Parallel closures and transport for toroidal plasmas in the collisionless limit

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Fluid equations and closures/transport

Maxwellian moment $(n_a, \mathbf{V}_a, T_a)$ equations

$$(0{,}0)\ d_t n_a + n_a \nabla\cdot\mathbf{V}_a = 0\,(d_t \equiv \partial_t + \mathbf{V}_a\cdot\nabla)$$

$$(0{,}1)\ \tfrac{3}{2} n_a d_t T_a + n_a T_a \nabla\cdot\mathbf{V}_a + \nabla\cdot\mathbf{h}_a + \nabla\mathbf{V}_a : \boldsymbol{\pi}_a = Q_a$$

$$(1{,}0)\ m_a n_a d_t \mathbf{V}_a - n_a q_a(\mathbf{E} + \mathbf{V}_a\times\mathbf{B}) + \nabla p_a + \nabla\cdot\boldsymbol{\pi}_a = \mathbf{R}_a$$

General moment equations $D\mathsf{n} + \Omega\mathbf{b}\breve{\times}\mathsf{n} = C\mathsf{n}$ ($\mathsf{n}^{lk} \to v^{l+2k}$ moment)

$$(1,1)\ d_t\mathbf{h} + \Omega\mathbf{b}\times\mathbf{h} + \tfrac{7}{5}(\nabla\cdot\mathbf{V})\mathbf{h} + \tfrac{7}{5}\mathbf{h}\cdot(\nabla\mathbf{V}) + \tfrac{2}{5}(\nabla\mathbf{V})\cdot\mathbf{h} + \frac{5p}{2m}\nabla T$$
$$+\frac{T}{m}\nabla\cdot\boldsymbol{\pi} + \frac{7}{2}\frac{\nabla T}{m}\cdot\boldsymbol{\pi} - \mathbf{a}\cdot\boldsymbol{\pi} + \nabla\cdot\boldsymbol{\theta} + \tfrac{1}{3}\nabla\mathsf{u}^{02} + \nabla\mathbf{V} : \mathsf{u}^{30}$$
$$= C^1_{10}\mathbf{V}_{\rm ei} + C^1_{11}\mathbf{h} + C^1_{12}\mathbf{r} + \cdots\ (\mathbf{h}\text{ heat flow})$$

$$(1,2)\ d_t\mathbf{r} + \Omega\mathbf{b}\times\mathbf{r} + \cdots = C^1_{10}\mathbf{V}_{\rm ei} + C^1_{21}\mathbf{h} + C^1_{22}\mathbf{r} + \cdots\quad (\mathbf{r}\text{ heat heat flow})$$

$$(2,0)\ d_t\boldsymbol{\pi} + \Omega\mathbf{b}\breve{\times}\boldsymbol{\pi} + (\nabla\cdot\mathbf{V})\boldsymbol{\pi} + \overline{2\boldsymbol{\pi}\cdot(\nabla\mathbf{V})} + p\mathsf{W} + \tfrac{4}{5}\overline{\nabla\mathbf{h}} + \nabla\cdot\mathsf{u}^{30}$$
$$= C^2_{00}\boldsymbol{\pi} + C^2_{01}\boldsymbol{\theta} + \cdots\ (\boldsymbol{\pi}\text{ viscosity})$$

$$(2,1)\ d_t\boldsymbol{\theta} + \Omega\mathbf{b}\breve{\times}\boldsymbol{\pi} + \cdots = C^2_{10}\boldsymbol{\pi} + C^2_{11}\boldsymbol{\theta} + \cdots\ (\boldsymbol{\theta}\text{ heat viscosity})$$

where $\mathbf{a} = \frac{q}{m}(\mathbf{E} + \mathbf{V}\times\mathbf{B}) - d_t\mathbf{V}$ and $\mathsf{W} = \nabla\mathbf{V} + (\nabla\mathbf{V})^{\mathsf{T}} - \frac{2}{3}\nabla\cdot\mathbf{V}\mathsf{I}$

Closures: express $\mathbf{h}_a(\mathsf{n}_a^{11})$, $\boldsymbol{\pi}_a(\mathsf{n}_a^{20})$, Q_a, $\mathbf{R}_a$ in terms of $n_a, \mathbf{V}_a, T_a$

Electron closures for high collisionality (Braginskii)

$\mathbf{h}_{\rm e} = (\beta)(\mathbf{V}_{\rm ei}) - (\kappa)(\nabla T_{\rm e})$, $\mathbf{R}_{\rm e} = -\mathbf{R}_{\rm i} = -(\alpha)(\mathbf{V}_{\rm ei}) - (\beta)(\nabla T_{\rm e})$

Transport: relate flux densities $\mathbf{h}_{\rm e}$, $\mathbf{J}$ to thermodynamic forces $\nabla T_{\rm e}$ and $\mathbf{E}$

$\mathbf{h}_{\rm e} = (\tilde{\alpha})\mathbf{E} - (\tilde{\kappa})(\nabla T_{\rm e})$, $\mathbf{J} = (\tilde{\sigma})\mathbf{E} - (\tilde{\alpha})(\nabla T_{\rm e})$

