A systematic derivation of general closures for electron-ion plasmas*

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Introduction: general structure of closures

- Closing 5 moment equations: $n_a, \mathbf{V}_a, T_a$

$$d_a n_a + n_a \nabla \cdot \mathbf{V}_a = 0$$

$$\frac{3}{2} n_a d_a T_a + n_a T_a \nabla \cdot \mathbf{V}_a + \nabla \cdot \mathbf{q}_a + \nabla \mathbf{V}_a : \boldsymbol{\pi}_a = Q_a$$

$$m_a n_a d_a \mathbf{V}_a - n_a e_a (\mathbf{E} + \mathbf{V}_a \times \mathbf{B}) + \nabla p_a + \nabla \cdot \boldsymbol{\pi}_a = \mathbf{R}_a$$

Higher order moment equations

  - Transport ordering: higher order moments are slowly varying
  - Solve kinetic equation to find
    $\mathbf{q}_a, \boldsymbol{\pi}_a \Leftarrow \nabla \mathbf{V}_a, \nabla T_a$ (thermodynamic drives)

- Closing 2 moment equations: n_a, T_a

  - $\mathbf{V}_a, \mathbf{q}_a, \boldsymbol{\pi}_a \Leftarrow \nabla n_a, \nabla T_a, \mathbf{E}$
  - Particle flux $\boldsymbol{\Gamma}_a = n_a \mathbf{V}_a$ will be found

Moment expansion of a distribution function

- Landau kinetic equation

$$\partial_t f_a + \mathbf{v} \cdot \nabla f_a + \frac{e_a}{m_a}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \partial_{\mathbf{v}} f_a = \sum_b C(f_a, f_b)$$

- Moment expansion: expansion coefficients, m^{lk}'s, are symmetric traceless fluid moments

$$\begin{aligned} f_a(t, \mathbf{x}, \mathbf{v}) &= f_a^{\mathrm{M}} \sum_{lk} \frac{1}{\sqrt{\sigma_k^l}} \mathsf{m}_a^{lk}(t, \mathbf{x}) \cdot \mathsf{p}_a^{lk} \\ \mathsf{n}_a^{lk} \equiv n_a \mathsf{m}_a^{lk}(t, \mathbf{x}) &= \int d\mathbf{v} \frac{1}{\sqrt{\sigma_k^l}} \mathsf{p}_a^{lk} f_a(t, \mathbf{x}, \mathbf{v}) \end{aligned}$$

- p^{lk}'s are orthogonal, irreducible, tensorial polynomials and form a complete set

$$\int d\mathbf{v} \mathsf{p}^{jp} \mathsf{p}^{lk} \cdot \mathsf{m}^{lk} f^{\mathrm{M}} = \delta_{jl} \delta_{pk} \sigma_p^j \mathsf{m}^{jp}$$

5 moment equations for electrons and ions

- For electrons

$$d_e n_e + n_e \nabla \cdot \mathbf{V}_e = 0$$

$$\frac{3}{2} n_e d_e T_e + T_e \nabla \cdot \mathbf{V}_e + \nabla \cdot \mathbf{q}_e + \nabla \mathbf{V}_e : \boldsymbol{\pi}_e = -\sqrt{\frac{3}{2}} T_e \hat{A}_{ei}^{01,0k} \mathsf{n}_e^{0k}$$

$$m_e n_e d_e \mathbf{V}_e - n_e e_e (\mathbf{E} + \mathbf{V}_e \times \mathbf{B}) + \nabla p_e + \nabla \cdot \boldsymbol{\pi}_e = \frac{1}{\sqrt{2}} m_e v_{Te} (\hat{A}_{ei}^{10,00} + \hat{A}_{ei}^{10,1k} \mathsf{n}_e^{1k})$$

- For ions

$$d_i n_i + n_i \nabla \cdot \mathbf{V}_i = 0$$

$$\frac{3}{2} n_i d_i T_i + T_i \nabla \cdot \mathbf{V}_i + \nabla \cdot \mathbf{q}_i + \nabla \mathbf{V}_i : \boldsymbol{\pi}_i = -\sqrt{\frac{3}{2}} T_i \hat{B}_{ie}^{01,0k} \mathsf{n}_e^{0k}$$

$$m_i n_i d_i \mathbf{V}_i - n_e e_i (\mathbf{E} + \mathbf{V}_i \times \mathbf{B}) + \nabla p_i + \nabla \cdot \boldsymbol{\pi}_i = \frac{1}{\sqrt{2}} m_i v_{Ti} (\hat{B}_{ie}^{10,00} + \hat{B}_{ie}^{10,1k} \mathsf{n}_e^{1k})$$

where

$$d_a = \frac{\partial}{\partial t} + \mathbf{V}_a \cdot \nabla$$

Moment equations for electrons

- Collision operators and moment equations calculated (Ji and Held 2006, 2008)

