Stability Considerations using a Chapman-Enskog-like (CEL) kinetic closure approach in NIMROD

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Thesis

- We have succeeded in eliminating an ”instability” caused by numerical errors in $\mathbf{b}_{\text{eq}} \cdot \nabla T_{i,\text{eq}}$.
- We are working on eliminating another numerical instability in coupled fluid-kinetic runs.
- A stability analysis is being undertaken for the coupled system that is similar to (Sovinec & King 2010)[1].

[1] C. Sovinec and J. King, “Analysis of a mixed semi-implicit/implicit algorithm for low-frequency two-fluid plasma modeling”, Journal of Computational Physics **229**, 5803–5819 (2010)

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Kinetics allows for more rigorous closure of the plasma fluid equations

- Without kinetics only heuristic closures can be specified; however, if one has the kinetic distribution function, rigorous closure is possible.
- As an example, to lowest order in $\delta \equiv \rho_{\mathrm{Larmor}}/L$ (with L = macroscopic length scale), the definition of the ion viscous stress tensor is $\mathbf{\Pi}_i = \pi_{i\|}(\mathbf{bb} - \mathbf{I}/3)$, where $\mathbf{I}$ is the identity tensor and:

$$\pi_{i\|} = m_i \int \mathrm{d}^3 v \, v^2 P_2 \left(v_\| / v\right) f_i = \left(p_{i\|} - p_{i\perp}\right) ,$$

  where $P_2(v_\|/v)$ is the second-order Legendre polynomial in $v_\|/v$, and f_i is the ion kinetic distribution function.
- We also note that the form of $\mathbf{q}_i$ to lowest order in δ is $\mathbf{q}_i = q_{i\|}\hat{\mathbf{b}} + \left(\frac{5 n_i T_i}{2eB}\mathbf{b} \times \nabla T_i\right)$, where

$$q_{i\|} = \int \mathrm{d}^3 v (m v^2/2) v_\| f_i \, .$$

The Chapman-Enskog-like approach avoids inconsistencies by assuming the distribution function is not far from a local Maxwellian

- Spirit of the Chapman-Enskog method [2]:
  1. In a magnetized fusion plasma the kinetic distribution function is not far from a local Maxwellian.
  2. The local Maxwellain depends on both space and time through the thermodynamic parameters n, T, and $\mathbf{v}$.
     - (this differs from traditional δf which assumes only non-time-evolving equilibrium fluid variables in the Maxwellian)
  3. This can be succinctly stated as: $f \approx f_M + f_1$ where f_M is a flow-shifted time-evolving Maxwellian ($f_M(\mathbf{x},t) = \frac{n(\mathbf{x},t)}{\pi^{3/2}(v_T(\mathbf{x},t))^3} exp(-\frac{(\mathbf{v}-\mathbf{u}(\mathbf{x},t))^2}{(v_T(\mathbf{x},t))^2})$, with $v_T(\mathbf{x},t) = \sqrt{2T(\mathbf{x},t)/m}$), and f_1 is a first-order (in δ) non-Maxwellian perturbation.

Full Chapman-Enskog-like kinetic equation can be derived by substituting in the fluid equations for $\partial f_{Mi}/\partial t$ [3]

$$\frac{\partial f_{NMi}}{\partial t} + \xi s v_{Ti} \mathbf{b} \cdot \nabla f_{NMi} - \frac{1-\xi^2}{2\xi}\left[\xi s v_{Ti}\left(\mathbf{b}\cdot\nabla \ln B - \frac{\mathbf{b}\cdot\nabla \ln n_i}{s^2}\right)\right]\frac{\partial f_{NMi}}{\partial \xi}$$
$$+\left[\frac{\xi v_{Ti}}{2}\left(\mathbf{b}\cdot\nabla \ln n - s^2 \mathbf{b}\cdot\nabla \ln T_i\right)\right]\frac{\partial f_{NMi}}{\partial s} - C\left(f_{NMi}\right) =$$
$$\left\{\frac{\xi s v_{Ti}}{T_i}\left(\frac{5}{2}-s^2\right)\mathbf{b}\cdot\nabla T_i + \frac{\xi s v_{Ti}}{n_i T_i}\left[\frac{2}{3}\mathbf{b}\cdot\nabla\left(p_{i\parallel}-p_{i\perp}\right) - \left(p_{i\parallel}-p_{i\perp}\right)\mathbf{b}\cdot\nabla \ln B\right] + P_2(\xi)\frac{2}{3}s^2\left(\nabla\cdot\mathbf{u}_i -\right.\right.$$
$$3\mathbf{b}\cdot\left[\mathbf{b}\cdot\nabla\mathbf{u}_i\right]) + \frac{2}{3n_i T_i}\left(s^2-\frac{3}{2}\right)\nabla\cdot\left(q_{i\parallel}\mathbf{b}\right) + \left[-\frac{2}{3}s^2 P_2(\xi)\left(\left(s^2-\frac{5}{2}\right)(2\kappa - \nabla \ln B) + \nabla \ln n_i\right) -\right.$$
$$\left.\left.\frac{4}{3}\left(\frac{1}{2}s^4 - \frac{5}{2}s^2 + \frac{15}{8}\right)(\kappa + \nabla \ln B)\right]\cdot\left(\frac{\nabla T_i \times \mathbf{b}}{eB}\right)\right\} f_{Mi}$$