Integral (nonlocal) parallel closures for arbitrary collisionality

- n_A responding to g_B:

$$n_{AB}(\eta) = \int d\eta' K_{AB}(\eta-\eta') g_B(\eta') \;\rightarrow\; n_{AB}(\eta) = \kappa_{AB}(\eta) g_B(\eta)$$

$$h_\parallel(\eta) = -\frac{1}{2}Tv_T \int d\eta' K_{hh}\frac{n}{T}\frac{dT}{d\eta'} + Tv_T\int d\eta' K_{hR} Zn\frac{V_{\mathrm{ei}\parallel}}{v_T} - Tv_T\int d\eta' K_{h\pi}(\frac{3}{4}n\tau_{\mathrm{ee}}W_\parallel)$$

$$R_\parallel(\eta) = -\frac{mn}{\tau_{\mathrm{ei}}}V_{\mathrm{ei}\parallel} + \frac{mv_T}{\tau_{\mathrm{ei}}}\int d\eta'[-K_{Rh}\frac{n}{2T}\frac{dT}{d\eta'} + K_{RR}Zn\frac{V_{\mathrm{ei}\parallel}}{v_T} - K_{R\pi}(\frac{3}{4}n\tau_{\mathrm{ee}}W_\parallel)]$$

$$\pi_\parallel(\eta) = -T\int d\eta' K_{\pi h}\frac{n}{T}\frac{dT}{d\eta'} + 2T\int d\eta' K_{\pi R}Zn\frac{V_{\mathrm{ei}\parallel}}{v_T} - T\int d\eta' K_{\pi\pi}(\frac{3}{4}n\tau_{\mathrm{ee}}W_\parallel)$$

Fitted kernel functions $K_{AB}(\eta) = -[d + a\exp(-b\eta^c)]\ln[1-\alpha\exp(-\beta\eta^\gamma)]$

- Fourier transform (FT) of integral closures: $\tilde{n}_{AB}(k) = \tilde{K}_{AB}(k)\tilde{g}_B(k)$

$$\tilde{h}_\parallel = -\frac{1}{2}n_0v_0\tilde{K}_{hh}ik\tilde{T}_1 + p_0\tilde{K}_{hR}\tilde{u}_{\mathrm{ei}} - p_0\tilde{K}_{h\pi}ik\tilde{u}$$

$$\tilde{R}_\parallel = -\frac{1}{2}\frac{mv_0}{\tau_{\mathrm{ee}}}\frac{n_0}{T_0}\tilde{K}_{Rh}ik\tilde{T}_1 - \frac{mn_0}{\tau_{\mathrm{ee}}}\left(1-\tilde{K}_{RR}\right)\tilde{u}_{\mathrm{ei}} - \frac{mn_0}{\tau_{\mathrm{ee}}}\tilde{K}_{R\pi}ik\tilde{u}$$

$$\tilde{\pi}_\parallel = -n_0\tilde{K}_{\pi h}ik\tilde{T}_1 + 2\frac{p_0}{v_0}\tilde{K}_{\pi R}\tilde{u}_{\mathrm{ei}} - \frac{p_0}{v_0}\tilde{K}_{\pi\pi}ik\tilde{u}$$

Fitted kernels in k space $\tilde{K}_{AB} = \dfrac{ak^\alpha}{1+d_1k^\delta+d_2k^{2\delta}+d_3k^{3\delta}+d_4k^{4\delta}+d_5k^{5\delta}+d_6k^{6\delta}}$

Obtain parallel closures in an inhomogeneous magnetic field: For arbitrary aspect ratio and collisionality

- Drift kinetic equation in the $(t, \mathbf{r}, w = \frac{1}{2}mv^2, \mu = mv_\perp^2/2B)$ coordinates

$$\frac{\partial \bar{f}}{\partial t} + (\mathbf{v}_\parallel + \mathbf{v}_\mathrm{D}) \cdot \nabla \bar{f} + q(\mathbf{v}_\parallel + \mathbf{v}_\mathrm{D}) \cdot \mathbf{E} \frac{\partial \bar{f}}{\partial w} = C(\bar{f})$$

$$\mathbf{v}_\mathrm{D} = \frac{1}{qB^2}(q\mathbf{E} - mv_\parallel^2 \boldsymbol{\kappa} - \mu \nabla B - mv_\parallel \frac{\partial \mathbf{b}}{\partial t}) \times \mathbf{B}$$

- Assume

$$f_0 = \frac{n_0}{\pi^{3/2} v_0^3} e^{-v^2/v_0^2} \text{ where } v_0 = \sqrt{\frac{2T_0(\psi)}{m}}$$

- Set $f_1 = f_1^\mathrm{M} + f_1^\mathrm{N}$ and
solve for $f_1^\mathrm{N}[h_\parallel, R_\parallel, \pi_\parallel]$ in terms of $f_0[\frac{d(p_0, T_0)}{d\psi_0}]$ and $f_1^\mathrm{M}[\partial_\parallel(T_1, V_{1\parallel}), V_{1\mathrm{e}\parallel} - V_{1\mathrm{i}\parallel}]$

$$v_\parallel \mathbf{b} \cdot \nabla f_1 + \mathbf{v}_\mathrm{d} \cdot \nabla f_0 + qv_\parallel E_\parallel \frac{\partial f_0}{\partial w} = C(f_1)$$

$$\mathbf{v}_\mathrm{d} = \frac{1}{\Omega} \mathbf{b} \times [(v_\parallel^2 + \frac{1}{2} v_\perp^2) \frac{\nabla B}{B}]$$

Parallel moment equations for axisymmetric geometry

$$[\psi]\left[\partial_{\|}N\right]+[\psi_B](\partial_{\|}\ln B)[N]=\frac{1}{\lambda_{\mathrm{C}}}[c][N]+\frac{1}{v_0}\left[g_{\|}\right]+\frac{B_0\partial_{\|}\ln B}{B}\{[g_p]p_\star+[g_T]T_\star\}$$

$$B=\frac{B_0}{1+\epsilon\cos\theta},\ \partial_{\|}=\mathbf{b}\cdot\nabla=\frac{B^\theta}{B}\frac{\partial}{\partial\theta}=\mathtt{qR}\frac{\partial}{\partial\theta},\ \mathtt{K}_0=\frac{\lambda_{\mathrm{C}}}{\mathtt{qR}}$$

$$\boxed{[\psi]\,\partial_\theta[n]+[\psi_B](\partial_\theta\ln B)[n]=\frac{1}{\mathtt{K}_0}[c][n]+[g_\theta]+\epsilon\sin\theta\{[g_p]p_\star+[g_T]T_\star\}}$$