$$\tau_{\rm ee}(\hat{D}_{\rm e}+\Omega_{\rm e}\mathbf{b}\times)\begin{bmatrix}\mathsf{n}_{\rm e}^0\\ \mathsf{n}_{\rm e}^1\\ \mathsf{n}_{\rm e}^2\\ \mathsf{n}_{\rm e}^3\\ \vdots\end{bmatrix}=(c_{\rm ee}+Za_{\rm ei})\begin{bmatrix}\mathsf{n}_{\rm e}^0\\ \mathsf{n}_{\rm e}^1\\ \mathsf{n}_{\rm e}^2\\ \mathsf{n}_{\rm e}^3\\ \vdots\end{bmatrix}+\tau_{\rm ee}\begin{bmatrix}0\\ \mathsf{G}_{\rm e}^1\\ \mathsf{G}_{\rm e}^2\\ 0\\ \vdots\end{bmatrix}$$

$$\mathsf{n}_{\rm e}^0=\begin{pmatrix}\mathsf{n}_{\rm e}^{02}\\ \mathsf{n}_{\rm e}^{03}\\ \mathsf{n}_{\rm e}^{04}\\ \vdots\end{pmatrix},\ \mathsf{n}_{\rm e}^1=\begin{pmatrix}\mathsf{n}_{\rm e}^{11}\\ \mathsf{n}_{\rm e}^{12}\\ \mathsf{n}_{\rm e}^{13}\\ \vdots\end{pmatrix},\ \mathsf{n}_{\rm e}^2=\begin{pmatrix}\mathsf{n}_{\rm e}^{20}\\ \mathsf{n}_{\rm e}^{21}\\ \mathsf{n}_{\rm e}^{22}\\ \vdots\end{pmatrix},\ \cdots$$

$$\mathsf{G}_{\rm e}^1=\begin{pmatrix}\sqrt{\sigma_1^1}n_{\rm e}v_{T\rm e}\nabla\ln T_{\rm e}+\hat{A}_{\rm ei}^{11,00}\\ \hat{A}_{\rm ei}^{12,00}\\ \hat{A}_{\rm ei}^{13,00}\\ \vdots\end{pmatrix},\ \mathsf{G}_{\rm e}^2=\begin{pmatrix}-\frac{1}{\sqrt{2}}n_{\rm e}\mathsf{W}_{\rm e}\\ 0\\ 0\\ \vdots\end{pmatrix}$$

$$\hat{A}_{\rm ei}^{1k,00}=Z\tau_{\rm ee}^{-1}\sqrt{2}a_{\rm ei}^{10,1k}\mathbf{v}_{\rm ei}$$

Moment equations for ions

$$\tau_{\rm ii}(\hat{D}_{\rm i} + \Omega_{\rm i}\mathbf{b}\times)\begin{bmatrix} \mathsf{n}_{\rm i}^0 \\ \mathsf{n}_{\rm i}^1 \\ \mathsf{n}_{\rm i}^2 \\ \mathsf{n}_{\rm i}^3 \\ \vdots \end{bmatrix} = (c_{\rm ii} + Z^{-1}a_{\rm ie})\begin{bmatrix} \mathsf{n}_{\rm i}^0 \\ \mathsf{n}_{\rm i}^1 \\ \mathsf{n}_{\rm i}^2 \\ \mathsf{n}_{\rm i}^3 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_{\rm ie}^2\mathsf{n}_{\rm e}^2 \\ 0 \\ \vdots \end{bmatrix} + \tau_{\rm ii}\begin{bmatrix} 0 \\ \mathsf{G}_{\rm i}^1 \\ \mathsf{G}_{\rm i}^2 \\ 0 \\ \vdots \end{bmatrix}$$

$$\mathsf{n}_{\rm i}^0 = \begin{pmatrix} \mathsf{n}_{\rm i}^{02} \\ \mathsf{n}_{\rm i}^{03} \\ \mathsf{n}_{\rm i}^{04} \\ \vdots \end{pmatrix}, \quad \mathsf{n}_{\rm i}^1 = \begin{pmatrix} \mathsf{n}_{\rm i}^{11} \\ \mathsf{n}_{\rm i}^{12} \\ \mathsf{n}_{\rm i}^{13} \\ \vdots \end{pmatrix}, \quad \mathsf{n}_{\rm i}^2 = \begin{pmatrix} \mathsf{n}_{\rm i}^{20} \\ \mathsf{n}_{\rm i}^{21} \\ \mathsf{n}_{\rm i}^{22} \\ \vdots \end{pmatrix}, \quad \cdots$$

$$b_{\rm ie}^2\mathsf{n}_{\rm e}^2 = \begin{pmatrix} b_{\rm ie}^{20,2k}\mathsf{n}_{\rm e}^{2k} \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad \mathsf{G}_{\rm i}^1 = \begin{pmatrix} \sqrt{\sigma_1^1}\, n_{\rm i} v_{T{\rm i}} \nabla \ln T_{\rm i} \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad \mathsf{G}_{\rm i}^2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} n_{\rm i}\mathsf{W}_{\rm i} \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$\mathsf{W} = \nabla\mathbf{V} + \widetilde{\nabla\mathbf{V}} - \frac{2}{3}\mathsf{I}\nabla\cdot\mathbf{V}$$

