FEM Formulation Comparison for Electron MHD

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Outline

Electron MHD is being used as a model problem for the Hall term in Ohm's Law

With the assumption $\omega_{ci} \ll \omega \ll \omega_{ce} \approx \omega_{pe}$ the linearized MHD equations can be written

$$\frac{\partial \tilde{\vec{B}}}{\partial t} = -\vec{\nabla} \times \tilde{\vec{E}} \qquad \text{(Faraday's Law)}$$

$$\tilde{\vec{E}} = \frac{\tilde{\vec{J}}}{ne} \times \vec{B}_0 \qquad \text{(Ohm's Law)}$$

$$\mu_0 \tilde{\vec{J}} = \vec{\nabla} \times \tilde{\vec{B}} \qquad \text{(Ampere's Law)}$$

$$\vec{\nabla} \cdot \tilde{\vec{B}} = 0 \qquad \text{(Divergence-Free Constraint)}$$

Using $H\left(\vec{\nabla}\times\right)$ conforming elements for $\tilde{\vec{B}}$

Find $\vec{b} \in H\left(\vec{\nabla}\times\right)$ such that,

$$\left\langle \frac{\partial \vec{b}}{\partial t}, \vec{c} \right\rangle = \left\langle \frac{\vec{B}_0}{ne\mu_0} \times \left(\vec{\nabla} \times \vec{b}\right), \vec{\nabla} \times \vec{c} \right\rangle \qquad \forall \vec{c} \in H\left(\vec{\nabla}\times\right)$$

- This formulation does not explicitly constrain $\vec{\nabla} \cdot \vec{b} = 0$
- The bilinear form on the right-hand side is not symmetric

Using H^1 conforming elements for $\tilde{\vec{B}}$

Find $\vec{b} \in H^1$ such that,

$$\left\langle \frac{\partial \vec{b}}{\partial t}, \vec{c} \right\rangle = \left\langle \frac{\vec{B}_0}{ne\mu_0} \times \left(\vec{\nabla} \times \vec{b} \right), \vec{\nabla} \times \vec{c} \right\rangle - \kappa \left\langle \vec{\nabla} \cdot \vec{b}, \vec{\nabla} \cdot \vec{c} \right\rangle \quad \forall \vec{c} \in H^1$$

- Constrains the divergence error with dissipation
- Satisfaction of the $\inf\text{-}\sup$ condition relies on appropriate κ
- Bilinear form is also not symmetric

$\vec{B}$ and $\vec{J}$ as fundamental quantities

Define residuals with time centering θ,

$$\vec{R}_B = \dot{\vec{b}} + \vec{\nabla} \times \left[\left(\vec{j} + \theta \Delta t \dot{\vec{j}} \right) \times \frac{\vec{B}_0}{ne} \right]$$

$$\vec{R}_J = \mu_0 \left(\vec{j} + \theta \Delta t \dot{\vec{j}} \right) - \vec{\nabla} \times \left(\vec{b} + \theta \Delta t \dot{\vec{b}} \right)$$

$$R_{dB} = \vec{\nabla} \cdot \left(\vec{b} + \theta \Delta t \dot{\vec{b}} \right)$$

$$R_{dJ} = \vec{\nabla} \cdot \left(\vec{j} + \theta \Delta t \dot{\vec{j}} \right)$$

Minimize I with appropriate weights C_B, C_J, C_{dB}, and C_{dJ},

$$I = \int \left(C_B \vec{R}_B \cdot \vec{R}_B + C_J \vec{R}_J \cdot \vec{R}_J + C_{dB} R_{dB}^2 + C_{dJ} R_{dJ}^2 \right) \mathrm{d}\Omega$$

where $\vec{b}, \vec{j} \in H^1$

$\vec{B}$ and $\vec{E}$ as fundamental quantities

Define residuals with time centering θ,

$$\vec{R}_B = \dot{\vec{b}} + \vec{\nabla} \times \left(\vec{e} + \theta \Delta t \dot{\vec{e}}\right)$$

$$\vec{R}_E = \vec{e} + \theta \Delta t \dot{\vec{e}} - \left[\vec{\nabla} \times \left(\vec{b} + \theta \Delta t \dot{\vec{b}}\right)\right] \times \frac{\vec{B}_0}{n e \mu_0}$$

$$R_{dB} = \vec{\nabla} \cdot \left(\vec{b} + \theta \Delta t \dot{\vec{b}}\right)$$

Minimize I with appropriate weights C_B, C_J, C_{dB}, and C_{dJ},

$$I = \int \left(C_B \vec{R}_B \cdot \vec{R}_B + C_E \vec{R}_E \cdot \vec{R}_E + C_{dB} R_{dB}^2\right) \mathrm{d}\Omega$$

where $\vec{b} \in H^1$ and $\vec{e} \in H\left(\vec{\nabla}\times\right)$

MFEM is a flexible finite element library developed by LLNL

- Supports H^1, $H\left(\vec{\nabla}\times\right)$, $H\left(\vec{\nabla}\cdot\right)$, and L_2 conforming finite elements of arbitrary order
- Ships with many bilinear and linear integrators pre-programmed
- Programming new integration terms is relatively simple
- Easy to get basic examples working
- Parallel capabilities with many solver options