Verification of NIMROD's Rosenbluth potentials*

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NIMROD team zoom meeting, Nov 1, 2021

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Verification of NIMROD's Rosenbluth potentials

Tests added to manuscript submitted to JCP in February:

1) Convergence of Bootstrap current

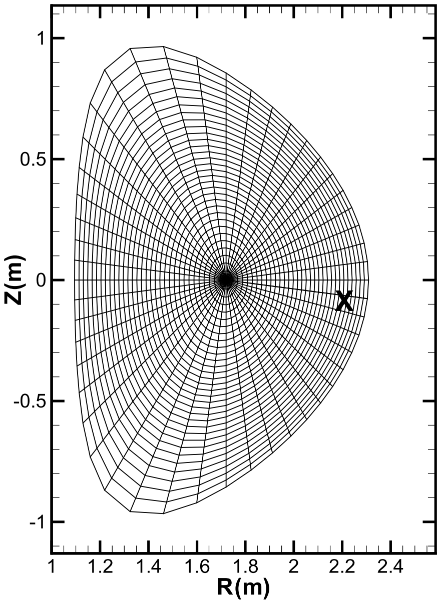
2) Poisson's equation using
method of manufactured solutions

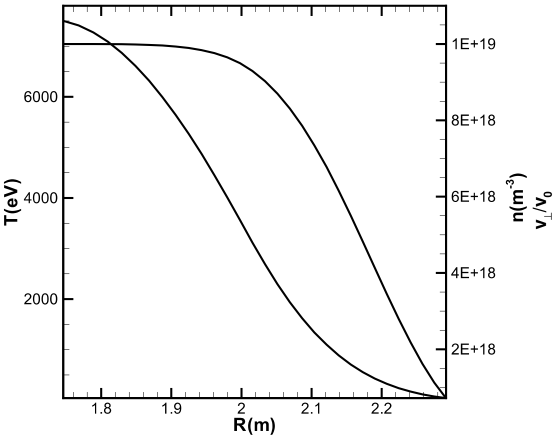
3) Sauter coefficient comparison (*tpb* GLL vs Legendre)
Landreman and Ernst, JCP 243 (2013)

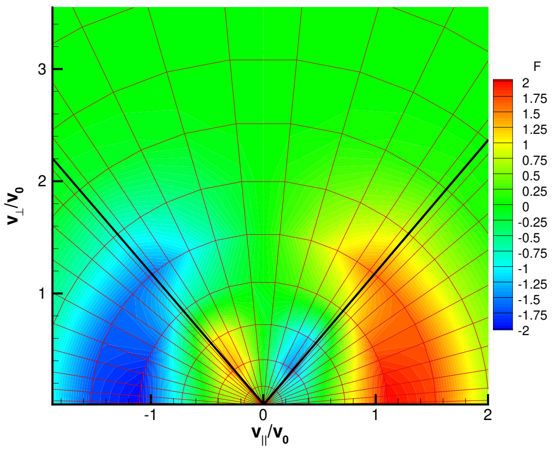
FE pitch-angle basis improves bootstrap current

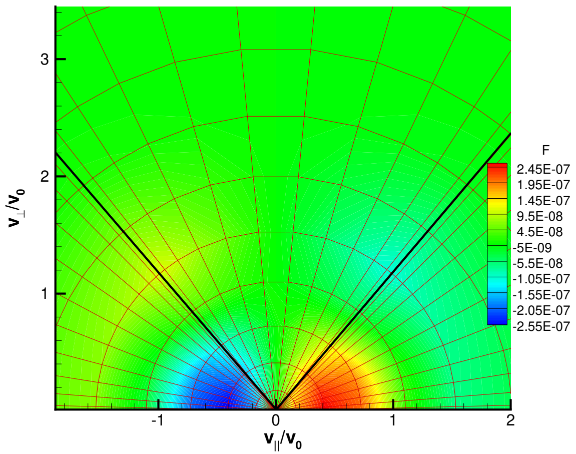
$$\mathbf{v}_{\|} \cdot \left[\nabla F - \frac{1-\xi^2}{2\xi} \nabla \ln B \frac{\partial F}{\partial \xi} \right] - C\left(F\right) = -\mathbf{v}_D \cdot \left(\nabla \ln n_0 - \left(\frac{3}{2} - s^2 \right) \nabla \ln T_0 \right) f_0$$