Kinetic equations vs. moment equations

- Kinetic equation $D_a f_a = \sum_b C(f_a, f_b)$

$$\partial_t f_a + \mathbf{v} \cdot \nabla f_a + \frac{e_a}{m_a}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \partial_{\mathbf{v}} f_a = \sum_b C(f_a, f_b)$$

- Chapman-Enskog like approach $f_a = f_a^{\mathrm{M}} + F_a$

$$\begin{aligned} f_a^{\mathrm{M}}(t, \mathbf{x}, \mathbf{v}) &= n_a(t, \mathbf{x}) \left(\frac{m_a}{2\pi T_a(t, \mathbf{x})} \right)^{3/2} \exp\left(-\frac{m[\mathbf{v} - \mathbf{V}_a(t, \mathbf{x})]^2}{2T_a(t, \mathbf{x})} \right) \\ F_a(t, \mathbf{x}, \mathbf{v}) &= f_a^{\mathrm{M}} \sum_{lk} \frac{1}{\sqrt{\sigma_k^l}} \mathsf{m}_a^{lk}(t, \mathbf{x}) \cdot \mathsf{p}_a^{lk} \end{aligned}$$

$$D_a F_a = \sum_b [C(f_a^{\mathrm{M}}, f_b^{\mathrm{M}}) + C(F_a, f_b^{\mathrm{M}}) + C(f_a^{\mathrm{M}}, F_b) + C(F_a, F_b)] - D_a f_a^{\mathrm{M}}$$

- Moment equation

$$(\hat{D}_a + \Omega_a \mathbf{b} \wedge) \mathsf{n}_a = \sum_b (\hat{A}_{ab} \mathsf{n}_a + \hat{B}_{ab} \mathsf{n}_b + C_{ab}^{(2)}) + \mathsf{G}_a$$

Solving linearized moment equations: two solvable cases

- Highly collisional plasma $C_a \gg D_a$ and general **B**
  - Braginskii's transport equations
    - $2 \times 2 \, (L_1^{(l+1/2)}, \, L_2^{(l+1/2)})$ calculation
  - Geometric method : coordinate-free expression
  - Corrections made for ions from $C_{\rm ie}$ (Ji and Held 2008)
  - Higher ϵ-order calculation: range of validity
    $\mathbf{q} = -3.950 \frac{n\tau T}{m} - 3843 \frac{n\tau^3}{m} (\nabla T)^2 \nabla T - 704.0 \frac{n\tau^3}{m} \nabla^2 T \nabla T \cdots$
- Strong magnetic field $\Omega_a \gg D_a, \, C_a$ with general collisionality
  - Parallel moments: integral closure (Ji, Held and Sovinec 2008)
  - Perpendicular moments: should be modified by parallel moment corrections
  - Geometric method
    - no flux surface average, no bounce average

Solving moment equations for collisional closures

$$\partial_t f_a + \mathbf{v} \cdot \nabla f_a + \frac{e_a}{m_a}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \partial_{\mathbf{v}} f_a = \sum_b C(f_a, f_b)$$

$$\begin{aligned} x_a &\equiv \Omega_a \tau_{aa} \\ c_\mathrm{e} &= c_\mathrm{ee} + Z a_\mathrm{ei} \\ c_\mathrm{i} &= c_\mathrm{ii} + Z^{-1} a_\mathrm{ie} \end{aligned}$$

$$\tau[\hat{D}] \begin{bmatrix} \mathsf{n}^0 \\ \mathsf{n}^1 \\ \mathsf{n}^2 \\ \mathsf{n}^3 \\ \vdots \end{bmatrix} + x\mathbf{b} \wedge \begin{bmatrix} \mathsf{n}^0 \\ \mathsf{n}^1 \\ \mathsf{n}^2 \\ \mathsf{n}^3 \\ \vdots \end{bmatrix} = [c] \begin{bmatrix} \mathsf{n}^0 \\ \mathsf{n}^1 \\ \mathsf{n}^2 \\ \mathsf{n}^3 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 \\ \mathsf{g}^1 \\ \mathsf{g}^2 \\ 0 \\ \vdots \end{bmatrix}$$