- Thermodynamic drives

$$p_\star=n_0\rho_0 I\frac{d\ln p_0}{d\psi}=\frac{2I}{qv_0B_0}\frac{dp_0}{d\psi}\text{ and }T_\star=n_0\rho_0 I\frac{d\ln T_0}{d\psi}=\frac{2n_0I}{qv_0B_0}\frac{dT_0}{d\psi}$$

$$\begin{aligned}
g_{ah}^{11} &= \frac{\sqrt{5}}{2}\frac{\partial}{\partial\theta}n_{a0}\hat{T}_a\\
g_{\mathrm{e}R}^{1k} &= \frac{B}{B^\theta}\frac{1}{v_{\mathrm{e}0}\tau_{\mathrm{ee}}}\sqrt{2}Za_{\mathrm{ei}}^{1k0}n_{\mathrm{e}0}\hat{V}_{\mathrm{ei}}\\
g_{a\pi}^{20} &= -\frac{2}{\sqrt{3}}\left[\partial_\theta+\frac{1}{2}(\partial_\theta\ln B)\right]n_{a0}\hat{V}_a
\end{aligned}$$

Fourier expansion method

- Fourier components of moments

$$[n] = [n_0^+] + \sum_{m=1}^{\mathtt{nfo}} \left([n_m^+] \cos m\theta + [n_m^-] \sin m\theta \right)$$

- Algebraic equations for Fourier coefficients of moments

$$[\psi] \, X(\theta) \, [n(\theta)] \to [A] \otimes [X] \, [n] \, , \; X = \frac{d}{d\theta}, \cos m\theta, \sin m\theta$$

$$\sum_{lk} \psi^{jp,lk} X(\theta) n^{lk} \to \sum_{lkq} \psi^{jp,lk} X^{mq} n_q^{lk}$$

- Closures $A, B = h \leftarrow \partial_{\|} T, \; R \leftarrow V_{\mathrm{e}\|} - V_{\mathrm{i}\|}, \pi \leftarrow (\nabla \mathbf{V})_{\|}$

$$\begin{bmatrix} n_1^A \\ n_2^A \\ n_3^A \\ n_4^A \end{bmatrix} = \sum_{B=h,R,\pi} \begin{bmatrix} \chi_{11}^{AB} & \chi_{12}^{AB} & \chi_{13}^{AB} & \chi_{14}^{AB} \\ \chi_{21}^{AB} & \chi_{22}^{AB} & \chi_{23}^{AB} & \chi_{24}^{AB} \\ \chi_{31}^{AB} & \chi_{32}^{AB} & \chi_{33}^{AB} & \chi_{34}^{AB} \\ \chi_{41}^{AB} & \chi_{42}^{AB} & \chi_{43}^{AB} & \chi_{44}^{AB} \end{bmatrix} \begin{bmatrix} g_1^B \\ g_2^B \\ g_3^B \\ g_4^B \end{bmatrix} + \sum_{\beta=p,T} \begin{bmatrix} \chi_1^{A\beta} \\ \chi_2^{A\beta} \\ \chi_3^{A\beta} \\ \chi_4^{A\beta} \end{bmatrix} \frac{d\beta_0}{d\psi}$$

with $(0, 1^-, 1^+, 2^-, 2^+, \cdots) \to (1, 2, 3, 4, 5, \cdots)$ and $(l, k) \to A$

Solving drift kinetic equation in the collisionless limit

- Adopt a simple model collision operator and take the $\nu \to 0$ limit

$$C(f_1) = -\nu f_1^{\mathrm{N}} = -\nu(f_1 - f_1^{\mathrm{M}})$$

- Drift kinetic equation with fluid equations removed

$$\Big(\frac{\partial}{\partial\theta} + \frac{\alpha\nu}{v_0 s_{\|}}\Big)F = G \equiv G^{p0} + G^{T0} + G^{h} + G^{\pi} + Dh + D\pi$$

where $f_1^{\mathrm{N}} = f_0 F$, $\hat{w} = \frac{1}{2}mv^2/T_0$, $\hat{\mu} = \frac{1}{2}mv_\perp^2/(T_0\hat{B})$, $\hat{B} = B/B_0$, and $s_{\|} = \pm\sqrt{\hat{w} - \hat{\mu}\hat{B}}$

$$\begin{aligned}
G^{p0} &= \frac{P^{20}}{3s_{\|}}\rho_0 I \frac{d\ln p_0}{d\psi}\frac{\partial \ln \hat{B}}{\hat{B}\partial\theta} \\
G^{T0} &= \frac{4P^{02} + P^{20} - P^{21}}{3s_{\|}}\rho_0 I \frac{d\ln T_0}{d\psi}\frac{\partial \ln \hat{B}}{\hat{B}\partial\theta} \\
G^{h} &= (\frac{5}{2} - \hat{w})\frac{1}{T_0}\frac{\partial T_1}{\partial\theta},\ G^{\pi} = -\frac{4P^{20}}{3s_{\|}}(\frac{\partial}{\partial\theta} + \frac{1}{2}\frac{\partial \ln \hat{B}}{\partial\theta})\frac{u}{v_0} \\
Dh &= -\frac{2P^{01}}{3s_{\|}}(\frac{\partial}{\partial\theta} - \frac{\partial \ln \hat{B}}{\partial\theta})\frac{h_{\|}}{p_0 v_0},\ D\pi = (\frac{\partial}{\partial\theta} - \frac{3}{2}\frac{\partial \ln \hat{B}}{\partial\theta})\frac{\pi_{\|}}{p_0}
\end{aligned}$$