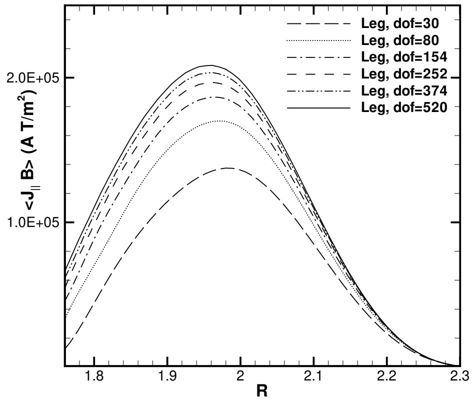
$$\langle J_{\|} B \rangle = \sum_b q_b \left\langle B \int d\mathbf{v} v_{\|} F_b \right\rangle$$

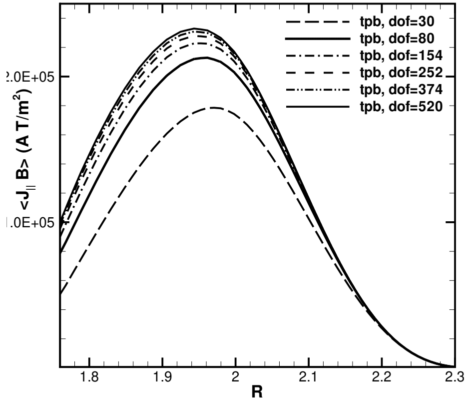


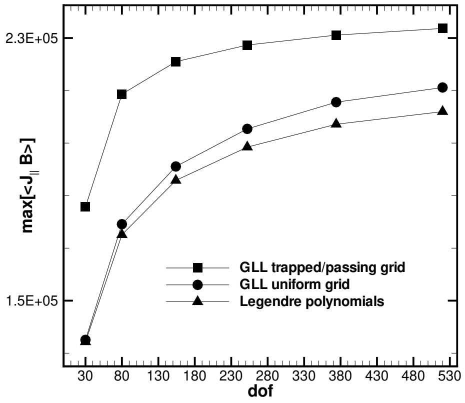


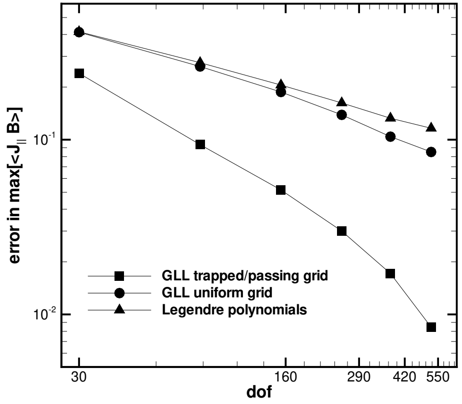












Solving Poisson's equation using method of manufactured solutions

The axisymmetric Coulomb collision field operator:

$$C\left[f_a^{\mathsf{M}}, F_b\right] = \Gamma_{ab} f_a^{\mathsf{M}} \left\{ 4\pi \frac{m_a}{m_b} F_b - \frac{2H_b}{v_{\mathsf{T}a}^2} + \left(\frac{m_a}{m_b} - 1\right) \frac{2v}{v_{\mathsf{T}a}^2} \frac{\partial H_b}{\partial v} + \frac{2v^2}{v_{\mathsf{T}a}^4} \frac{\partial^2 G_b}{\partial v^2} \right\}$$

