Closures and transport in the collisionless limit: linear response theory and beyond

Jeong-Young Ji and Eric Held

Department of Physics, Utah State University

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Closures

- Maxwellian moment $(n_a, \mathbf{V}_a, T_a)$ equations

$$d_a n_a + n_a \nabla \cdot \mathbf{V}_a = 0 \, (d_a \equiv \partial_t + \mathbf{V}_a \cdot \nabla)$$

$$\tfrac{3}{2} n_a d_a T_a + n_a T_a \nabla \cdot \mathbf{V}_a + \nabla \cdot \mathbf{h}_a + \nabla \mathbf{V}_a : \boldsymbol{\pi}_a = Q_a$$

$$m_a n_a d_a \mathbf{V}_a - n_a q_a (\mathbf{E} + \mathbf{V}_a \times \mathbf{B}) + \nabla p_a + \nabla \cdot \boldsymbol{\pi}_a = \mathbf{R}_a$$

Solve moment equations for n_a (non-Maxwellian moments)

$$D \mathsf{n}_a + \Omega_a \mathbf{b} \times \mathsf{n}_a = C \mathsf{n}_a + \mathsf{G}_a$$

Express $\mathbf{h}_a(\mathsf{n}_a^{11})$, $\boldsymbol{\pi}_a(\mathsf{n}_a^{20})$, Q_a, $\mathbf{R}_a$ in terms of $n_a, \mathbf{V}_a, T_a$

$$\mathbf{R}_\mathrm{e} = -(\alpha)(\mathbf{V}_\mathrm{ei}) - (\beta)(\nabla T_\mathrm{e}), \;\; \mathbf{h}_\mathrm{e} = (\beta)(\mathbf{V}_\mathrm{ei}) - (\kappa)(\nabla T_\mathrm{e})$$