- Velocity defined to be in macroscopic flow reference frame.
- $s \equiv v/v_{Ti}$, $\xi \equiv v_{\parallel}/v$, and C is the full linearized Fokker-Planck Coulomb collision operator
- $\kappa = \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}$
- Full CEL approach requires coupling to fluid evolution equations for n_i, T_i, and $\mathbf{u}_i$, (and an Ohm's law for $\mathbf{B}$).

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An initial test of the full CEL approach involves the full coupled system in axisymmetric geometry with no equilibrim flow

- Requires coupling the kinetic equation to the fluid evolution equations for n_i, $\mathbf{u}_i$, T_i, and T_e.
- Resistive MHD Ohm's law: $\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \mathbf{J}$.
- Temperature-dependent resistivity: $\eta = \eta_0 \left(T_{\text{ref}} / T_{\text{e,n=0}} \right)^{3/2}$, where η_0 is a reference resistivity, T_{ref} is a constant reference temperature, and $T_{e,n=0}$ is the axisymmetric electron temperature.
- We also use a small amount of isotropic diffusion, $D\nabla^2$, in all equations.
- For a first test, we initialize the equilibrium flows to 0 and then perturb the flow slightly at $t = 0$.

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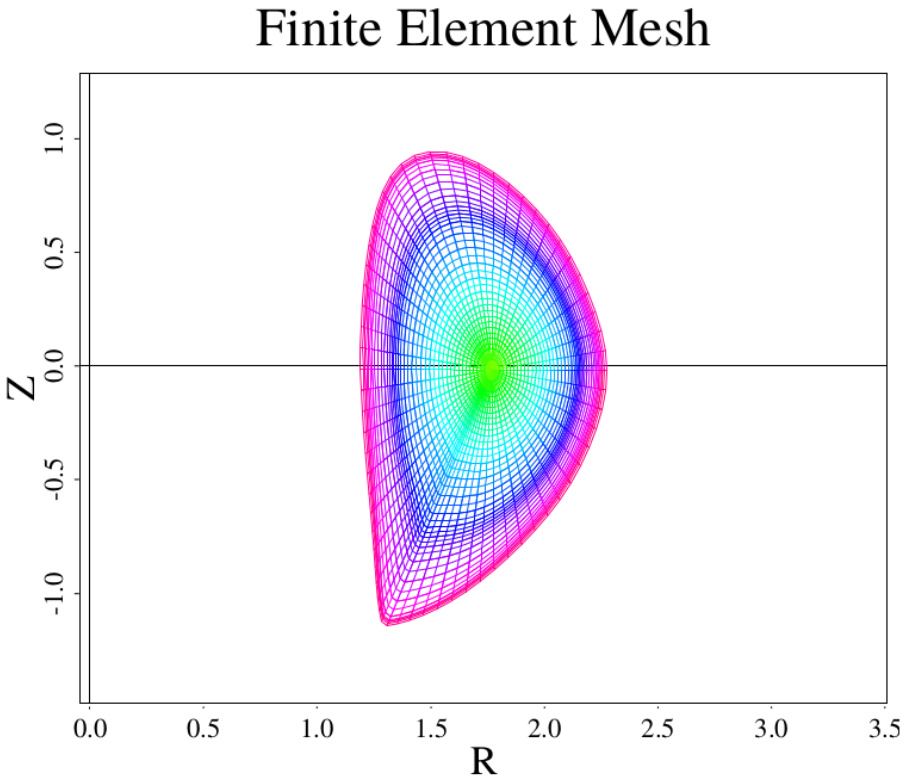
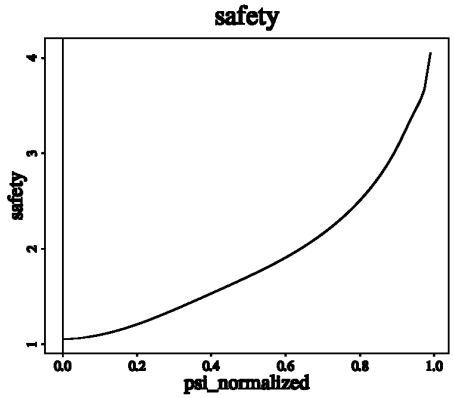
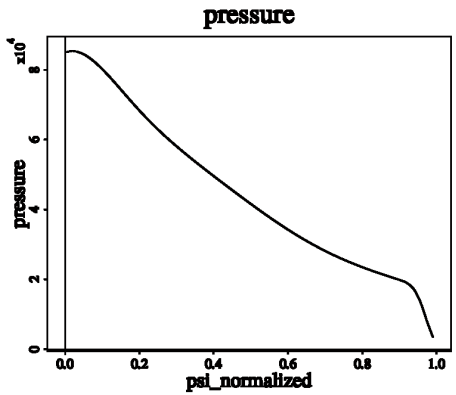
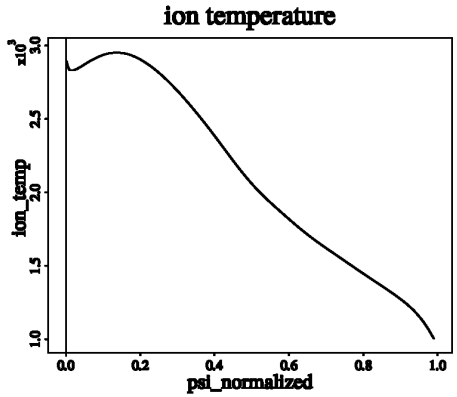
We use a DIII-D equilibrium (DIII-D Iter-Baseline-Scenario (IBS) discharge 174446 at 3390 ms) to test the CEL formulation









