Energy Conservation and Power Flows in MHD

James D. Hanson[1], Eric Howell[2], Omar Lopez Ortiz[1], Dave Maurer[1]

[1]Auburn University [2]Tech-X

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Energy in Buckets

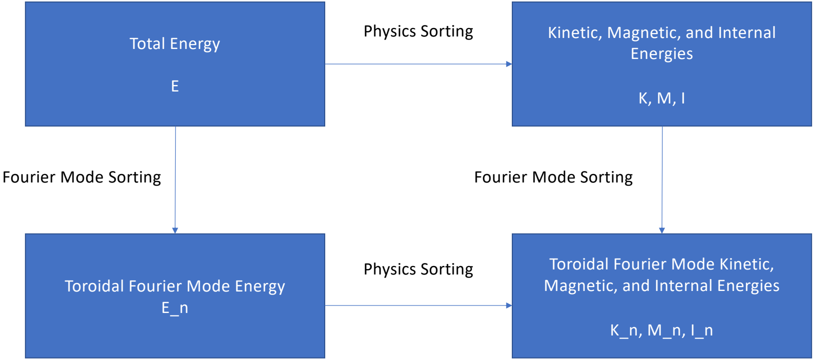
- MHD energy density:

$$\rho_E(\boldsymbol{r}, t) = \frac{|\boldsymbol{B}(\boldsymbol{r}, t)|^2}{2\mu_o} + \frac{\rho(\boldsymbol{r}, t)|\boldsymbol{V}(\boldsymbol{r}, t)|^2}{2} + \frac{p(\boldsymbol{r}, t)}{\gamma - 1} \tag{1}$$

- Tracking this energy density in space and through time - too much information
- One strategy - group energy into a finite number of buckets, and track the energy in the buckets
- Group by type of energy - magnetic, kinetic, internal
- Group by toroidal mode number - toroidal Fourier decomposition

$$E = \sum_n E_n = \sum_n \int d^2A(\frac{1}{2\pi} \int e^{-in\phi} \rho_E(R, Z, \phi)) \tag{2}$$

Multiple Ways to Sort Energy into Buckets



Bird's Eye View - Two-Index Dynamics

- Use an index (α) to label the buckets

$$E = \sum_{\alpha} E_{\alpha} \tag{3}$$

- Two-index dynamics - time evolution of energy in one bucket involves a second index (another bucket)

$$\frac{dE_{\alpha}}{dt} = \sum_{\beta} F_{\alpha\beta} \tag{4}$$

- One way to achieve energy conservation - terms cancel in pairs - $F_{\alpha\beta} + F_{\beta\alpha} = 0$

- Interpretation - energy moved from bucket β to bucket α is accounted for twice, as a gain by bucket α, ($F_{\alpha\beta}$) and as a loss by bucket β ($F_{\beta\alpha} = -F_{\alpha\beta}$)

Bird's Eye View - Three-Index Dynamics

- Three-index dynamics - time evolution of energy in one bucket involves two more indices (two other buckets)

$$\frac{dE_\alpha}{dt} = \sum_{\beta\gamma} G_{\alpha\beta\gamma} \tag{5}$$

- One way to achieve energy conservation - terms cancel in triples - $G_{\alpha\beta\gamma} + G_{\beta\gamma\alpha} + G_{\gamma\alpha\beta} = 0$
- Interpretation is not as straightforward as for two-index dynamics
- Without other considerations, can't construct satisfactory two-index dynamics from three-index dynamics. For fixed α, β, γ, given $G_{\alpha\beta\gamma}, G_{\beta\gamma\alpha}$ and $G_{\gamma\alpha\beta} = -(G_{\alpha\beta\gamma} + G_{\beta\gamma\alpha})$, there is no unique way to determine $F_{\alpha\beta}, F_{\alpha\gamma}, F_{\beta\gamma}$ (along with $F_{\beta\alpha} = -F_{\alpha\beta}$, $F_{\gamma\alpha} = -F_{\alpha\gamma}$, and $F_{\gamma\beta} = -F_{\beta\gamma}$)