- Transport theory

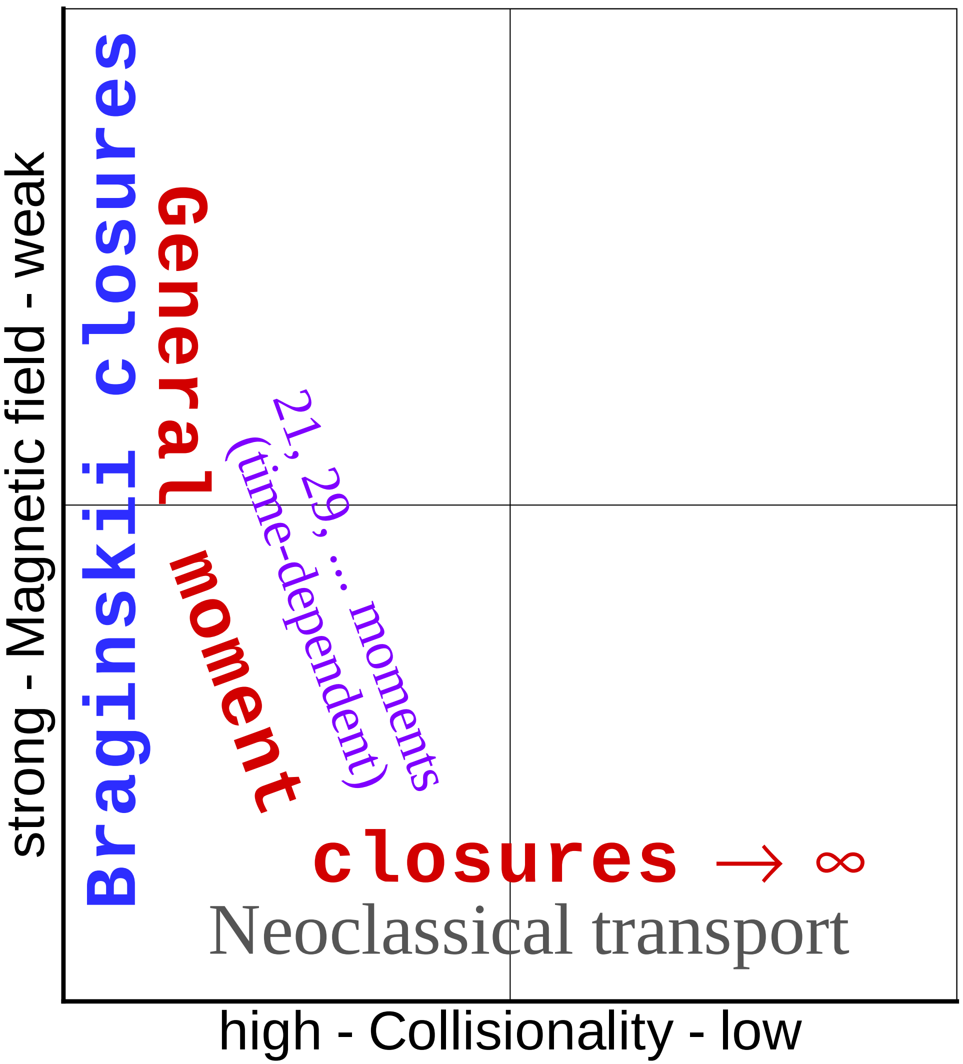
Solve momentum balance equation

$$ne\mathbf{E}' + ne\mathbf{V}_\mathrm{ei} \times \mathbf{B} = \mathbf{R}_\mathrm{e} \text{ where } \mathbf{E}' = \mathbf{E} + \mathbf{V}_\mathrm{i} \times \mathbf{B} + (ne)^{-1} \nabla p_\mathrm{e}$$

Express fluxes in terms of thermodynamic drives

$$\mathbf{J} = (\sigma)(\mathbf{E}') - (\alpha')(\nabla T_\mathrm{e}), \;\; \mathbf{h}_\mathrm{e} = (\alpha')(\mathbf{E}') - (\kappa')(\nabla T_\mathrm{e})$$

Closures/transport theory



General moment closures/transport	Neoclassical transport
exact Landau (FP) collision operator	model operator Lorentz+restoring
general (unified) collisionality	high (PS) low (banana)
general magnetic field	flux surfaces (averaged)
collisionless limit: ∞ moments	solving DKE

Drift kinetic equation

- Landau-Fokker-Planck kinetic equation

$$\frac{\partial f_a}{\partial t} + \mathbf{v}\cdot\nabla f_a + \frac{q_a}{m_a}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_a}{\partial \mathbf{v}} = \sum_b C(f_a, f_b)$$

- Drift kinetic equation

$$(\mathbf{x}, w, \mu): \quad \frac{\partial \bar{f}}{\partial t} + (\mathbf{v}_\parallel + \mathbf{v}_\mathrm{D})\cdot\nabla\bar{f} + q(\mathbf{v}_\parallel + \mathbf{v}_\mathrm{D})\cdot\mathbf{E}\frac{\partial \bar{f}}{\partial w} = C(\bar{f})$$

$$(\mathbf{x}, U, \mu): \quad \frac{\partial \bar{f}}{\partial t} + (\mathbf{v}_\parallel + \mathbf{v}_\mathrm{D})\cdot\nabla\bar{f} + q\mathbf{v}_\parallel\cdot\frac{\partial \mathbf{A}}{\partial t}\frac{\partial \bar{f}}{\partial U} = C(\bar{f})$$

where $\mu = \frac{mv_\perp^2}{2B}$, $w = \frac{1}{2}mv^2$, $U = \frac{1}{2}mv^2 + q\Phi$, and

$$\begin{aligned}\mathbf{v}_\mathrm{D} &= \frac{1}{\Omega}\mathbf{b}\times\left(-\frac{q\mathbf{E}}{m} + \frac{\mu}{m}\nabla B + v_\parallel^2\boldsymbol{\kappa}\right)\\ &= \frac{1}{\Omega}\mathbf{b}\times\left[(v_\parallel^2 + \frac{1}{2}v_\perp^2)\frac{\nabla B}{B} + \frac{q}{m}\nabla\Phi\right] + \frac{v_\parallel^2}{\Omega}\frac{\mu_0\mathbf{J}_\perp}{B}\end{aligned}$$

Parallel drives $\mathbf{b}\cdot\nabla f^{\mathrm{M}}_{n,T,V_\parallel} \sim O(\delta^0)$

- Set $\bar{f}_0 = f^{\mathrm{M}}_{n,T,V_\parallel} + F$ and solve the 0^{th} order DKE for F

$$\frac{\partial}{\partial t}\bar{f}_0 + v_\parallel \mathbf{b}\cdot\nabla \bar{f}_0 + q v_\parallel E_{0\parallel}\frac{\partial \bar{f}_0}{\partial w} = C(\bar{f}_0)$$