Normalize speed $s \equiv v/v_{\mathsf{T}a}$, speed collocation method and pitch-angle FEM:

$$\nabla_{\mathbf{v}}^2 H_b = -4\pi F_b$$

$$F_a\left(v_{\mathsf{T}a}s, \xi'\right) = \sum_{l'} F_{a,l'}\left(v_{\mathsf{T}a}s\right) Q_{l'}\left(\xi'\right)$$

$$F_{a,l'}\left(v_{\mathsf{T}a}s\right) = \sum_{k=1}^{N_s} F_{a,l',k} L_k\left(s\right) \mathsf{e}^{-s^2}$$

$$\int_0^\infty ds L_j\left(s\right) L_k\left(s\right) \mathsf{e}^{-s^2} = \delta_{jk}$$

$$F_{a,l'}\left(v_{\mathsf{T}a}s\right) = \sum_{k=1}^{N_s} \sum_j w_j L_{k'}\left(s_j\right) F_{a,l'}\left(v_{\mathsf{T}a}s_j\right) L_k\left(s\right) \mathsf{e}^{-s^2}$$

$$\int_{-1}^{1} d\xi H_b\left(v_{\mathsf{T}a}s_i, \xi\right) Q_l\left(\xi\right) = v_{\mathsf{T}a}^2 s_i^2 \sum_{l'} \sum_j \mathsf{H}_{a,b,l,i,l',j} F_{b,l'}\left(v_{\mathsf{T}b}s_j\right)$$

$$\int_{-1}^{1} d\xi \frac{\partial H_b}{\partial v}\left(v_{\mathsf{T}a}s_i, \xi\right) Q_l\left(\xi\right) = v_{\mathsf{T}a} s_i \sum_{l'} \sum_j \mathsf{H}'_{a,b,l,i,l',j} F_{b,l'}\left(v_{\mathsf{T}b}s_j\right)$$

$$\int_{-1}^{1} d\xi \frac{\partial^2 G_b}{\partial v^2}\left(v_{\mathsf{T}a}s_i, \xi\right) Q_l\left(\xi\right) = v_{\mathsf{T}a}^2 s_i^2 \sum_{l'} \sum_j \mathsf{G}''_{a,b,l,i,l',j} F_{b,l'}\left(v_{\mathsf{T}b}s_j\right)$$

where $\mathsf{H}_{a,b,l,i,l',j}$, $\mathsf{H}'_{a,b,l,i,l',j}$, and $\mathsf{G}''_{a,b,l,i,l',j}$ are precomputed coupling arrays

Solving Poisson's equation using method of manufactured solutions

1) Starting with exact solution we want to reproduce:

$$H\left(v,\xi\right)=\exp\left\{-\left[\theta_{\parallel}^{-1}\xi^{2}+\theta_{\perp}^{-1}\left(1-\xi^{2}\right)\right]\left(\frac{v}{v_{\mathsf{T}}}\right)^{2}\right\}$$

2) Let operator $\nabla_{\mathbf{v}}^{2}H=-4\pi F$ act on solution to produce RHS:

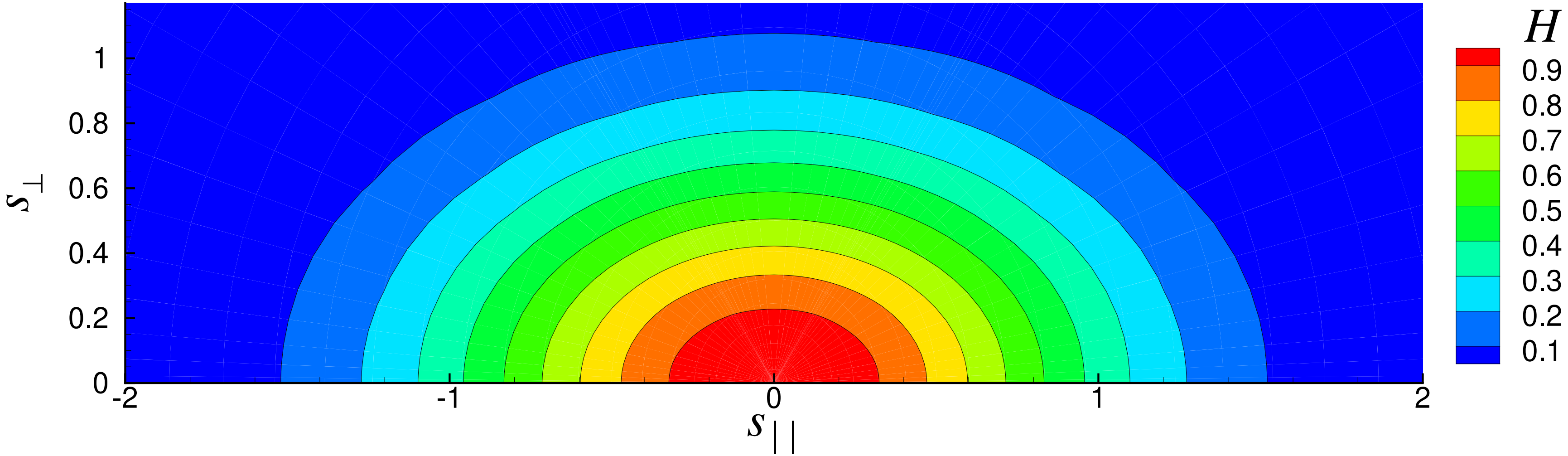
$$F\left(v,\xi\right)=\frac{1}{2\pi v_{\mathsf{T}}^{2}}\left[\theta_{\parallel}^{-1}+2\theta_{\perp}^{-1}-2\theta_{\parallel}^{-2}\xi^{2}\left(\frac{v}{v_{\mathsf{T}}}\right)^{2}-2\theta_{\perp}^{-2}\left(1-\xi^{2}\right)\left(\frac{v}{v_{\mathsf{T}}}\right)^{2}\right]H\left(v,\xi\right)$$