Focus on Two Terms from MHD - Lorentz Power Flux

- Using a Fourier sorting for the energy, two of the terms are

$$\frac{dE_n}{dt} = \int_A (\boldsymbol{V}_n^* \cdot (\boldsymbol{J} \times \boldsymbol{B})_n + \boldsymbol{J}_n^* \cdot (\boldsymbol{V} \times \boldsymbol{B})_n) + c.c. + ... \qquad (6)$$

- We called these two terms the *Lorentz power flux*.
- Lorentz power fluxes depend on three indices.
- At first, we went down a rabbit hole, and looked at three-index dynamics relations, $G_{\alpha\beta\gamma} + G_{\beta\gamma\alpha} + G_{\gamma\alpha\beta} = 0$.
- Recently, we found a better way, with two-index dynamics. The trick was to *further* split our Fourier buckets into energy types.

Power into Fourier Kinetic Energy - Lorentz Force

- Energy into K_n

$$\frac{dK_n}{dt} - ... = \int_A (\boldsymbol{V}_n^* \cdot (\boldsymbol{J} \times \boldsymbol{B})_n + c.c.) = \int_A (\boldsymbol{V}_n^* \cdot \sum_{n'} (\boldsymbol{J}_{n'} \times \boldsymbol{B}_{n-n'}) + c.c.) \tag{7}$$

- Define: $R_{n(n-n')n'} \equiv \boldsymbol{V}_n^* \cdot (\boldsymbol{J}_{n'} \times \boldsymbol{B}_{n-n'}) + c.c.$

$$\frac{dK_n}{dt} = \int_A \sum_{n'} R_{n(n-n')n'} + ... \tag{8}$$

- $R_{n(n-n')n'}$ is a power density into K_n associated with the Lorentz force

Power into Fourier Magnetic Energy - Lorentz Force

- Energy into M_n

$$\frac{dM_n}{dt} - ... = \int_A (\boldsymbol{J}_n^* \cdot (\boldsymbol{V} \times \boldsymbol{B})_n + c.c.) = \int_A (\boldsymbol{J}_n^* \cdot \sum_{n'} (\boldsymbol{V}_{n'} \times \boldsymbol{B}_{n-n'}) + c.c.) \tag{9}$$

- Define: $S_{n(n-n')n'} \equiv \boldsymbol{J}_n^* \cdot (\boldsymbol{V}_{n'} \times \boldsymbol{B}_{n-n'}) + c.c.$

$$\frac{dM_n}{dt} = \int_A \sum_{n'} S_{n(n-n')n'} + ... \tag{10}$$

- $S_{n(n-n')n'}$ is a power density into M_n associated with the Lorentz force

Insight - Energy Transfer from One Bucket to Another

- Using $\boldsymbol{B}^*_{n-n'} = \boldsymbol{B}_{n'-n}$, one can easily show:

$$R_{n(n-n')n'} = -S_{n'(n'-n)n} \tag{11}$$

- We have identified an energy transfer between two meaningful buckets!

- Combine the Magnetic, Kinetic (and Internal) Fourier buckets

$$E_n = K_n + M_n + I_n \tag{12}$$

- Define: $F_{n(n-n')n'} \equiv R_{n(n-n')n'} + S_{n(n-n')n'}$

$$F_{n(n-n')n'} \equiv \boldsymbol{J}^*_n \cdot (\boldsymbol{V}_{n'} \times \boldsymbol{B}_{n-n'}) + \boldsymbol{V}^*_n \cdot (\boldsymbol{J}_{n'} \times \boldsymbol{B}_{n-n'}) + c.c. \tag{13}$$

Interpretation

- (Repeat the definition)

$$F_{n(n-n')n'} \equiv \boldsymbol{J}_n^* \cdot (\boldsymbol{V}_{n'} \times \boldsymbol{B}_{n-n'}) + \boldsymbol{V}_n^* \cdot (\boldsymbol{J}_{n'} \times \boldsymbol{B}_{n-n'}) + c.c. \quad (14)$$

