Boundary integral methods for NIMROD resistive wall

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Outline

1. Introduction
2. External Vacuum Solver
3. Accuracy and convergence tests

Goal: To achieve higher-order convergence in NIMROD with a boundary-integral method (BIM) for problems that use a (thin) resistive wall.

- Use free-space Green's functions to calculate the vacuum fields to update the normal component of **B** (B_n); external to NIMROD†
- The implementation mirrors the existing NIMROD–GRIN‡ interface, except it:
  - requires NO additional interpolation in NIMROD.
  - provides the matrix necessary to compute the tangential magnetic field, $\mathbf{B}_{Tx}$, required to update B_n.
- An alternative to Carl Sovinec's coding of the vacuum region in NIMROD.
- Originally built by D. Barnes to model the Tibbar electrical transformer (finite cylinder).

† All the work is being performed with the developer version of NIMROD, NIMDEVEL

‡ Becera, Sovinec, Hegna, Tech-X

Progress to date

- The solver has demonstrated $\mathcal{O}(10^{-9} - 10^{-10})$ accuracy for $n = 0$ and $n = 1$.
- Solver tests on a circular torus indicate a higher-order (algebraic) convergence for $n = 0$ and $n = 1$.
- Successful interface with NIMROD.
  - Debugging the incorporation of the derivative matrix to compute $\mathbf{B}_{Tx}$.
- H–refinement tests in NIMROD produced the same growth rates as those from GRIN for the $(2, 1)$ RWM, with better convergence rates.
- Analytic verification *via* toroidal Bessel functions: ring functions.

Outline

Given a normal **B**, B_n, on the boundary, calculate the magnetic potential Φ_M and the tangential field $\mathbf{B}_{Tx}$: Neumann problem

$$\mathbf{B}_x = \nabla\Phi_M, \nabla^2\Phi_M = 0$$

$$\text{continuous } B_n = \partial_n \Phi_M \Rightarrow \Phi_M$$

$$\Rightarrow \mathbf{B}_{Tx} = \left(\Phi_M{}', \frac{in}{R}\Phi_M \right)$$

$$\Rightarrow [[\mathbf{B}_T]] \Rightarrow \mathbf{J}_T^* \Rightarrow \mathbf{E}_T \Rightarrow \dot{B}_n$$

$$\Phi_M = \int dS \frac{\sigma(S)}{\|\mathbf{r}-\mathbf{r}_S\|} \approx\approx D\sigma$$

$$\partial_n \Phi_M = \int dS \sigma(S) \partial_n \frac{1}{\|\mathbf{r}-\mathbf{r}_S\|} \approx\approx N\sigma$$

$$S = DN^{-1}$$

where S is the response matrix, *i.e.*, $\Phi_M = SB_n$.

- A derivative matrix, S', required for the poloidal component of $\mathbf{B}_{Tx}$: $\Phi'_M = S'B_n$, is also computed.

Components of the boundary integral approach

- Thin wall approximation: continuous B_n and a jump in $\mathbf{B}_t$ at $r = r_w$.
- Solve $\nabla^2 \Phi_M = 0$, using the free-space Green's function.
- Approximate integrals due to a (2-D) logarithmic singularity.
  - $n = 0$ & $n = 1$ use elliptic integrals.
  - $n > 1$ by recursion.
- Matrix inversion and algebra:
  - $\Phi_M = SB_n$
  - $\Phi'_M = S'B_n$ to calculate $\mathbf{B}_{Tx}$.
- All of this is done in `nimset` time scales, independent of the time-advance.
- S and S' are never recalculated during the NIMROD time-advance.

Vacuum solver challenges

- The Nyström collocation method (Young & Martinsson) is used to discretize the boundary.
- This requires an interpolation from the collocation points to the NIMROD finite-element (FE) nodes.

  This interpolation is performed in the vacuum solver, external to NIMROD$\Rightarrow$ no additional interpolation in NIMROD!
- Special quadratures to handle the logarithmic singularities:
  - Far segments use Gauss-Legendre (GL) quadrature.
  - Next door segments use special modified GL (Jim Bremer).
  - Same segments use yet different modified GL.