Analytic solution

- For $-\theta_0 \leq \theta \leq \theta_0$, $F_+ : \; v_\| \geq 0$, $F_- : \; v_\| \leq 0$

$$F_\pm(\theta) = e^{\mp\beta(\mp\theta_0,\theta)} F_\pm(\mp\theta_0) + \int_{\mp\theta_0}^{\theta} e^{\mp\beta(\phi,\theta)} G_\pm(\phi) d\phi$$

where $\beta(\theta_1, \theta_2) = \dfrac{\alpha\nu}{v_0} [K(\theta_2) - K(\theta_1)]$, $K(\theta) = \int_0^\theta \frac{1}{s_\|} d\theta' = \int_0^\theta \frac{1}{\sqrt{\hat{w} - \hat{\mu}\hat{B}(\theta')}} d\theta'$,
$\beta_0 = \beta(-\theta_0, \theta_0)$

- Passing particle solution for $\hat{\mu} \leq \hat{w}(1-\epsilon)$: $\theta_0 = \pi$

$$F_\pm(\pi) = F_\pm(-\pi) = \frac{\pm \int_{-\pi}^{\pi} e^{\mp\beta(\phi,\pm\pi)} G_\pm(\phi) d\phi}{1 - e^{-\beta_0}}$$

- Trapped particle solution for $\hat{w}(1-\epsilon) \leq \hat{\mu} \leq \hat{w}(1+\epsilon)$: $\cos\theta_0 = \dfrac{1}{\epsilon}(\dfrac{\hat{\mu}}{\hat{w}} - 1)$

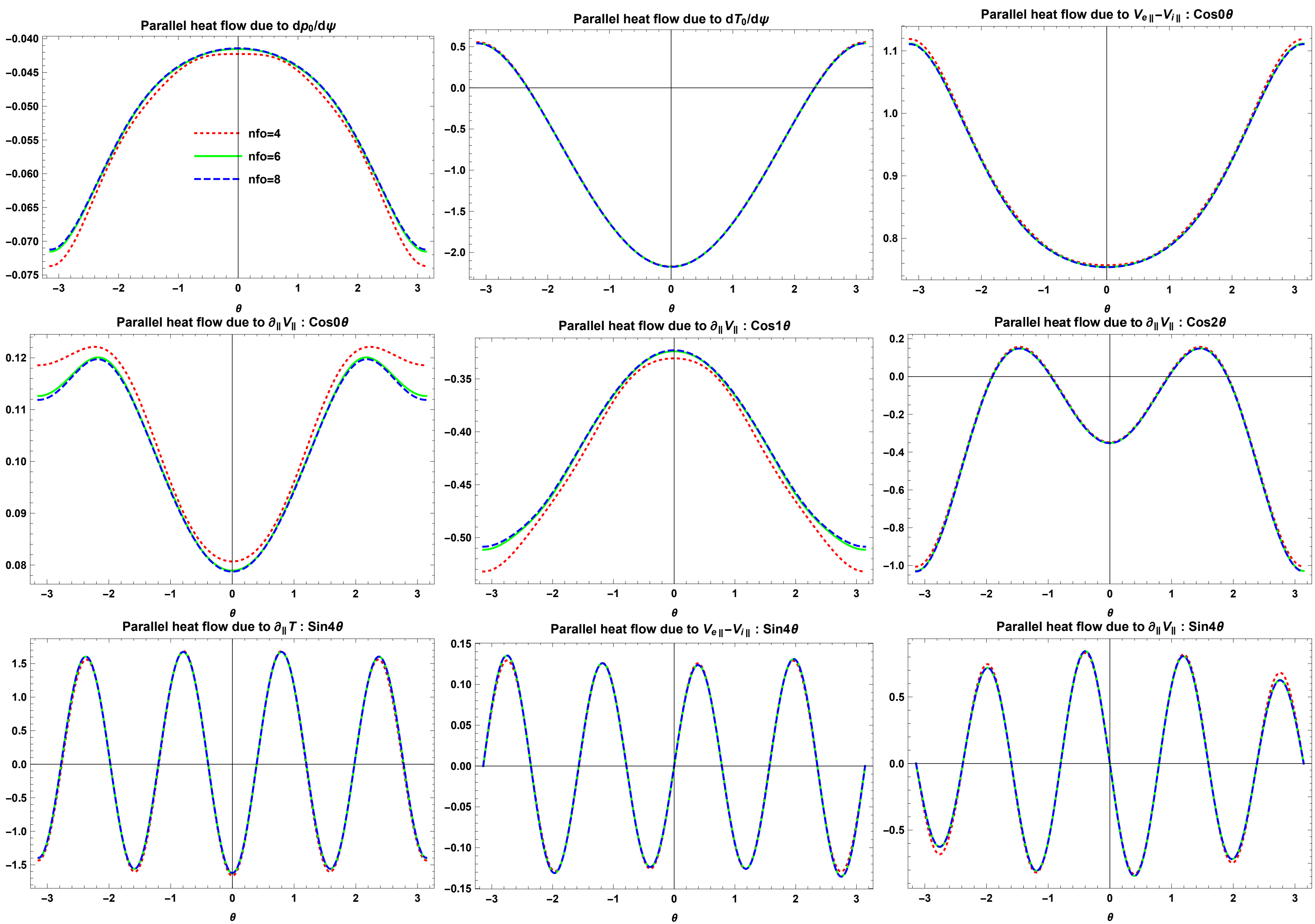
$$F_+(\pm\theta_0) = F_-(\pm\theta_0) = \frac{\int_{-\theta_0}^{\theta_0} \left[\mp e^{-\beta_0} e^{\pm\beta(\phi,\mp\theta_0)} G_\mp(\phi) \pm e^{\mp\beta(\phi,\pm\theta_0)} G_\pm(\phi)\right] d\phi}{1 - e^{-2\beta_0}}$$