- $l = 0$ scalar moment equations

$$0 = c^0 \mathsf{n}^0 \quad \Rightarrow \quad \mathsf{n}^0 = 0$$

Solving $l = 1$ moment equations: geometric method

- Vector equation

$$x\mathbf{b} \times \mathsf{n}^1 = c^1 \mathsf{n}^1 + \mathsf{g}^1$$

$$\begin{pmatrix} x\mathbf{b} \times \mathsf{n}^{11} \\ x\mathbf{b} \times \mathsf{n}^{12} \\ x\mathbf{b} \times \mathsf{n}^{13} \\ x\mathbf{b} \times \mathsf{n}^{14} \\ \vdots \end{pmatrix} = \begin{pmatrix} c^1_{11} & c^1_{12} & c^1_{13} & c^1_{14} & \cdots \\ c^1_{21} & c^1_{22} & c^1_{23} & c^1_{24} & \cdots \\ c^1_{31} & c^1_{32} & c^1_{33} & c^1_{34} & \cdots \\ c^1_{41} & c^1_{42} & c^1_{43} & c^1_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \mathsf{n}^{11} \\ \mathsf{n}^{12} \\ \mathsf{n}^{13} \\ \mathsf{n}^{14} \\ \vdots \end{pmatrix} + \begin{pmatrix} \mathsf{g}^{11} \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

- Vector decomposition

$$\mathsf{n}^1 = \mathsf{n}^1_{\parallel} + \mathsf{n}^1_{\perp} = \mathbf{bb} \cdot \mathsf{n}^1 + (\mathsf{I} - \mathbf{bb}) \cdot \mathsf{n}^1 = \mathbf{bb} \cdot \mathsf{n}^1 - \mathbf{b} \times (\mathbf{b} \times \mathsf{n}^1)$$

- Parallel moment:

$$0 = c^1 \mathsf{n}^1_{\parallel} + \mathsf{g}^1_{\parallel} \quad \Rightarrow \quad \mathsf{n}^1_{\parallel} = -(c^1)^{-1} \mathsf{g}^1_{\parallel}$$

Solving $l = 1$ moment equations: geometric method

- Perpendicular moment:

$$x\mathbf{b} \times \mathsf{n}^1 = c^1 \mathsf{n}^1 + \mathsf{g}^1$$

$$\begin{aligned} x\mathbf{b} \times (\mathbf{b} \times \mathsf{n}^1) &= c^1 \mathbf{b} \times \mathsf{n}^1 + \mathbf{b} \times \mathsf{g}^1 \\ x\mathbf{b} \times [\mathbf{b} \times (\mathbf{b} \times \mathsf{n}^1)] &= c^1 \mathbf{b} \times (\mathbf{b} \times \mathsf{n}^1) + \mathbf{b} \times (\mathbf{b} \times \mathsf{g}^1) \end{aligned}$$

- $\mathsf{n}^1_\times \equiv \mathbf{b} \times \mathsf{n}^1$, $\mathsf{n}^1_\perp \equiv -\mathbf{b} \times (\mathbf{b} \times \mathsf{n}^1)$

$$\begin{bmatrix} c^1 & x \\ -x & c^1 \end{bmatrix} \begin{bmatrix} \mathsf{n}^1_\times \\ \mathsf{n}^1_\perp \end{bmatrix} = \begin{bmatrix} -\mathsf{g}^1_\times \\ -\mathsf{g}^1_\perp \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \mathsf{n}^1_\times \\ \mathsf{n}^1_\perp \end{bmatrix} = \begin{bmatrix} c^1 & x \\ -x & c^1 \end{bmatrix}^{-1} \begin{bmatrix} -\mathsf{g}^1_\times \\ -\mathsf{g}^1_\perp \end{bmatrix}$$

$$\Rightarrow \quad \mathsf{n}^1_\perp = -\frac{1}{(c^1)^2 + (x)^2} (x\mathsf{g}^1_\times + c^1 \mathsf{g}^1_\perp)$$

Solving $l = 2$ moment equations: geometric method

- Rank-2 symmetric tensor equation

$$x\mathbf{b} \wedge \mathsf{n}^2 = c^2 \mathsf{n}^2 + \mathsf{g}^2$$

- Rank-2 tensor decomposition:

$$\mathsf{n}^2 = \mathsf{n}^2_{\parallel\parallel} + \mathsf{n}^2_{\parallel\perp} + \mathsf{n}^2_{\perp\parallel} + \mathsf{n}^2_{\perp\perp}$$

- Parallel moment:

$$0 = c^2 \mathsf{n}^2_{\parallel\parallel} + \mathsf{g}^2_{\parallel\parallel} \quad \Rightarrow \quad \mathsf{n}^2_{\parallel\parallel} = -(c^2)^{-1} \mathsf{g}^2_{\parallel\parallel}$$

Solving $l = 2$ moment equations: geometric method

- Parallel-perpendicular moments

$$\begin{bmatrix} c^2 & x \\ -x & c^2 \end{bmatrix} \begin{bmatrix} \mathsf{n}^2_{\parallel\times} \\ \mathsf{n}^2_{\parallel\perp} \end{bmatrix} = - \begin{bmatrix} \mathsf{g}^2_{\parallel\times} \\ \mathsf{g}^2_{\parallel\perp} \end{bmatrix} \quad \Rightarrow \quad \mathsf{n}^2_{\parallel\perp} = -\frac{1}{(c^2)^2 + (x)^2}(x\mathsf{g}^2_{\parallel\times} + c^2\mathsf{g}^2_{\parallel\perp})$$