Outline

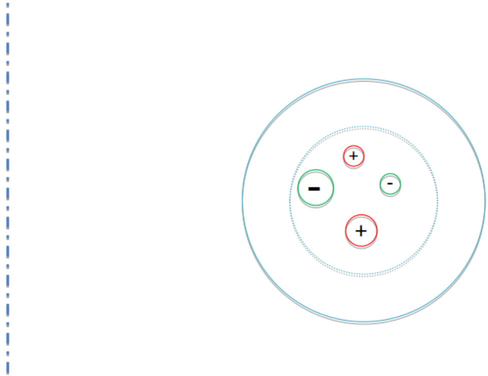
Testing

- Test the accuracy and convergence rates of the vacuum solver externally with a manufactured solution. ✓
- Test the accuracy and convergence rates in NIMROD on an established resistive wall mode (RWM) test problem[§] ✓
- Test analytically by solving $\nabla^2 \Phi_M = 0$. ✗

[§] Andi Becera's master thesis

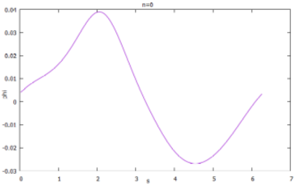
Testing of the vacuum solver

- Manufacture a solution using several random sources (rings of charge density) inside the torus.

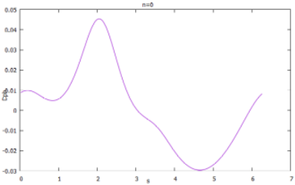


- Given $\partial_n \Phi_M$, solve for Φ_M and compare to the manufactured solution.

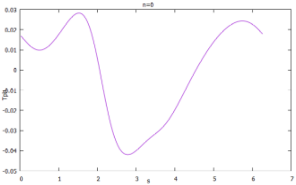
Success: Numerical solution agrees well with the manufactured solution (for both $n = 0$ and $n = 1$)



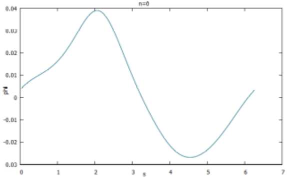
Potential



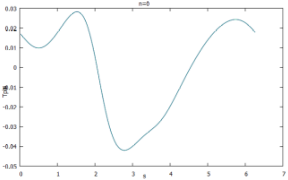
Normal Derivative



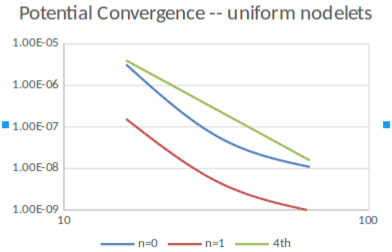
Tangential Derivative

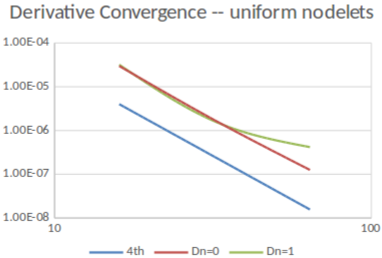


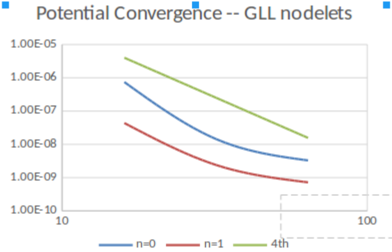
$n = 0$

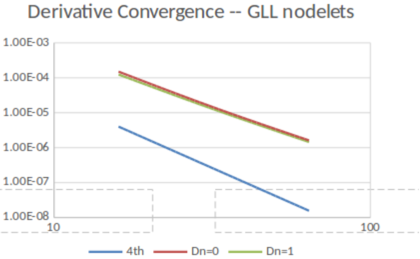


External h–refinement tests demonstrate higher-order convergence.









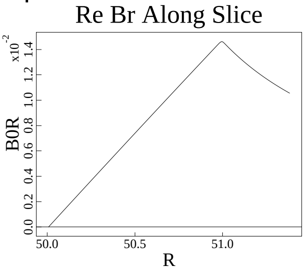
Preliminary tests in NIMROD on a very large-aspect ratio (50) torus that is unstable to (2, 1) RWM¶