- Take closure moments: numerical integration requires accuracy in $\hat{\nu}$

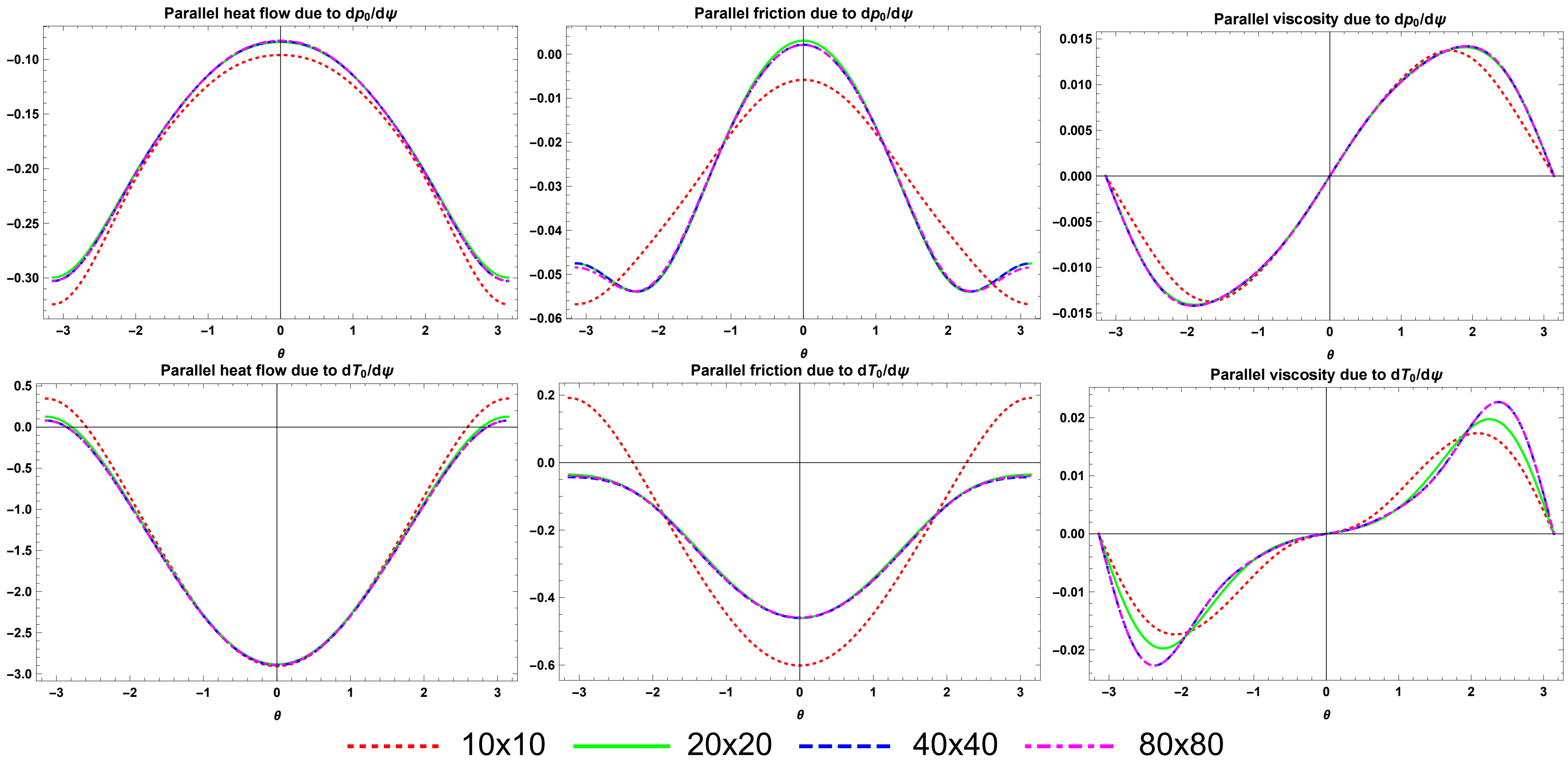
$$(h_\|, \pi_\|) = (h_\|, \pi_\|) + \hat{\nu}[(h_\|, \pi_\|) - \mathtt{K}_{(h_\|, \pi_\|)} G]$$