- Perpendicular-perpendicular moments

$$\begin{bmatrix} c^2 & 2x & \\ -x & c^2 & x \\ & -2x & c^2 \end{bmatrix} \begin{bmatrix} \mathsf{n}^2_{\times\times} \\ \mathsf{n}^2_{\times\perp} \\ \mathsf{n}^2_{\perp\perp} \end{bmatrix} = \begin{bmatrix} -\mathsf{g}^2_{\times\times} \\ -\mathsf{g}^2_{\times\perp} \\ -\mathsf{g}^2_{\perp\perp} \end{bmatrix}$$

- Further decomposition by $\mathsf{n}_\pm \equiv \frac{1}{2}(\mathsf{n}_{\perp\perp} \pm \mathsf{n}_{\times\times})$

$$c^2\mathsf{n}^2_+ = -\mathsf{g}^2_+ \quad \Rightarrow \quad \mathsf{n}^2_+ = -(c^2)^{-1}\mathsf{g}^2_+$$

$$\begin{bmatrix} c^2 & 2x \\ -2x & c^2 \end{bmatrix} \begin{bmatrix} \mathsf{n}^2_{\times\perp} \\ \mathsf{n}^2_{-} \end{bmatrix} = - \begin{bmatrix} \mathsf{g}^2_{\times\perp} \\ \mathsf{g}^2_{-} \end{bmatrix} \quad \Rightarrow \quad \mathsf{n}^2_{-} = -\frac{1}{(c^2)^2 + (2x)^2}(2x\mathsf{g}^2_{\times\perp} + c^2\mathsf{g}^2_{-})$$

Solution $l = 2$: matrix convention

$$\mathsf{W}_{\parallel\parallel} \doteq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & W_{zz} \end{pmatrix}, \quad \mathsf{W}_{\parallel\times} \doteq \begin{pmatrix} 0 & 0 & -\frac{1}{2}W_{yz} \\ 0 & 0 & \frac{1}{2}W_{xz} \\ -\frac{1}{2}W_{yz} & \frac{1}{2}W_{xz} & 0 \end{pmatrix}$$

$$\mathsf{W}_{\parallel\perp} \doteq \begin{pmatrix} 0 & 0 & \frac{1}{2}W_{xz} \\ 0 & 0 & \frac{1}{2}W_{yz} \\ \frac{1}{2}W_{xz} & \frac{1}{2}W_{yz} & 0 \end{pmatrix}, \quad \mathsf{W}_{\times\times} \doteq \begin{pmatrix} W_{yy} & -W_{xy} & 0 \\ -W_{xy} & W_{xx} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathsf{W}_{\perp\perp} \doteq \begin{pmatrix} W_{xx} & W_{xy} & 0 \\ W_{xy} & W_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{W}_{\times\perp} \doteq \begin{pmatrix} -W_{xy} & \frac{1}{2}(W_{xx} - W_{yy}) & 0 \\ \frac{1}{2}(W_{xx} - W_{yy}) & W_{xy} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Braginskii's: $\boldsymbol{\pi} = \eta_0 \mathsf{W}_0 + \eta_1 \mathsf{W}_1 + \eta_2 \mathsf{W}_2 + \eta_3 \mathsf{W}_3 + \eta_4 \mathsf{W}_4$

$$\mathsf{W}_0 = \frac{1}{2}(\mathsf{W}_{\times\times} + \mathsf{W}_{\perp\perp}) + \mathsf{W}_{\parallel\parallel}, \ \mathsf{W}_1 = \frac{1}{2}(\mathsf{W}_{\perp\perp} - \mathsf{W}_{\times\times}), \ \mathsf{W}_2 = 2\mathsf{W}_{\parallel\perp}, \ \mathsf{W}_3 = 2\mathsf{W}_{\times\perp} \ \mathsf{W}_4 = 2\mathsf{W}_{\parallel\times}$$