3) Project RHS onto basis and solve equation to obtain numerical approximate to solution.

$$H\left(v_{\mathsf{T}}s_{i},\xi_{k}\right)=\sum_{l}\left[\mathsf{M}^{-1}\right]_{kl}\boxed{\int_{-1}^{1}d\xi H\left(v_{\mathsf{T}}s_{i},\xi\right)Q_{l}\left(\xi\right)}\quad\text{where}\quad\mathsf{M}_{kl}\equiv\int_{-1}^{1}d\xi Q_{k}\left(\xi\right)Q_{l}\left(\xi\right)$$

Solving Poisson's equation using method of manufactured solutions

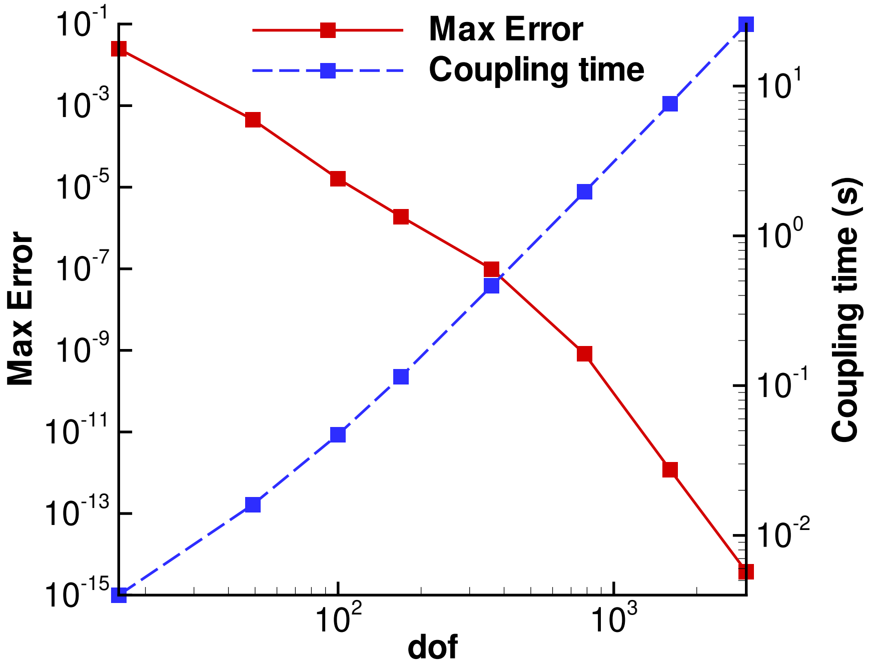
$$H = \text{exp}\left[-\left(s_{\parallel}^2 + 2s_{\perp}^2\right)\right]$$