Difficult to resolve the solution at passing-trapped boundary

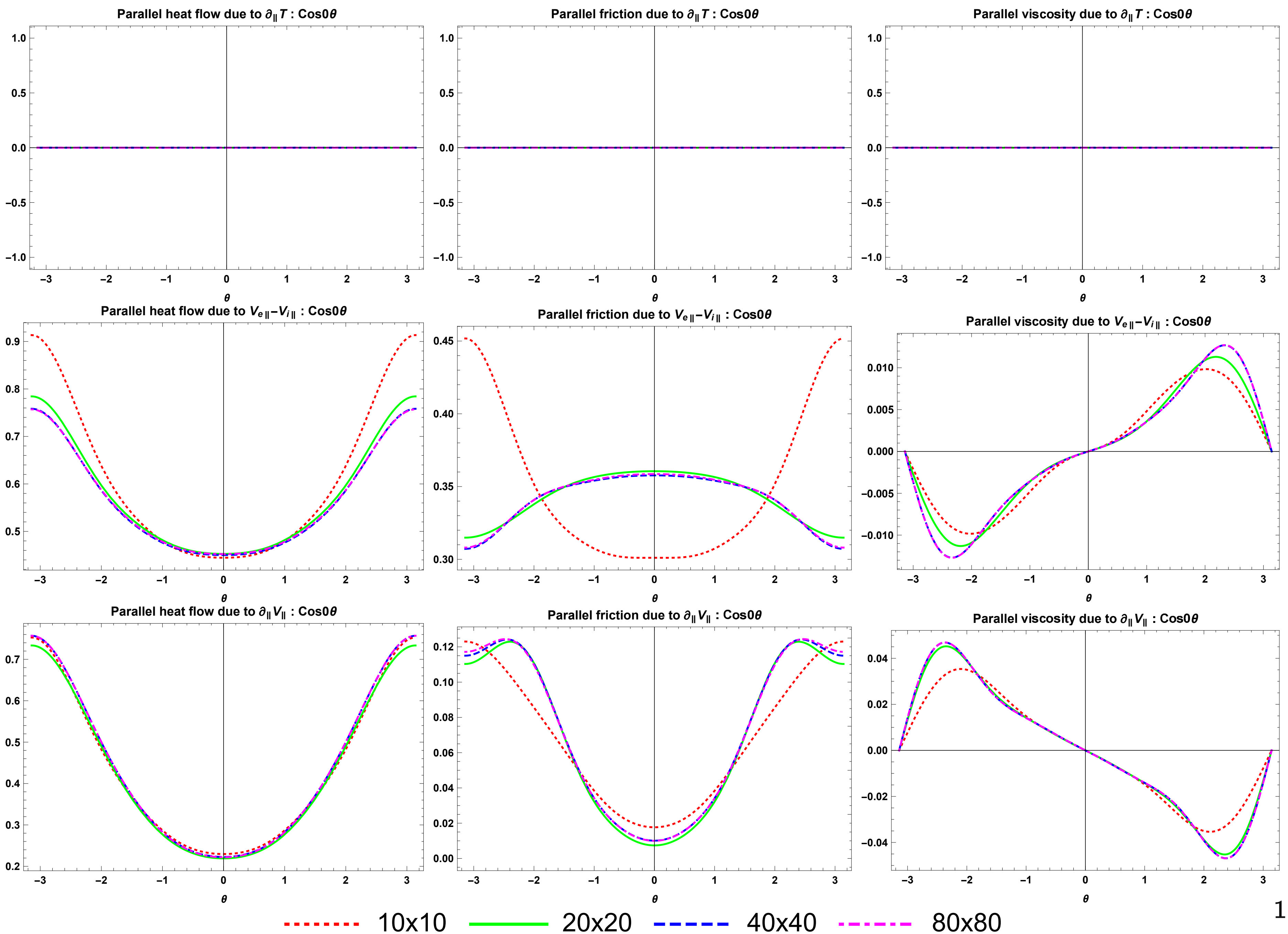
Convergence with increasing Fourier modes ($\epsilon = 0.5$, $K_0 = 10$)



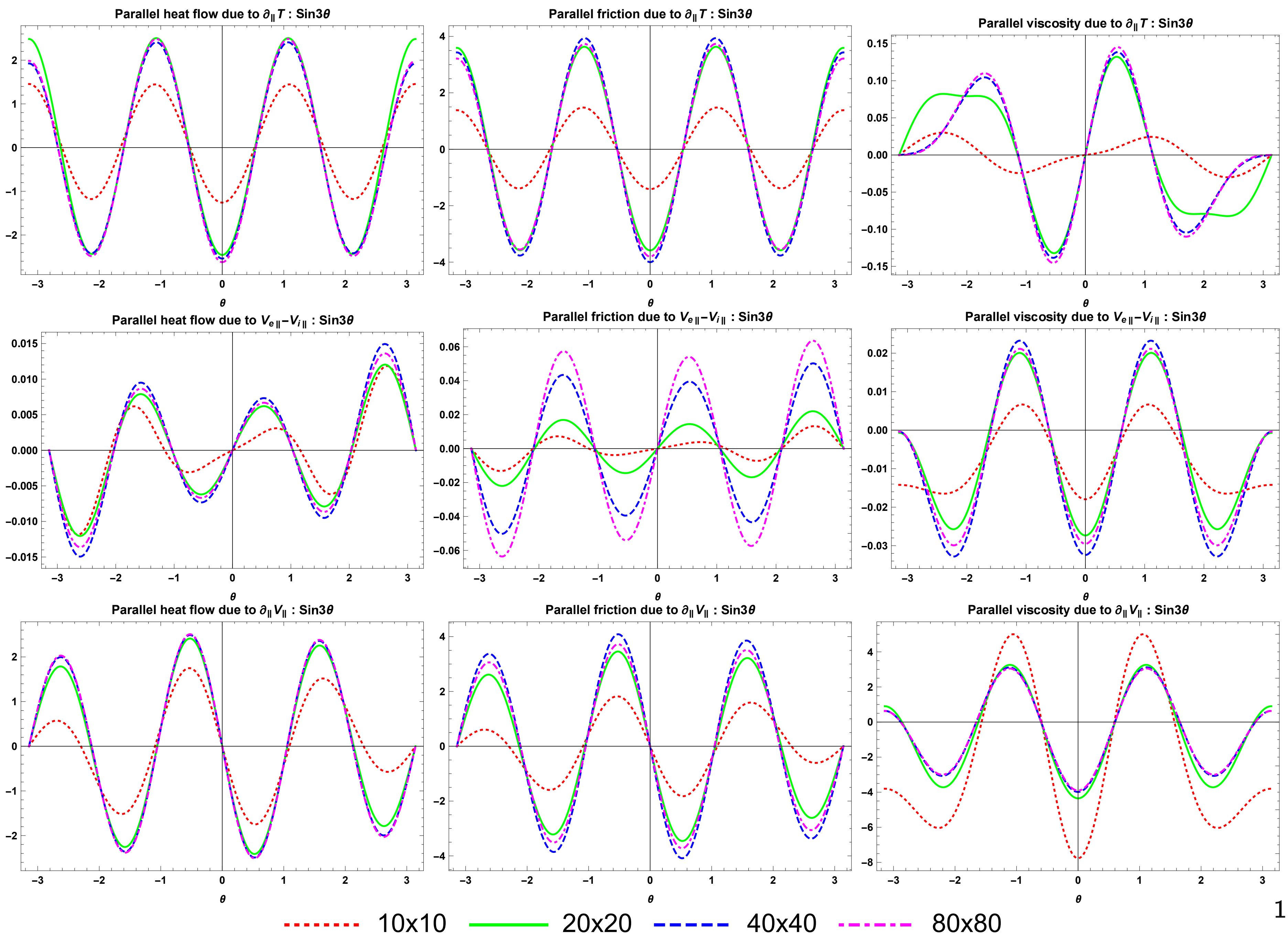
Convergence with increasing moments ($\epsilon = 0.5$, $K_0 = 100$)



