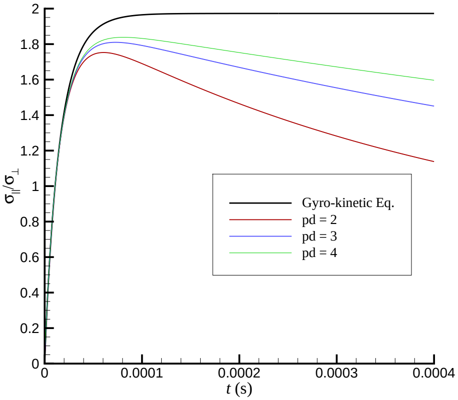

Figure 2: $\frac{\sigma_{\parallel}}{\sigma_{\perp}}$ with $n_{\mathrm{e}} = n_{\mathrm{i}} = 10^{19}\,\mathrm{m}^{-3}$, and initial $T_{\mathrm{e}} = T_{\mathrm{i}} = 200\,\mathrm{eV}$.

Grid Packing to improve F_a

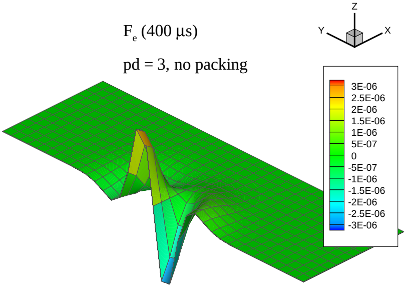

Figure 3: F_e after $400\,\mu$s with no grid packing. poly_degree = 3, on a 4×12 grid.

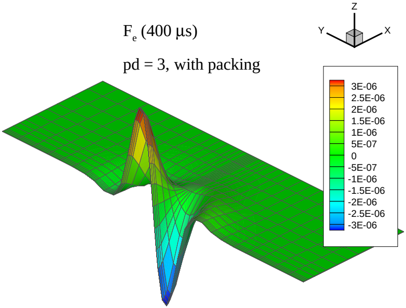

Figure 4: F_e after $400\,\mu$s with grid packing. poly_degree = 3, on a 4×12 grid.

Rosenbluth Potentials

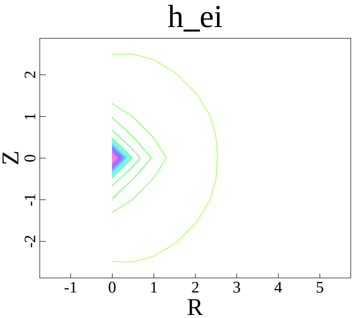

Figure 5: h_{ei} with no grid packing. poly_degree = 2, on a 4×12 grid.

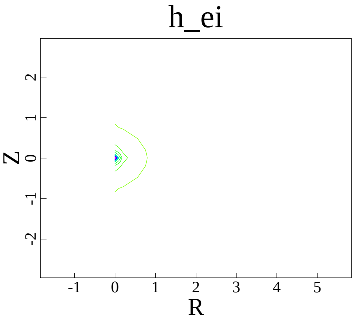

Figure 6: h_{ei} with grid packing. poly_degree = 2, on a 4×12 grid.

Grid Packing

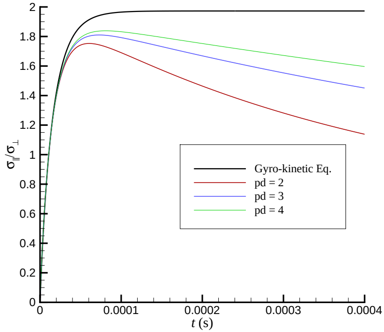

Figure 7: Evolution of $\frac{\sigma_\parallel}{\sigma_\perp}$ with no grid packing, on a 4×12 grid.

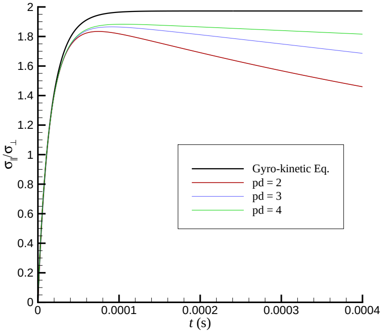

Figure 8: Evolution of $\frac{\sigma_\parallel}{\sigma_\perp}$ with grid packing,on a 4×12 grid.

Conservation of momentum

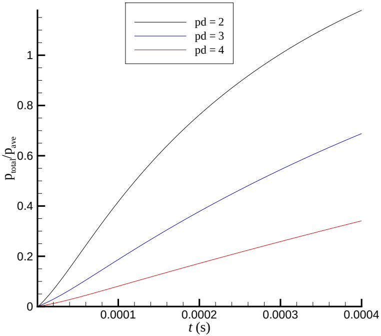

Figure 9: $p_{\mathrm{total}}/p_{\mathrm{ave}}$ with no grid packing.

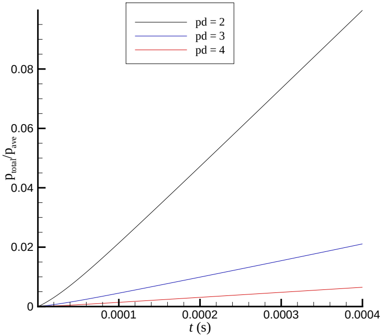

Figure 10: $p_{\mathrm{total}}/p_{\mathrm{ave}}$ with grid packing.

Summary & Future work

- We've used NIMROD to solve the Kinetic Equation with *test particle* and *field* terms.
- Electron speeds are generally much larger than ion speeds which lead to resolution issues.
- Future work under consideration:
  - More testing
    - Steady state for Spitzer conductivity
    - Collisional heating problem
  - Implicit time advance of field terms
  - Add spatial dimensions
  - Couple to NIMROD's fluid equations