- For $\partial/\partial t = 0,\ C = 0$:

$$v_\parallel \partial_\ell (f^{\mathrm{M}}_{n,T,V_\parallel} + F) = 0 \Rightarrow F(\ell) = -f^{\mathrm{M}}_{n,T,V_\parallel} + g(\ell\text{-indep.}) + \text{missing?}$$

- Linearized DKE $T = T_0 + T_1,\ n = n_0 + n_1,\ V_\parallel = V_{1\parallel} = u$

$$\frac{\partial F}{\partial t} + v_\parallel \frac{\partial F}{\partial \ell} + \nu F = \Bigg[-\Big(\frac{\partial n_1}{\partial t} + n_0\frac{\partial u}{\partial \ell}\Big) + \Big(\frac{n_0}{T_0}\frac{\partial T_1}{\partial t} + \frac{2}{3}n_0\frac{\partial u}{\partial \ell}\Big)P^{01}$$

$$-\frac{v_0}{T_0}\Big(m n_0 \frac{\partial u}{\partial t} + \frac{\partial p_1}{\partial \ell} - n_0 q E_\parallel\Big)P^{10} + \frac{n_0 v_0}{T_0}\frac{\partial T_1}{\partial \ell}P^{11} - \frac{4}{3}n_0\frac{\partial u}{\partial \ell}P^{20}\Bigg]\hat{f}_0$$

where $\hat{f}_0 = (n_0/\pi^{3/2}v_0^3)e^{-s^2}$, $s = v/\sqrt{2T_0/m}$ and $P^{20} = P_2(v_\parallel/v)L_k^{(l+1/2)}(s^2)$:

$$P^{01} = \tfrac{3}{2} - s^2, \quad P^{02} = \tfrac{1}{8}(s^4 - 20s^2 + 15)$$

$$P^{10} = s_\parallel, \quad P^{11} = s_\parallel(\tfrac{5}{2} - s^2)$$

$$P^{20} = s_\parallel^2 - \tfrac{1}{2}s_\perp^2, \quad P^{21} = (s_\parallel^2 - \tfrac{1}{2}s_\perp^2)(\tfrac{7}{2} - s^2)$$

How to get nontrivial solution

- Linearized DKE with $\partial/\partial t = 0 \Rightarrow$ trivial solutions

$$\frac{\partial F}{\partial t} + v_\parallel \frac{\partial F}{\partial \ell} + \nu F = \Bigg[-\left(\frac{\partial n_1}{\partial t} + n_0 \frac{\partial u}{\partial \ell}\right) + \left(\frac{n_0}{T_0}\frac{\partial T_1}{\partial t} + \frac{2}{3} n_0 \frac{\partial u}{\partial \ell}\right) P^{01}$$

$$- \frac{v_0}{T_0}\left(m n_0 \frac{\partial u}{\partial t} + \frac{\partial p_1}{\partial \ell} - n_0 q E_\parallel \right) P^{10} + \frac{n_0 v_0}{T_0}\frac{\partial T_1}{\partial \ell} P^{11} - \frac{4}{3} n_0 \frac{\partial u}{\partial \ell} P^{20} \Bigg] \hat{f}_0$$

$$\stackrel{\text{F.T.}}{\Longrightarrow} \; \tilde{F} = \frac{1}{ikv_\parallel + \nu} \Bigg[-n_0 ik\tilde{u} + \frac{2}{3} n_0 ik\tilde{u} P^{01}$$

$$- \left(\frac{v_0}{T_0} ik\tilde{p}_1 - n_0 q \tilde{E}_\parallel \right) P^{10} + \frac{n_0 v_0}{T_0} ik\tilde{T}_1 P^{11} - \frac{4}{3} n_0 ik\tilde{u} P^{20} \Bigg]$$

$$\stackrel{\nu \to 0}{\Longrightarrow} \; \tilde{h}_\parallel = -T_0 v_0 \int d\mathbf{v} P^{11} \tilde{F} = 0, \;\; \tilde{\pi}_\parallel = T_0 \int d\mathbf{v} \tfrac{4}{3} P^{20} \tilde{F} = 0$$