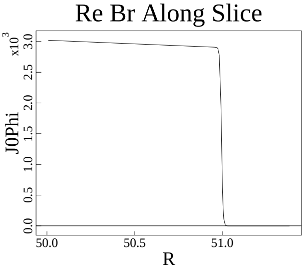
- Smoothed top-hat equilibrium current profile

$$J(\psi) = \frac{1}{2} J_0 [tanh(\psi_v - \psi) w_J + 1] \quad (1)$$

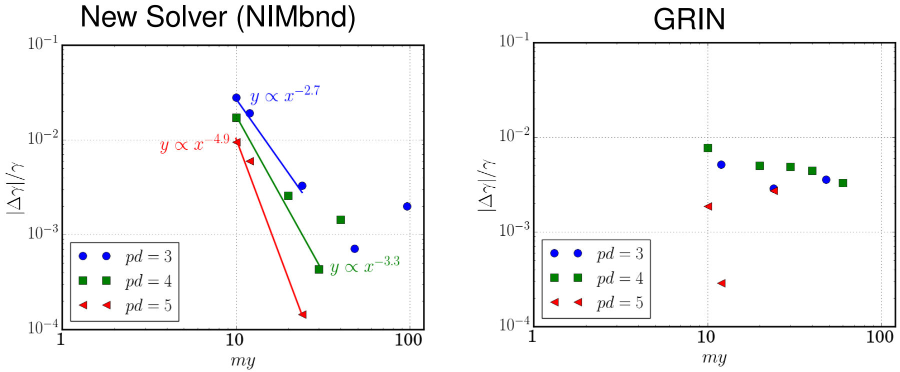
yields $q_0 = 1.06$.

- Resistive wall of several mm. in thickness located at $r_w = 1.4$
- Growth rate converges to that of a periodic cylinder for very large aspect ratios.





H-refinement tests using the new vacuum solver indicate high-order convergence



- Better performance than the NIMROD–GRIN implementation (right), especially for even-order FE's.
- External tests indicate a convergence rate of *pd* or better.

Analytic verification

- Impose a B_n and solve for Φ_M *via* $\Phi_M = SB_n$.
- The analytic solution of $\nabla^2 \chi = 0$ is:

$$\chi(\rho, \theta) = \sqrt{\cosh \rho - \cos \theta} \sum_m \alpha_m Q^n_{m-1/2}(\cosh \rho)\, e^{im\theta}, \qquad (2)$$

where $Q^n_{m-1/2}$'s are the ring functions for the exterior of the torus.

- Solve for α_m's by setting $\Phi_M = \chi$ at $\rho = \rho_0$.
- Compute and compare the resulting B_n to that of the vacuum solver:

$$B_n(\theta) = \frac{\cosh \rho_0 - \cos \theta}{a} \sinh \rho_0 \sum_m \alpha_m e^{im\theta} \times \left(\frac{1}{2\sqrt{\cosh \rho_0 - \cos \theta}} Q^n_{m-1/2}(\cosh \rho_0) + \sqrt{\cosh \rho_0 - \cos \theta}\, Q^{n\,\prime}_{m-1/2}(\cosh \rho_0) \right).$$

- If B_n is good, check $\mathbf{B}_{Tx}$ by computing and comparing χ' to Φ'_M.

Conclusions: A new vacuum solver based on the free-space Green's function has been developed and plugged into NIMROD.

- The new solver
  - uses special quadratures to handle logarithmic singularities,
  - interpolates natively to output the solution at the NIMROD nodes $\Rightarrow$ no additional interpolation in NIMROD!,
  - calculates the tangential **B** ($\mathbf{B}_{Tx}$) internally, outside NIMROD,
  - reproduces a manufactured solution to $\mathcal{O}(10^{-9} - 10^{-10})$,
  - has demonstrated higher-order convergence external to NIMROD.
- H–refinement tests for the $(2, 1)$ RWM problem in NIMROD yield a $\sim$cubic convergence rate, independent of FE polynomial order.
- The fixed convergence rate suggests improving the interpolation.
- Interfacing the derivative matrix required for $\mathbf{B}_{Tx}$ could also improve the convergence rates.

Future Work

- Finish the incorporation of the derivative matrix required for the tangential **B** into NIMROD.

  Then rerun the h-refinement tests.
- Explore other interpolation methods for the vacuum solver.

  Then rerun the h-refinement tests.
- Calculate the response and derivative matrices for $n > 1$.
- Finish the analytic verification, using the toroidal ring functions.
- Publish in PoP or JCP (or both?)