$$(N_s, p) = (55, 18)$$

Solving Poisson's equation using method of manufactured solutions

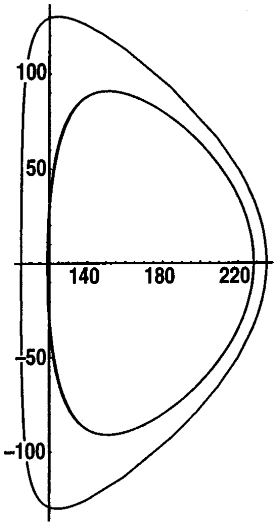
$$\text{Max Error} = \max_{i,l} \left| s_i^2 \sum_{l'} \sum_j \mathsf{H}_{l,i,l',j} v_\mathsf{T}^2 F_{l'}\left(v_\mathsf{T} s_j\right) - \int_{-1}^{1} d\xi \, \exp\left\{-\left[\xi^2 s_i^2 + 2\left(1-\xi^2\right) s_i^2\right]\right\} Q_l\left(\xi\right) \right|$$

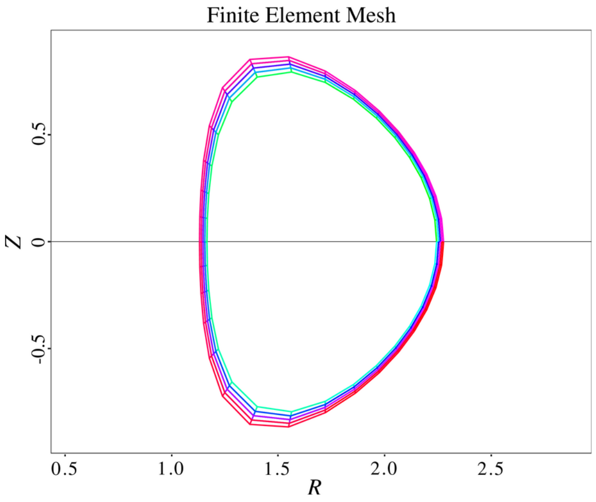


$$\left(N_s, p\right) = \left(4,1\right), \left(7,2\right), \left(10,3\right), \left(13,4\right), \left(19,6\right), \left(28,9\right), \left(40,13\right), \left(55,18\right)$$

L_{32} Sauter coefficient (*tpb* GLL vs Legendre)

$$\langle \mathbf{j}\cdot\mathbf{B}\rangle = -cRB_\phi p_\mathrm{e}\left[\mathcal{L}_{31}\frac{1}{p_\mathrm{e}}\left(\frac{dp_\mathrm{e}}{d\psi}+\frac{dp_\mathrm{i}}{d\psi}\right)+\frac{\mathcal{L}_{32}}{T_\mathrm{e}}\frac{dT_\mathrm{e}}{d\psi}+\frac{\mathcal{L}_{34}\alpha}{ZT_\mathrm{e}}\frac{dT_\mathrm{i}}{d\psi}\right]$$