- Hammett & Perkins (1990): $h_\parallel = \int d\mathbf{v} \frac{1}{2} m w^2 w_\parallel \neq \int d\mathbf{v} \frac{1}{2} m w_\parallel^3$, $\mathbf{w} = \mathbf{v} - \mathbf{V}$
- Chang & Callen (1992): substitute Maxwellian moment equations, k-space
- Hazeltine (1998): introduce external source, 1D

Parallel closures in the collisionless limit

- Linearized DKE with Maxwellian moment equations replaced

$$\frac{\partial F}{\partial t} + v_\parallel \frac{\partial F}{\partial \ell} + \nu F = \left[-\frac{2}{3}\frac{1}{T_0}\frac{\partial h_\parallel}{\partial \ell} P^{01} + \frac{v_0}{T_0}\frac{\partial \pi_\parallel}{\partial \ell} P^{10} + \frac{n_0 v_0}{T_0}\frac{\partial T_1}{\partial \ell} P^{11} - \frac{4}{3} n_0 \frac{\partial u}{\partial \ell} P^{20} \right] \hat{f}_0$$

$$\overset{\text{F.T.}}{\Longrightarrow}$$

$$\tilde{F} = \frac{1}{ikv_\parallel + \nu - i\omega} \left[-\frac{2}{3}\frac{1}{T_0} ik\tilde{h}_\parallel P^{01} + \frac{v_0}{T_0} ik\tilde{\pi}_\parallel P^{10} + \frac{n_0 v_0}{T_0} ik\tilde{T}_1 P^{11} - \frac{4}{3} n_0 ik\tilde{u} P^{20} \right] \hat{f}_0$$

$$\overset{\nu,\omega\to 0}{\Longrightarrow} \begin{pmatrix} \dfrac{\tilde{h}_\parallel}{v_0 T_0} \\ \dfrac{\tilde{\pi}}{T_0} \end{pmatrix} = \begin{pmatrix} \dfrac{9}{5\sqrt{\pi}}\dfrac{k}{i|k|} & -\dfrac{2}{5} \\ -\dfrac{2}{5} & \dfrac{4\sqrt{\pi}}{5}\dfrac{k}{i|k|} \end{pmatrix} \begin{pmatrix} \dfrac{n_0 \tilde{T}_1}{T_0} \\ \dfrac{n_0 \tilde{u}}{v_0} \end{pmatrix}$$

$$\overset{\text{inverse F.T.}}{\Longrightarrow} \quad h_\parallel(\ell) = \frac{9}{5\pi^{3/2}} n_0 v_0 \int_{-\infty}^{\infty} d\ell' \frac{T_1(\ell')}{\ell - \ell'} - \frac{2}{5} n_0 T_0 u(\ell)$$

$$\pi_\parallel(\ell) = -\frac{2}{5} n_0 T_1(\ell) + \frac{4}{5\sqrt{\pi}} \frac{n_0 T_0}{v_0} \int_{-\infty}^{\infty} d\ell' \frac{u(\ell')}{\ell - \ell'}$$

Comments on neoclassical transport in the banana regime

- δ^0 : $f_{a0} = \frac{n_0(\psi)}{\pi^{3/2} v_0^3} e^{-s^2},\ s = \frac{v}{v_0},\ v_0 = \sqrt{\frac{2T_{a0}(\psi)}{m}}$
- δ^1 : $\mathbf{v}_\| \cdot \nabla \bar{f}_{a1} + \mathbf{v}_\mathrm{D} \cdot \nabla f_{a0} = C(\bar{f}_{a1})$

  For $\mathbf{B} = I(\psi)\nabla\varphi + \nabla\varphi \times \nabla\psi \quad \Longrightarrow \quad \mathbf{v}_\mathrm{D} \cdot \nabla f_{a0} = v_\| \partial_\| k_a$

  where $k_a = -\frac{I v_\|}{\Omega_a} \frac{\partial f_{a0}}{\partial \psi} = -\frac{I}{\Omega_a v_0} \left(\frac{dp_0}{d\psi} P^{10} - \frac{dT_0}{d\psi} P^{11} \right) \hat{f}_{a0}$

