Unexpected linear resistive mode in RE+MHD

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Linear fluid RE + MHD model equations.

Equilibrium quantities: $N_r, \boldsymbol{B}, \boldsymbol{U}_r, \boldsymbol{J}$; Perturbations: $n_r, \boldsymbol{b}, \boldsymbol{u}_r, \boldsymbol{v}, \boldsymbol{j}$. $0-\beta$, incompressible MHD.

$$\partial_t n_r + \boldsymbol{\nabla} \cdot (N_r \boldsymbol{u}_r + n_r \boldsymbol{U}_r) = 0 \tag{1}$$

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) - \boldsymbol{\nabla} \times (\eta \boldsymbol{j} + e n_r \boldsymbol{U}_r + e N_r \boldsymbol{u}_r) \tag{2}$$

$$\partial_t \boldsymbol{v} = \boldsymbol{j} \times \boldsymbol{B} + \boldsymbol{J} \times \boldsymbol{b} \tag{3}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{b} = 0, \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \tag{4}$$

The linear fluid RE velocity is essentially different than the nonlinear model.

The equilibrium velocity is $\boldsymbol{U}_r = -c_r \boldsymbol{B}/B$. The full perturbed RE velocity including drift effects is given by

$$\boldsymbol{u}_{r,\perp}(1+\alpha^2) = \boldsymbol{v}_\perp - c_r \frac{\boldsymbol{b}_\perp}{B} + \eta\alpha \boldsymbol{j}_\perp + \left(\eta \frac{\boldsymbol{j}}{B} + \alpha \boldsymbol{v} - \alpha c_r \frac{\boldsymbol{b}}{B}\right) \times \frac{\boldsymbol{B}}{B} \tag{5}$$

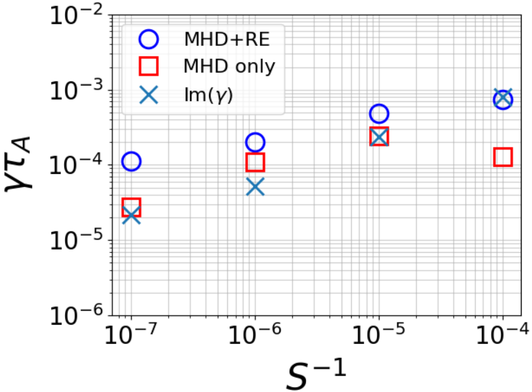
$$\alpha \equiv \frac{\eta e N_r}{B} \tag{6}$$

At present, consider only the perturbation of the parallel component.

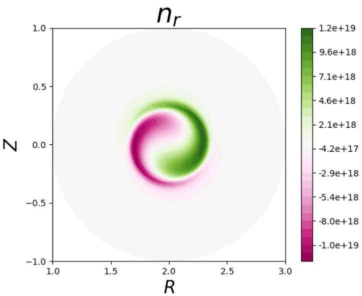
$$\boldsymbol{u}_r = -c_r \frac{\boldsymbol{b}_\perp}{B} \tag{7}$$

Results from cylindrical geometry tearing mode case reproduce Liu et. al analysis.

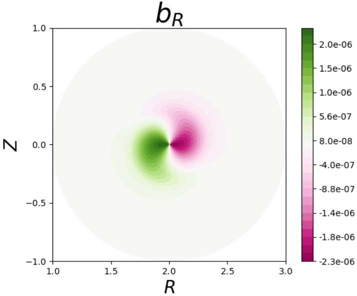
$\beta = 0$ screw pinch equilibrium in a cylinder $q = 1.15\left(1 + \frac{(r/a)^2}{0.6561}\right)$
Alfvén velocity $V_A \equiv B/\sqrt{\rho\mu_0} = 1$
Lundquist number: $S \equiv aV_A/(\eta/\mu_0)$
$q \equiv rB_\phi/RB_\theta$ where $R = L/2\pi$.



Solutions of the linear equation show odd behavior at lower values of S, high η.

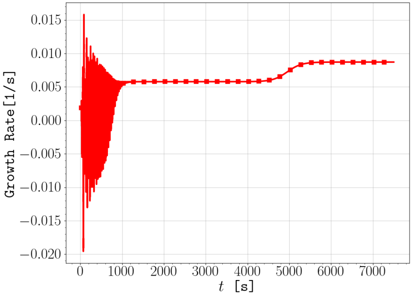


(a) Runaway density perturbation of growing linear mode at $S = 10^3$

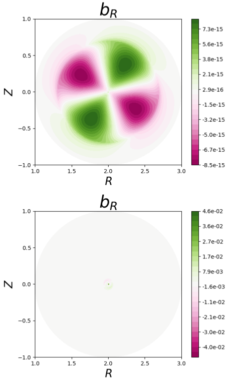


(b) Radial magnetic perturbation of growing linear mode at $S = 10^3$

The growth of the anomalous mode competes with the tearing mode at intermediate values of S.