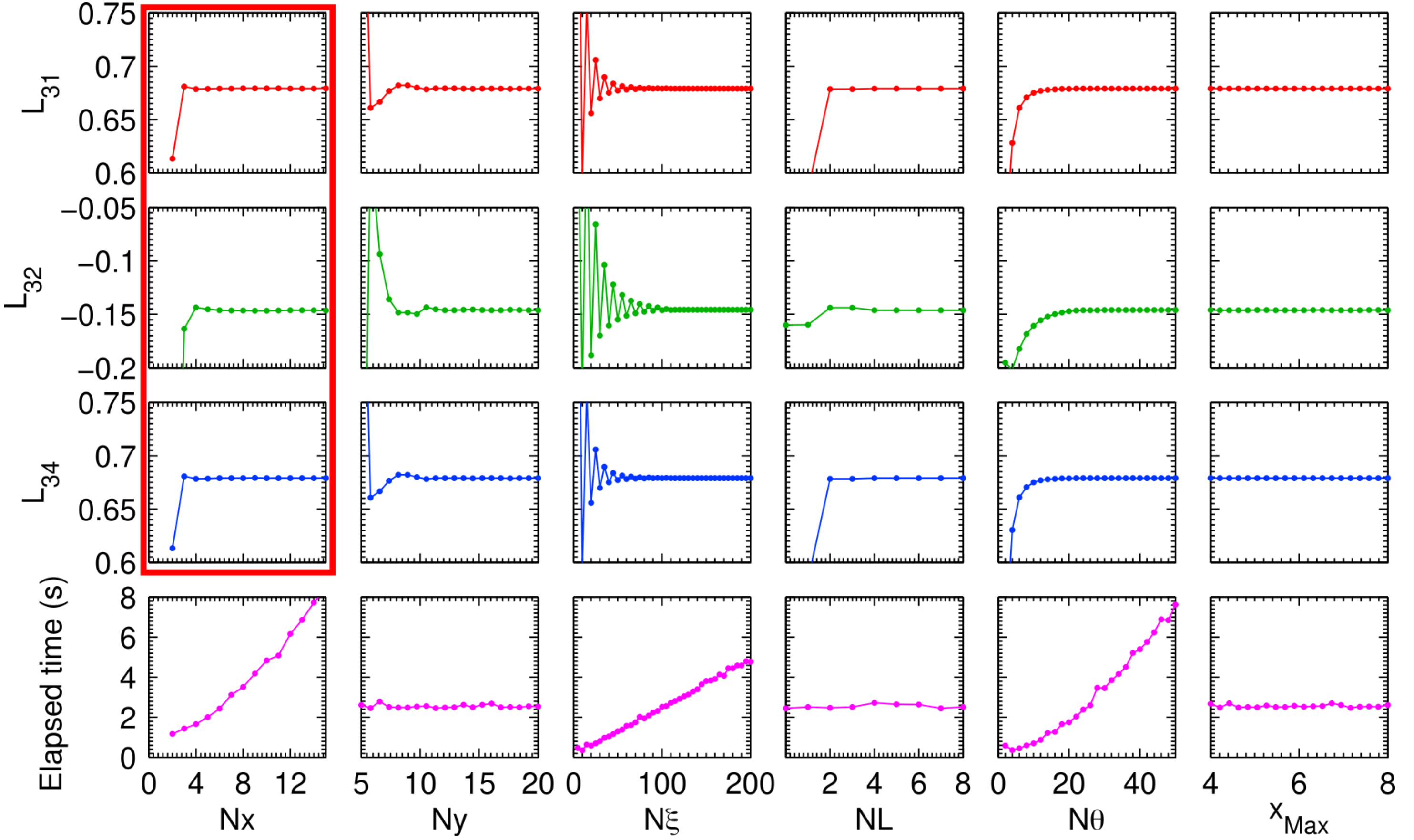


$$\mathcal{L}_{32} = -4\pi^{-1/2}\left\langle \hat{B}\int_{-1}^{1} d\xi \int_0^\infty ds\, s^3 \xi f_{\mathrm{e}1}^{(T_\mathrm{e})}\right\rangle$$

$$s\xi\frac{\partial f_{\mathrm{e}1}^{(T_\mathrm{e})}}{\partial\theta} - \frac{\left(1-\xi^2\right)s}{2\hat{B}}\left(\frac{\partial\hat{B}}{\partial\theta}\right)\frac{\partial f_{\mathrm{e}1}^{(T_\mathrm{e})}}{\partial\xi} - \frac{v'}{qR_0\mathbf{b}\cdot\nabla\theta}\hat{C} = \frac{1+\xi^2}{2}s^2\left(s^2-\frac{5}{2}\right)e^{-s^2}\frac{1}{\hat{B}^2}\frac{\partial\hat{B}}{\partial\theta}$$

See M. Landreman, D. R. Ernst, J. Comput. Phys. 243 (2013).
R. L. Miller, M. S. Chu, J. M. Greene, Y. R. Lin-Liu, and R. E. Waltz, Phys. Plasmas 5, 973 (1998).