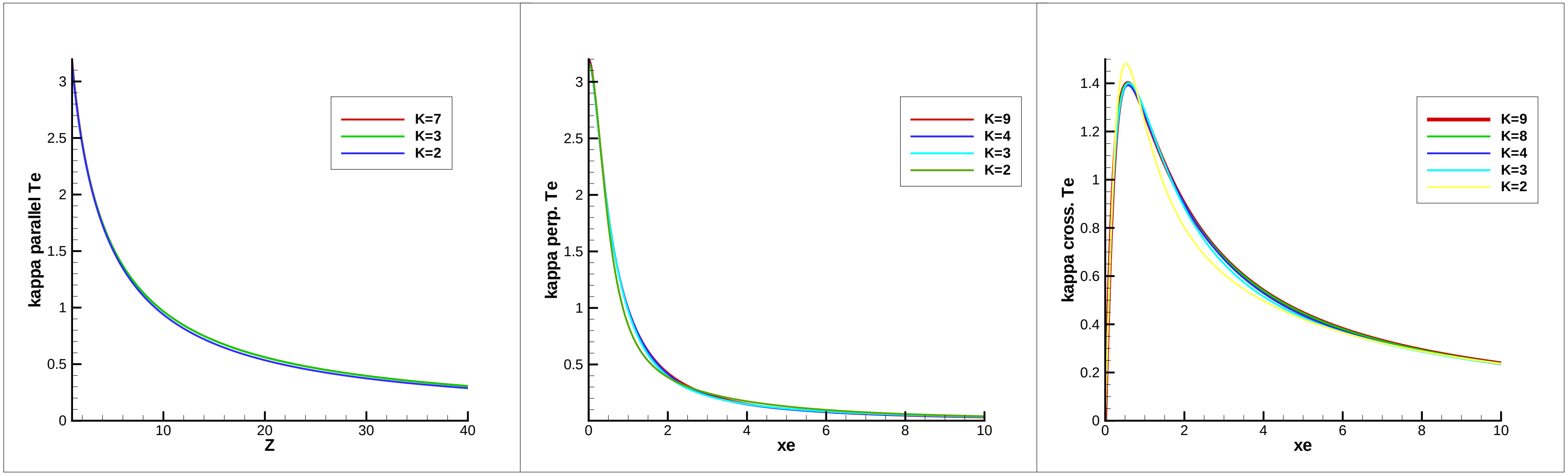
- Chew-Goldberger-Low pressure tensor: $\mathbf{b} \times \mathsf{n}^2 = 0$

$$\mathsf{n}^2_{\mathrm{CGL}} \doteq \begin{pmatrix} -\frac{1}{2}n_{zz} & 0 & 0 \\ 0 & -\frac{1}{2}n_{zz} & 0 \\ 0 & 0 & n_{zz} \end{pmatrix}$$

Heat flux for electrons

$$\mathbf{q}_\mathrm{e} = -\underline{\kappa^T_{\mathrm{e}\|}}\nabla_\| T_\mathrm{e} - \underline{\kappa^T_{\mathrm{e}\perp}}\nabla_\perp T_\mathrm{e} + \underline{\kappa^T_{\times}}\mathbf{b}\times\nabla T_\mathrm{e}$$
$$-\kappa^V_{\mathrm{e}\|}\mathbf{V}_{\mathrm{ei}\|} - \kappa^V_{\mathrm{e}\perp}\mathbf{V}_{\mathrm{ei}\perp} + \kappa^V_{\mathrm{e}\times}\mathbf{V}_{\mathrm{ei}\times}$$

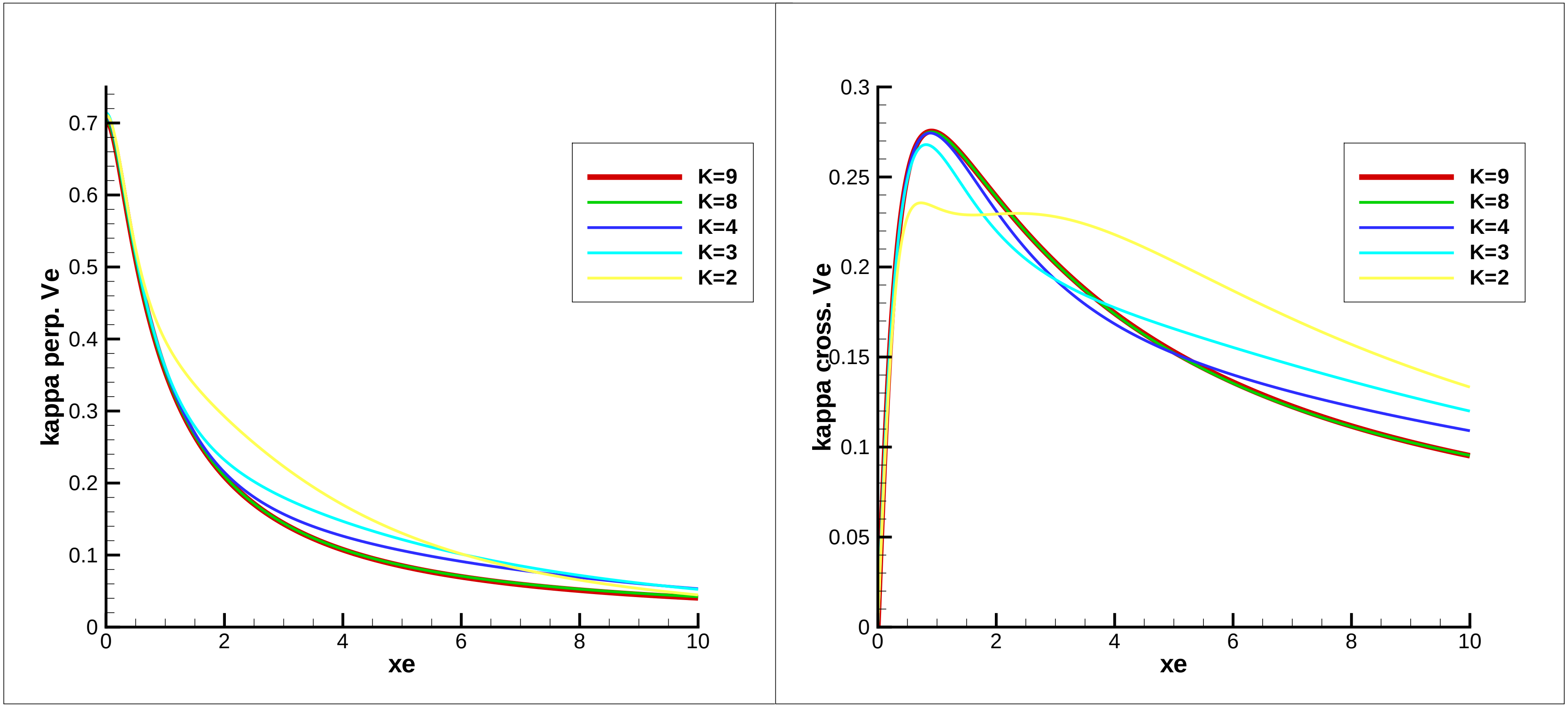
$$\kappa^T_{\mathrm{e}A} = \frac{n_\mathrm{e} T_\mathrm{e} \tau_\mathrm{e}}{m_\mathrm{e}}\hat{\kappa}^T_{\mathrm{e}A} \quad (A = \|, \perp, \times)$$