Convergence with increasing moments ($\epsilon = 0.5$, $K_0 = 100$)



Convergence with increasing moments ($\epsilon = 0.5$, $K_0 = 100$)



Closures and neoclassical transport

- Closures expressed by Fourier coefficients

$$(h) = \left(\chi^{hh}\right)(T) + \left(\chi^{hR}\right)(u_{ei}) + \left(\chi^{h\pi}\right)(u) + \left(\chi^{hp}\right)\frac{dp_0}{d\psi} + \left(\chi^{hT}\right)\frac{dT_0}{d\psi}$$

$$(R) = \left(\chi^{Rh}\right)(T) + \left(\chi^{RR}\right)(u_{ei}) + \left(\chi^{R\pi}\right)(u) + \left(\chi^{Rp}\right)\frac{dp_0}{d\psi} + \left(\chi^{RT}\right)\frac{dT_0}{d\psi}$$

$$(\pi) = \left(\chi^{\pi h}\right)(T) + \left(\chi^{\pi R}\right)(u_{ei}) + \left(\chi^{\pi\pi}\right)(u) + \left(\chi^{\pi p}\right)\frac{dp_0}{d\psi} + \left(\chi^{\pi T}\right)\frac{dT_0}{d\psi}$$

- Combine with the momentum balance equation to find (u)

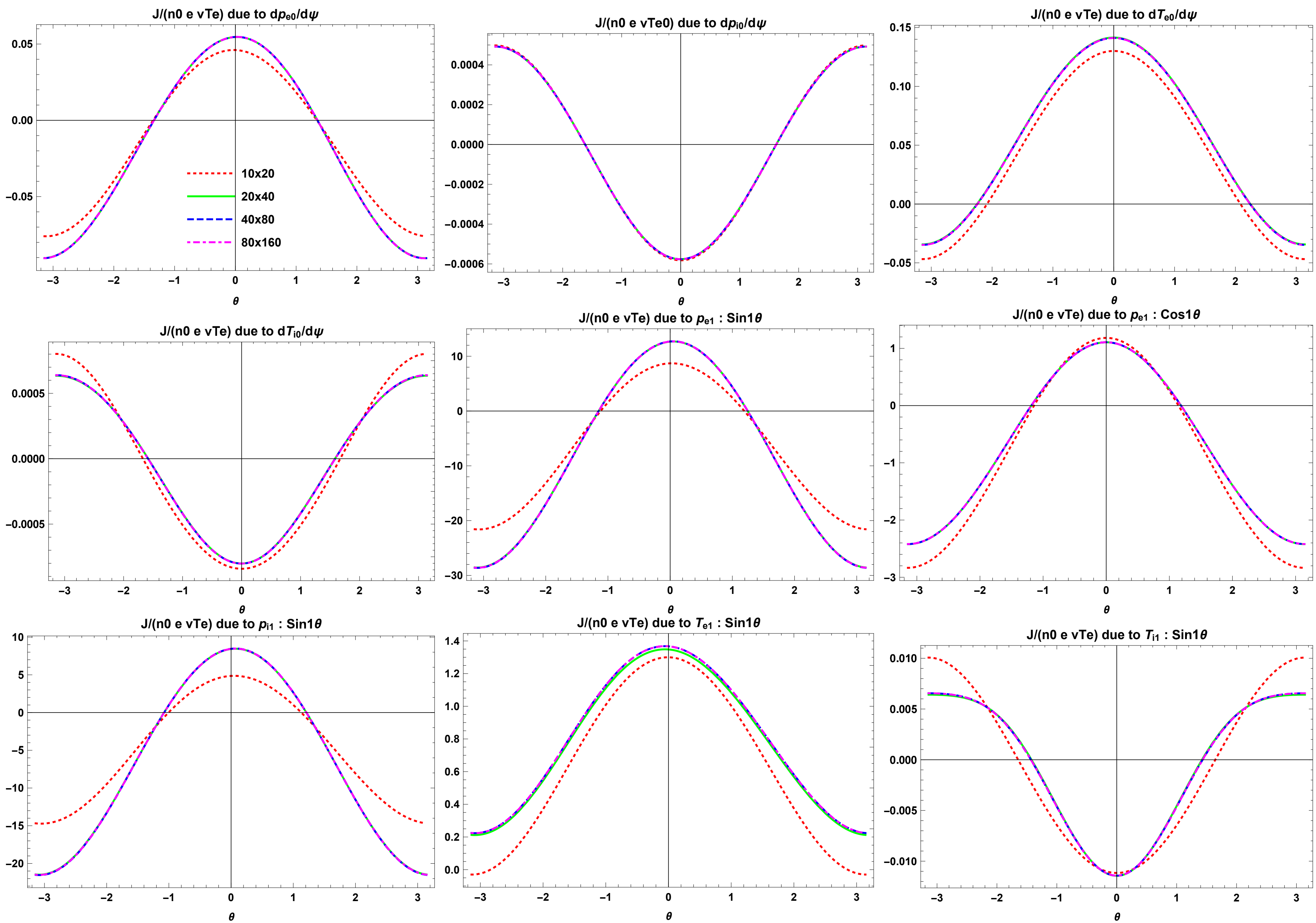
$$mn(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\mathbf{V} + \underline{\nabla p + \nabla\cdot\boldsymbol{\pi} - nq(\mathbf{E}} + \mathbf{V}\times\mathbf{B}) = \underline{\mathbf{R}}$$