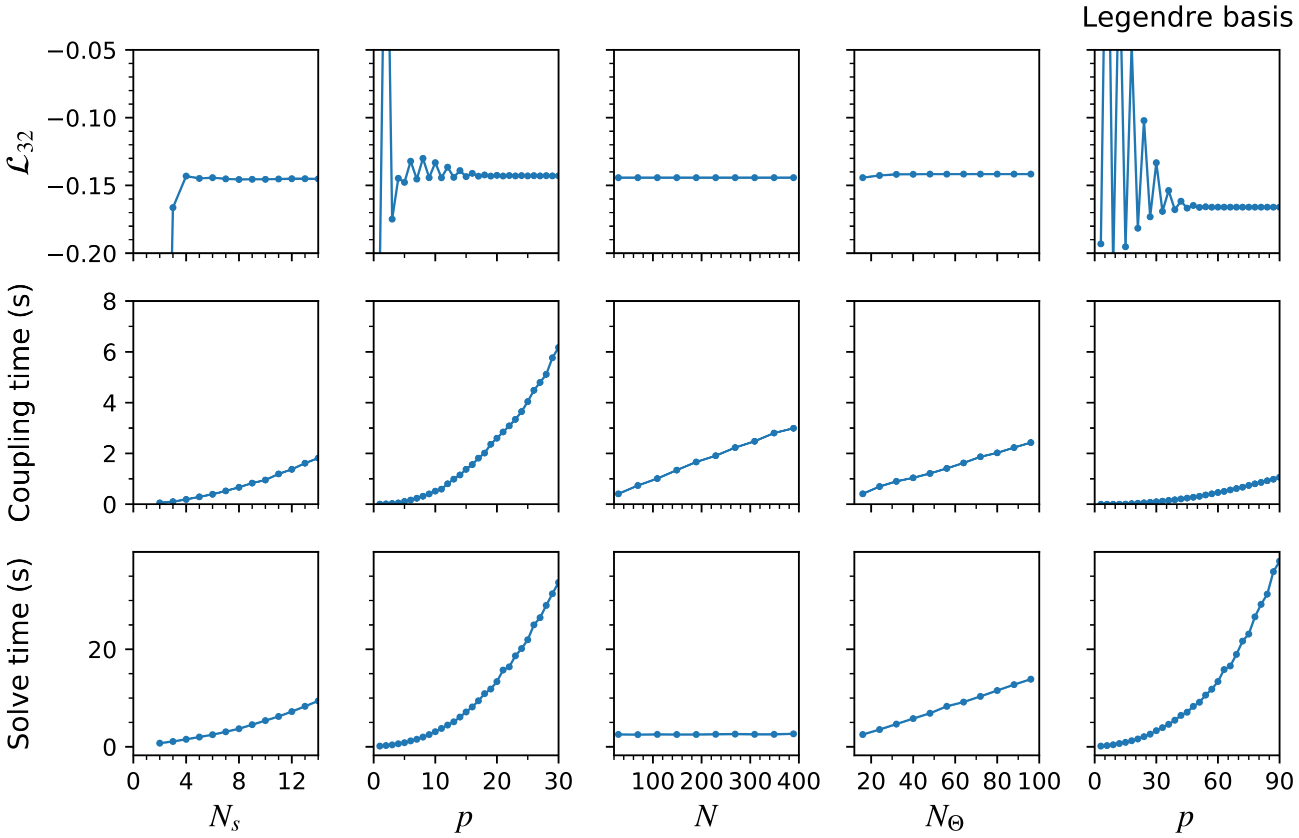
L_{32} Sauter coefficient (*tpb* GLL vs Legendre)



Except for quantities being scanned, $Nx = 6$, $Ny = 20$, $N\xi = 100$, $NL = 4$, $N\theta = 25$, and $x_{\text{Max}} = 5$.

Figure taken from M. Landreman, D. R. Ernst, J. Comput. Phys. 243 (2013).

L_{32} Sauter coefficient (*tpb* GLL vs Legendre)



Except for quantities being scanned, $N_s = 6$, $p = 9$, $N = 29$, $N_\Theta = 16$.

Summary

We've improved NIMROD's axisymmetric Coulomb collision field operator

Tests:

- Low order collisional moments
- Bootstrap current using different pitch-angle bases adds to the story in 2015 PoP paper†

Two new verification tests added

- "Poison solve", H, tested with method of manufactured solutions
- L_{32} Sauter coefficient calculation

Bootstrap current computed with higher resolution and convergence shown with error plot