$$v_\| \frac{\partial}{\partial \ell} (\bar{f}_{a1} - k_a) = C(\bar{f}_{a1})$$

- Auxiliary expansion $f_{a1} = f_{a1}^{(0)} + f_{a1}^{(1)} + \cdots$ based on low collisionality

$$0^{\mathrm{th}} : \ v_\| \frac{\partial}{\partial \ell} (f_{a1}^{(0)} - k_a) = 0$$

$$\Rightarrow \ f_{a1}^{(0)}(\ell, \cdots) = k_a(\ell, \cdots) + g_a(\ell\text{−indep.}, \cdots) + \text{missing?}$$

$$1^{\mathrm{st}} : \ v_\| \frac{\partial}{\partial \ell} f_{a1}^{(1)} = C(f_{a1}^{(0)}) \ \Rightarrow \left\langle \frac{B}{v_\|} C_a(k_a + g_a) \right\rangle = 0 \ \Rightarrow g_a$$

- Particular solution: $v_\| \frac{\partial}{\partial \ell} f_{a1}^{(0)} = G_a,\ G_a = v_\| \frac{\partial k_a}{\partial \ell} \Rightarrow \ f_{a1}^{(0)} = f(\{G_a\})$

Parallel transport on flux surfaces $\mathbf{v}_\mathrm{D} \cdot \nabla f_0 \sim \mathbf{v}_\| \cdot \nabla f_1 \sim O(\delta^1)$

- δ^1 : $(\mathbf{x}, w, \mu)$: $\dfrac{\partial \bar{f}_1}{\partial t} + v_\| \mathbf{b} \cdot \nabla \bar{f}_1 + \mathbf{v}_\mathrm{D} \cdot \nabla f_0 + q(v_\| E_{1\|} + \mathbf{v}_\mathrm{D} \cdot \mathbf{E}_0) \dfrac{\partial \bar{f}_0}{\partial w} = C(\bar{f}_1)$

  $\bar{f}_1 = f_0 + F \quad \Rightarrow \quad \dfrac{\partial F}{\partial t} + v_\| \dfrac{\partial F}{\partial \ell} + \nu F = G_\mathrm{t}$

  where $G_\mathrm{t} = (G_\mathrm{t}^{00} + G_\mathrm{t}^{01} P^{01} + G_\mathrm{t}^{02} P^{02} + G_\mathrm{t}^{10} P^{10} + G_\mathrm{t}^{20} P^{20} + G_\mathrm{t}^{21} P^{21}) \hat{f}_0$

  $$G_\mathrm{t}^{00} = -2\alpha \frac{dp_{0*}}{d\psi},\ G_\mathrm{t}^{01} = \frac{2}{3}\alpha(2\frac{dp_{0*}}{d\psi} + 5n_0 \frac{dT_0}{d\psi}),\ G_\mathrm{t}^{02} = -\frac{8}{3}\alpha n_0 \frac{dT_0}{d\psi},$$

  $$G_\mathrm{t}^{10} = \frac{n_0 q v_0 E_{1\|}}{T_0},\ G_\mathrm{t}^{20} = -\frac{2}{3}\alpha(\frac{dp_{0*}}{d\psi} + n_0 \frac{dT_0}{d\psi}),\ G_\mathrm{t}^{21} = \frac{2}{3}\alpha n_0 \frac{dT_0}{d\psi}$$

  with $\alpha = \dfrac{1}{qB^2} \mathbf{b} \times \nabla B \cdot \nabla \psi$ and $p_{0*} = p_0 + n_0 q \Phi_0$

- Axisymmetry $\alpha = \dfrac{I}{q} \dfrac{dB^{-1}}{d\ell}$: $n_0 u(\ell) = 2 \dfrac{I B_1(\ell)}{q B_0^2} \dfrac{dp_{0*}}{d\psi}$ and $h_\|(\ell) = 5 p_0 \dfrac{I B_1(\ell)}{q B_0^2} \dfrac{dT_0}{d\psi}$

  $$n_1(\ell) = -\frac{I}{q B_0^2 v_0 \sqrt{\pi}} \int d\ell' \frac{B_1(\ell')}{\ell - \ell'} (\frac{dp_0}{d\psi} - \frac{n_0}{2} \frac{dT_0}{d\psi}) - \frac{n_0 q \Phi_1(\ell)}{T_0}$$

  $$T_1(\ell) = -\frac{T_0 I}{q B_0^2 v_0 \sqrt{\pi}} \int d\ell' \frac{B_1(\ell')}{\ell - \ell'} (\frac{1}{2n_0} \frac{dp_0}{d\psi} + \frac{7}{4} \frac{dT_0}{d\psi})$$