- x-point packing makes the grid slightly non-flux aligned

We have found it is important to remove the $\mathbf{b}_{\rm eq} \cdot \nabla T_{i,{\rm eq}}$ part of the moment terms on the RHS of the kinetic equation

- In the $\mathbf{b} \cdot \nabla \{ (p_{i\parallel} - p_{i\perp}) , q_{i\parallel} \}$ drives on the RHS of the kinetic equation, there are subtle $\mathbf{b}_{\rm eq} \cdot \nabla T_{i,{\rm eq}}$ parts included because both $(p_{i\parallel} - p_{i\perp})$ and $q_{i\parallel}$ depend on T_i (and we are doing currently the equivalent of a transfer_eq=.true. for the kinetic equation).
- Explicitly omiting the $\mathbf{b}_{\rm eq} \cdot \nabla T_{i,{\rm eq}}$ parts of those terms prevents large numerical errors from washing out the solution after about 100 time-steps.
- I am able to run ~ 10 times as far if I omit the $\mathbf{b}_{\rm eq} \cdot \nabla T_{i,{\rm eq}}$ parts from these drive terms.

Stability analysis for the coupled system is now underway

- Analysis assumes that all equilibrium quantities are uniform in space, with a uniform background magnetic field pointing in the y-z plane.
- We also assume only small perturbations about equilibrium for the perturbed variables.
- We then Fourier transform all linearized equations, and assume $\hat{\mathbf{k}} = \hat{\mathbf{y}}$ (i.e. that the wavevector is in the y direction).
- To deal with with the velocity space dependence of the kinetic equation, we first expand $f_{i1,k}$ as

$$f_{i1,k} = \sum_{l,m} \bar{f}_{lm,k} P_l(\xi) L_m(s) e^{-s^2} ,$$

  where $P_l(\xi)$ are Legendre polynomials and $L_m(s)$ are speed polynomials orthonormal with respect to the weight function e^{-s^2}.
- We then integrate the kinetic equation by $P_{l'}(\xi)$ and $L_{m'}(s)$. This then gives us a set of equations for $\bar{f}_{lm,k}$ which we truncate at some max l and m.

continued . . .

Stability analysis for the coupled system is now underway (continued)

- Solving for the eigenvalues of the timestep operator then allows one to observe numerical growth and damping rates. Specifically we assume $\mathbf{u}_k^{j+1} = \lambda \mathbf{u}_K^j$, where j represents the timestep, and $\mathbf{u}_k$ is a vector of all the perturbed quantities.
- This is currently implemented in matlab.

Conclusions/Future Work

Conclusions:

- Substantial progress has been made in incorporating the Chapman-Enskog-like approach into NIMROD for Tokamak applications.
- The solely kinetic aspects of the CEL approach have been successfully benchmarked.

Current/Future Work:

- Von Nuemann stability analysis of the full coupled system is currently underway.
- Verifying the full implementation through comparison with other codes/analytics/experiment, etc.
- Incorporate poloidal flow damping physics into a non-trivial nonlinear extended MHD calculation, continuing previous work done in NIMROD on the time-dependent field error induced reconnection problem [4], (i.e. Move to full toroidal geometry, and use our improved closure scheme for $\mathbf{\Pi}_i$).

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References

[1] C. Sovinec and J. King, "Analysis of a mixed semi-implicit/implicit algorithm for low-frequency two-fluid plasma modeling", Journal of Computational Physics **229**, 5803–5819 (2010).

[2] J. P. Wang and J. D. Callen, "Fluid/kinetic hybrid moment description of plasmas via a chapman–enskog-like approach", Physics of Fluids B: Plasma Physics **4**, 1139–1151 (1992).

[3] J. J. Ramos, Phys. Plasmas **17**, 082502 (2010).

[4] M. T. Beidler, J. D. Callen, C. C. Hegna, and C. R. Sovinec, "Mode penetration induced by transient magnetic perturbations", Physics of Plasmas **25**, 082507 (2018).