- Note that $F_{n(n-n')n'}$ obeys a two-index energy conservation relation!

$$F_{n(n-n')n'} + F_{n'(n'-n)n} = 0 \quad (15)$$

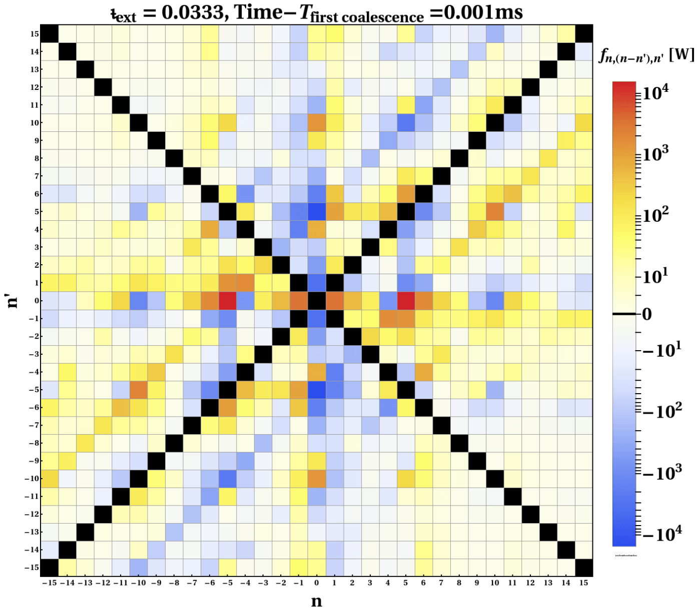
- The magnetic field ($\boldsymbol{B}_{n-n'}$) acts as a *catalyst* for the Lorentz power transfer process from the n' bucket to the n bucket.

- Integrating over the cross sectional area, we define

$$f_{n(n-n')n'} \equiv \int_A F_{n(n-n')n'} \quad (16)$$

- We consistently place the middle index of F and f in parentheses, to emphasize that it is not independent of the first and last indices.

Plot of $f_{n(n-n')n'}$ for a Sawtooth Simulation of CTH



Features in the plot

- Figure symmetric on reflection through the origin

  Complex conjugate in definition of F ensures
  $f_{n(n-n')n'} = f_{-n(-n+n')-n'}$

- Figure symmetric (with sign change) on reflection through $n = n'$ line

  Energy conservation relation $f_{n(n-n')n'} = -f_{n'(n'-n)n}$

- Structure along line $n' = n + 5p$ (p any integer)

  CTH has stellarator fields with 5-fold periodicity
  $\boldsymbol{B}_{5p}$ is relatively large
  So, expect $f_{n(5p)n-5p}$ large

Summary and Conclusions

- We looked at two general types of energy flow dynamics, two-index and three-index dynamics.
- Two-index dynamics is simple to interpret - energy moves from one bucket to another
- Three-index dynamics is more complicated and harder to interpret. In general, it can not (unambiguously) be simplified to two-index dynamics.
- MHD Lorentz power flow between toroidal modes does not fit into either simple model of two-index or three-index dynamics.
- MHD Lorentz power flow involves three toroidal mode indices, but can be interpreted as a direct flow of energy from mode n' to mode n, catalyzed by the magnetic field ($\boldsymbol{B}_{n-n'}$)
- Plots of $f_{n(n-n')n'}$ vs. n and n' reflect the expected symmetries, energy conservation, and structure of the equilibrium magnetic field.
- We are still gaining experience with the display and interpretation of our results.

EXTRA SLIDES FOLLOW

Our Rabbit Hole

- We started with the Lorentz power flow

$$\frac{dE_n}{dt} = \int_A (\boldsymbol{V}_n^* \cdot (\boldsymbol{J} \times \boldsymbol{B})_n + \boldsymbol{J}_n^* \cdot (\boldsymbol{V} \times \boldsymbol{B})_n + c.c.) + ... \qquad (17)$$

- We defined a three-index power flow term and symmetrized on the second and third indices.

- When we found a three-index energy conservation relation, we thought we were on the right track.