Parallel closures on flux surfaces $\mathbf{v}_\mathrm{D}\cdot\nabla f_0 \sim \mathbf{v}_\parallel\cdot\nabla f_1 \sim O(\delta^1)$

- δ^1 : $(\mathbf{x}, w, \mu)$: $\dfrac{\partial \bar f_1}{\partial t} + v_\parallel \mathbf{b}\cdot\nabla \bar f_1 + \mathbf{v}_\mathrm{D}\cdot\nabla f_0 + q(v_\parallel E_{1\parallel} + \mathbf{v}_\mathrm{D}\cdot\mathbf{E}_0)\dfrac{\partial \bar f_0}{\partial w} = C(\bar f_1)$

  Set $\bar f_1 = \bar f_1^\mathrm{M} + F$, $\bar f_1^\mathrm{M} = f_0\left[\dfrac{n_1}{n_0} + \left(\dfrac{v^2}{v_T^2} - \dfrac{3}{2}\right)\dfrac{T_1}{T_0} + \dfrac{m}{T_0}v_\parallel u\right]$

$$\frac{\partial F}{\partial t} + v_\parallel \frac{\partial F}{\partial \ell} + \nu F = \Bigg[-\frac{2}{3}\frac{1}{T_0}\frac{\partial h_\parallel}{\partial \ell}P^{01} + \frac{v_0}{T_0}\frac{\partial \pi_\parallel}{\partial \ell}P^{10} + G_\mathrm{t}^{02}P^{02} + G_\parallel^{11}P^{11} + (G_\parallel^{20} + G_\mathrm{t}^{20})P^{20} + G_\mathrm{t}^{21}P^{21}\Bigg]\hat f_0$$

$$\begin{pmatrix} \dfrac{\tilde h_\parallel}{v_0 T_0} \\ \dfrac{\tilde \pi_\parallel}{T_0} \end{pmatrix} = \frac{1}{ik}\begin{pmatrix} \dfrac{9}{5\sqrt{\pi}}\dfrac{|k|}{ik} & \dfrac{3}{10} & \dfrac{33}{40} & -\dfrac{3}{4} \\ -\dfrac{2}{5} & \dfrac{3\sqrt{\pi}}{5}\dfrac{|k|}{ik} & \dfrac{\sqrt{\pi}}{10}\dfrac{|k|}{ik} & -\dfrac{\sqrt{\pi}}{2}\dfrac{|k|}{ik} \end{pmatrix}\begin{pmatrix} \tilde G_\parallel^{11} \\ \tilde G_\parallel^{20} + \tilde G_\mathrm{t}^{20} \\ \tilde G_\mathrm{t}^{02} \\ \tilde G_\mathrm{t}^{21} \end{pmatrix}$$

$$h_\parallel(\ell) = \frac{9 n_0 v_0}{5\pi^{3/2}}\int d\ell' \frac{T_1(\ell')}{\ell - \ell'} - \frac{2}{5}n_0 T_0 u(\ell) - \frac{v_0 T_0 I}{q B_0^2(\ell)}\Big(\frac{1}{5}\frac{dp_{0*}}{d\psi} + \frac{29}{10}\frac{dT_0}{d\psi}\Big)$$

$$\pi_\parallel(\ell) = -\frac{2}{5}n_0 T_1(\ell) + \frac{4 p_0}{5\sqrt{\pi} v_0}\int d\ell' \frac{u(\ell')}{\ell - \ell'} + \frac{T_0 I}{q}\int d\ell' \frac{B^{-1}(\ell')}{\ell - \ell'}\Big(\frac{2}{5}\frac{dp_{0*}}{d\psi} - \frac{n_0}{5}\frac{dT_0}{d\psi}\Big)$$

Future work

- Find closures in the moderate collisionality regime for finite perturbations $(n_1,\,T_1,\,B_1)$
- Interpolate closures to the collisionless regime
- Develop 21 or 29 moment equations (dynamic closures) in NIMROD