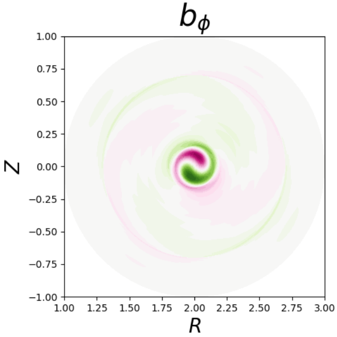
(a) Growth rate over time at $S = 10^4$

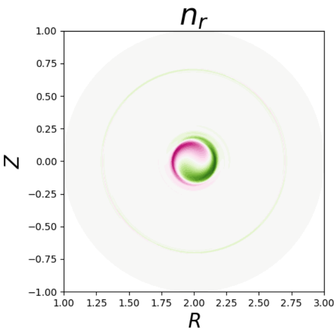


(b) Radial magnetic perturbation of growing linear mode at $t = 2000$ and $t = 5500$ $S = 10^4$

Small time steps don't get rid of the mode

$\Delta t = 10^{-3}$





The mode persists when velocity is commented out from the NIMROD advance

Then the equations we need to consider are

$$\partial_t n_r - c_r \cdot \nabla \frac{n_r}{B} = c_r \boldsymbol{\nabla} \cdot \left(\frac{N_r}{B} \boldsymbol{b} - \frac{N_r(\boldsymbol{b} \cdot \boldsymbol{B}}{B^3} \boldsymbol{B} \right) \tag{8}$$

$$\partial_t \boldsymbol{b} - \frac{\eta}{\mu_0} \nabla^2 \boldsymbol{b} = \eta e c_r \boldsymbol{\nabla} \times \left(\frac{n_r}{B} \boldsymbol{B} + \frac{N_r}{B} \left(\boldsymbol{b} - \frac{\boldsymbol{b} \cdot \boldsymbol{B}}{B^2} \boldsymbol{B} \right) \right) \tag{9}$$

Slab geometry analysis suggests the possibility of an instability

$$\boldsymbol{b} \sim \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t) \tag{10}$$

$$-i\omega n - ic_r\frac{\boldsymbol{k}\cdot\boldsymbol{B}}{B}n = -ic_r\frac{N}{B^3}(\boldsymbol{k}\cdot\boldsymbol{B})(\boldsymbol{b}\cdot\boldsymbol{B}) \tag{11}$$

$$-i\omega\boldsymbol{b} + \frac{\eta}{\mu_0}k^2\boldsymbol{b} = i\eta e c_r\left(\frac{n}{B}\boldsymbol{k}\times\boldsymbol{B} + \frac{N}{B^3}\boldsymbol{k}\times(\boldsymbol{B}\times(\boldsymbol{b}\times\boldsymbol{B})\right) \tag{12}$$

Combining:

$$\left(\omega + i\frac{\eta}{\mu_0}k^2\right)\boldsymbol{b} + \frac{\eta e c_r N}{B^3}\left(\frac{c_r(\boldsymbol{k}\cdot\boldsymbol{B})}{(B\omega + c_r\boldsymbol{k}\cdot\boldsymbol{B})} - 1\right)(\boldsymbol{B}\cdot\boldsymbol{b})\boldsymbol{k}\times\boldsymbol{B} + \frac{\eta e c_r N}{B}\boldsymbol{k}\times\boldsymbol{b} = 0 \tag{13}$$

Slab analysis cont.

Define $D \equiv \omega - i\eta k^2/\mu_0$, and $H \equiv \eta e c_r N/B$. Then consider $k \parallel \boldsymbol{B} \implies \boldsymbol{b} \cdot \boldsymbol{B} = 0$,

$$D\boldsymbol{b} + H\boldsymbol{k} \times \boldsymbol{b} = 0. \tag{14}$$

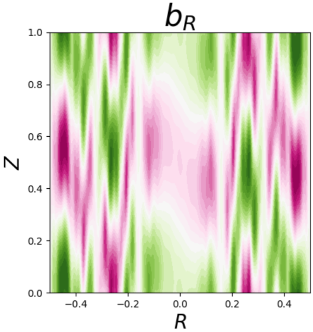
The resulting dispersion relation is

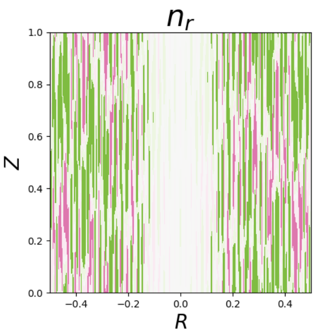
$$D^2 = -H^2 k^2 \implies \omega = i\left(Hk - \frac{\eta}{\mu_0} k^2\right) \tag{15}$$

$$\tag{16}$$

In this case there is an instability for $k < H\mu_0/\eta$ or $k < e c_r \mu_0 N/B$.

Behavior of slab cases suggest it is more likely a numerical issue





Semi-discrete form of the equations have a relatively simple form without perturbed drift.

$$\Delta \boldsymbol{b} = -\Delta t \boldsymbol{\nabla} \times (\boldsymbol{e}_{ideal}) - \Delta t \boldsymbol{\nabla} \times \eta \left(\boldsymbol{\nabla} \times \boldsymbol{b} - ecn_r \frac{\boldsymbol{B}}{B} - ecN_r \boldsymbol{b}_\perp \right) \tag{17}$$

$$\Delta n_r - \frac{\Delta t}{2} c_r \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{B}}{B} \Delta n_r \right) = \Delta t c_r \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{b}}{B} N_r + \frac{\boldsymbol{B}}{B} n_r \right) \tag{18}$$

But $\boldsymbol{b}$ in (18) is evaluated as $\boldsymbol{b} + \Delta \boldsymbol{b}/2$

Summary

- Linear RE simulations show unexpected behavior at large resistivity
- Some evidence of numerical instability
- Slab wave analysis suggests there is a potential for instability in the equations with growth rate scaling with η.

Future Work

Simple Von-Neumann numerical stability analysis for the time stepping scheme with REs + magnetic field should be done.

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