Parallel momentum equation $\quad (\mathtt{D})(p) + \left(\mathtt{D}^{1+}\right)(\pi) - nq\,(E) = (\mathbf{R})$

$$(u_e) = (\chi^{up}_{ee})\frac{dp_{e0}}{d\psi} + \left(\chi^{uT}_{ee}\right)\frac{dT_{e0}}{d\psi} + (\kappa^{uu}_{ee})(p_e) + \left(\kappa^{uh}_{ee}\right)(T_e)$$
$$+ (\chi^{up}_{ei})\frac{dp_{i0}}{d\psi} + \left(\chi^{uT}_{ei}\right)\frac{dT_{i0}}{d\psi} + (\kappa^{uu}_{ei})(p_i) + \left(\kappa^{uh}_{ei}\right)(T_i) + \left(\kappa^{uE}_{e}\right)\left(\hat{E}\right)$$

$$(h_e) = \left(\chi^{hp}_{ee}\right)\frac{dp_{e0}}{d\psi} + \left(\chi^{hT}_{ee}\right)\frac{dT_{e0}}{d\psi} + \left(\kappa^{hu}_{ee}\right)(p_e) + \left(\kappa^{hh}_{ee}\right)(T_e)$$
$$+ (\chi^{up}_{ei})\frac{dp_{i0}}{d\psi} + \left(\chi^{uT}_{ei}\right)\frac{dT_{i0}}{d\psi} + \left(\kappa^{hu}_{ei}\right)(p_i) + \left(\kappa^{hh}_{ei}\right)(T_i) + \left(\kappa^{hE}_{e}\right)\left(\hat{E}\right)$$

Current responding to various drives ($\epsilon = 0.1$, $\mathrm{K}_0 = 100$)



Solving density and temperature equations

- Find (n) and (T) from

$$\frac{\partial n}{\partial t} + \underline{\nabla \cdot (n\mathbf{V})} = 0 \Rightarrow \nabla \cdot (n\mathbf{V}) \Rightarrow \; n_0 u = \frac{I}{qB}\frac{dp_0}{d\psi} + \gamma_u(\psi) B$$

$$\frac{3}{2} n (\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla) T + \underline{p\nabla\cdot\mathbf{V} + \nabla\cdot\mathbf{h}} + \nabla\mathbf{V} : \boldsymbol{\pi} = Q$$

$$\Rightarrow \nabla\cdot\mathbf{h} = 0 \; \Rightarrow h_{\parallel} = \frac{5}{2} n_0 T_0 \frac{I}{qB}\frac{dT_0}{d\psi} + \gamma_h(\psi) B$$

- How to determine the integration constants $\gamma_u(\psi)$ and $\gamma_h(\psi)$

$$\frac{\partial}{\partial t}\begin{pmatrix} \hat{n} \\ \hat{T} \end{pmatrix} + \begin{pmatrix} \kappa^{nn} & \kappa^{nT} \\ \kappa^{Tn} & \kappa^{TT} \end{pmatrix}\begin{pmatrix} \hat{n} \\ \hat{T} \end{pmatrix} = \left(\frac{dp_0}{d\psi}, \frac{dT_0}{d\psi}, \hat{E}\right)$$

Diagonalize $\frac{\partial}{\partial t}[L_i] + \lambda_i [L_i] = [G_i]$: eigenvalues λ_i determine the steady state

$$[L_i(t)] = e^{-\lambda_i t}\left([L_i(0)] - \frac{1}{\lambda_i}[G_i]\right) + \frac{1}{\lambda_i}[G_i]$$

$$\lambda_i = 0 \quad : \quad [L_i(t)] = [L_i(0)]$$

$$\lambda_i > 0 \quad : \quad \lim_{t\to\infty} L_i(t) = \frac{1}{\lambda_i}[G_i]$$

$\lambda_i < 0 \quad : \quad$ unstable (saturated by nonlinearity? suppressed?)

Future work

- Closures in the collisionless limit, high mode number: $\mathtt{K_{eff}} \to \infty$
  - Solve the drift kinetic equation with a Krook type model operator $-\nu \overline{f}^{\mathrm{N}}$
    - Analytic treatment of the passing-trapped solution near the bouncing points
  - Solve the moment equation with the Krook operator
    - Use a sparse solver (much larger number of moments can be used)
- Up to nearly collisionless plasmas
  - Use 25600 moments (converge for $\mathtt{K}_0 \lesssim 1000$)
  - Fourier numbers (converge for $(\mathrm{K_{eff}} \lesssim 4000)$
  - Parallel computation
- For general collisionality $\mathtt{K_{eff}}$ and aspect ratio ϵ
  - Compute closures for various mode numbers
  - Find fitted functions of $\mathtt{K_{eff}}$ and ϵ