Heat flux for electrons

$$\begin{aligned}\mathbf{q}_{\mathrm{e}} &= -\kappa^{T}_{\mathrm{e}\parallel}\nabla_{\parallel}T_{\mathrm{e}} - \kappa^{T}_{\mathrm{e}\perp}\nabla_{\perp}T_{\mathrm{e}} + \kappa^{T}_{\times}\mathbf{b}\times\nabla T_{\mathrm{e}} \\ &\quad -\kappa^{V}_{\mathrm{e}\parallel}\mathbf{V}_{\mathrm{ei}\parallel} - \underline{\kappa^{V}_{\mathrm{e}\perp}}\mathbf{V}_{\mathrm{ei}\perp} + \underline{\kappa^{V}_{\mathrm{e}\times}}\mathbf{V}_{\mathrm{ei}\times}\end{aligned}$$

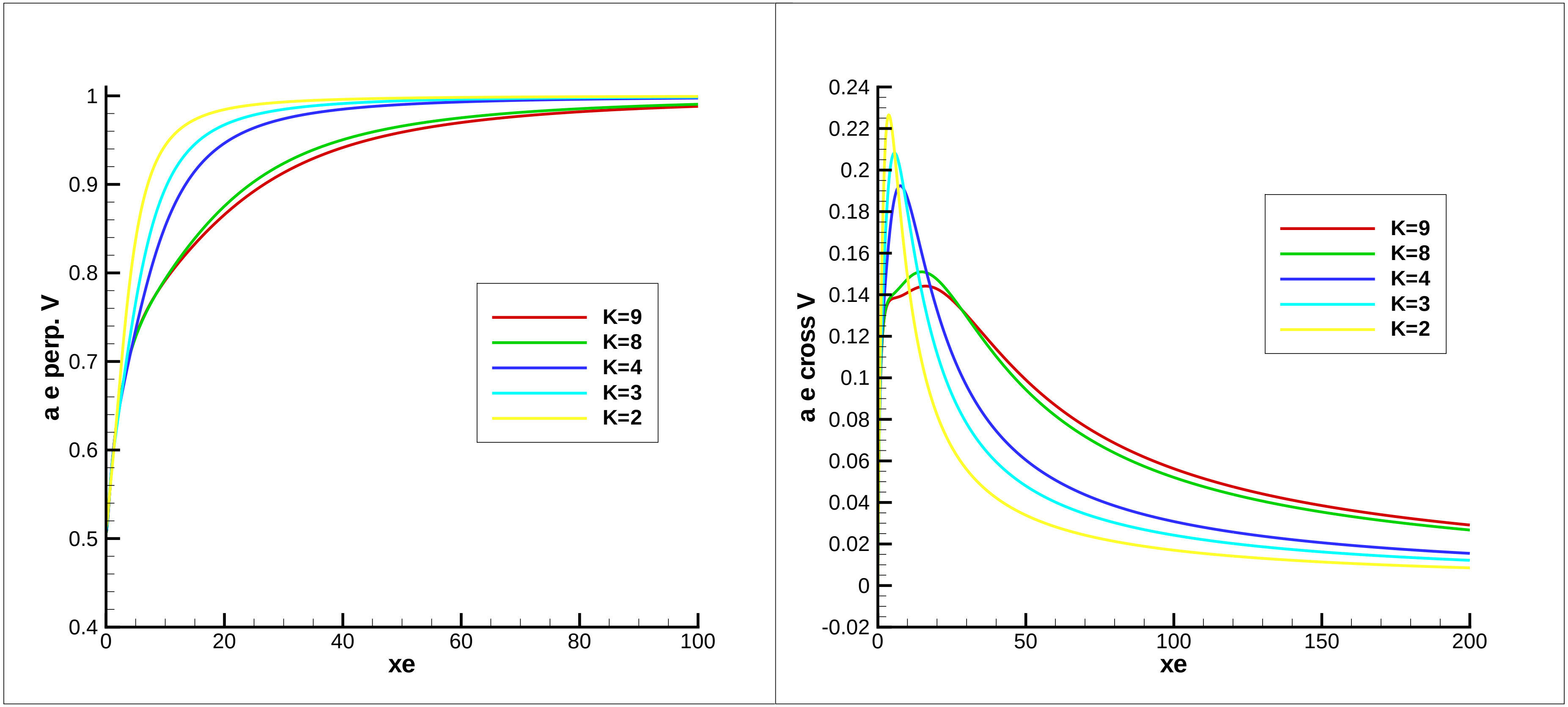
$$\kappa^{V}_{\mathrm{e}A} = n_{\mathrm{e}}T_{\mathrm{e}}\hat{\kappa}^{V}_{\mathrm{e}A} \quad (A = \parallel, \perp, \times)$$



Collisional friction (force density)

$$\begin{aligned}\mathbf{R}_{\mathrm{e}\perp} &= -\hat{\alpha}^{T}_{\mathrm{e}\|} n_{\mathrm{e}} \nabla_{\|} T_{\mathrm{e}} - \hat{\alpha}^{T}_{\mathrm{e}\perp} n_{\mathrm{e}} \nabla_{\perp} T_{\mathrm{e}} - \hat{\alpha}^{T}_{\mathrm{e}\times} n_{\mathrm{e}} \nabla_{\times} T_{\mathrm{e}} \\ & \quad -\hat{\alpha}^{V}_{\mathrm{e}\|} \frac{m_{\mathrm{e}} n_{\mathrm{e}}}{\tau_{\mathrm{ei}}} \mathbf{V}_{\mathrm{ei}\|} - \underline{\hat{\alpha}^{V}_{\mathrm{e}\perp}} \frac{m_{\mathrm{e}} n_{\mathrm{e}}}{\tau_{\mathrm{ei}}} \mathbf{V}_{\mathrm{ei}\perp} + \underline{\hat{\alpha}^{V}_{\mathrm{e}\times}} \frac{m_{\mathrm{e}} n_{\mathrm{e}}}{\tau_{\mathrm{ei}}} \mathbf{V}_{\mathrm{ei}\times}\end{aligned}$$

$$\hat{\alpha}^{T}_{\mathrm{e}A} = \hat{\kappa}^{V}_{\mathrm{e}A}$$

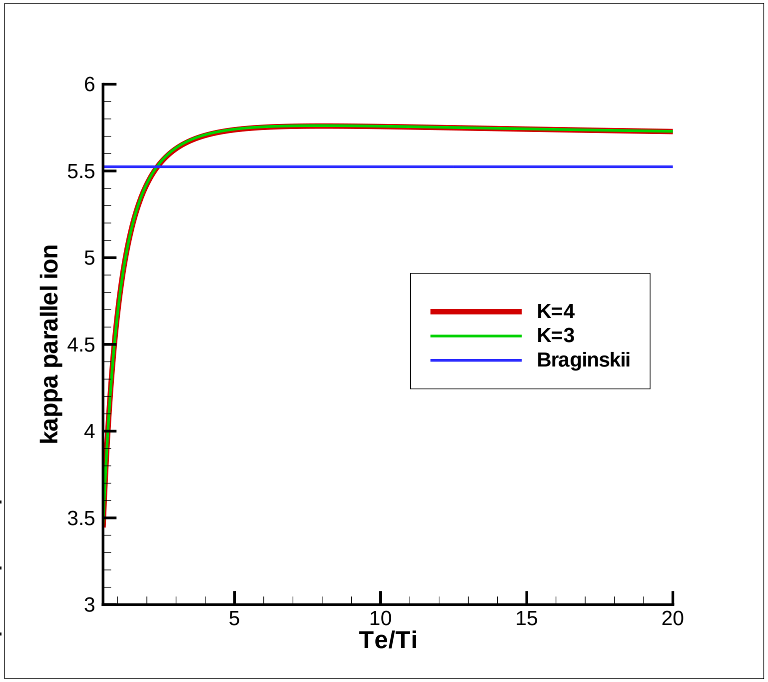


Heat flux for ions

- Heat flux from 2×2 calculation for $Z = 1,\ T_\mathrm{i} = T_\mathrm{e}$

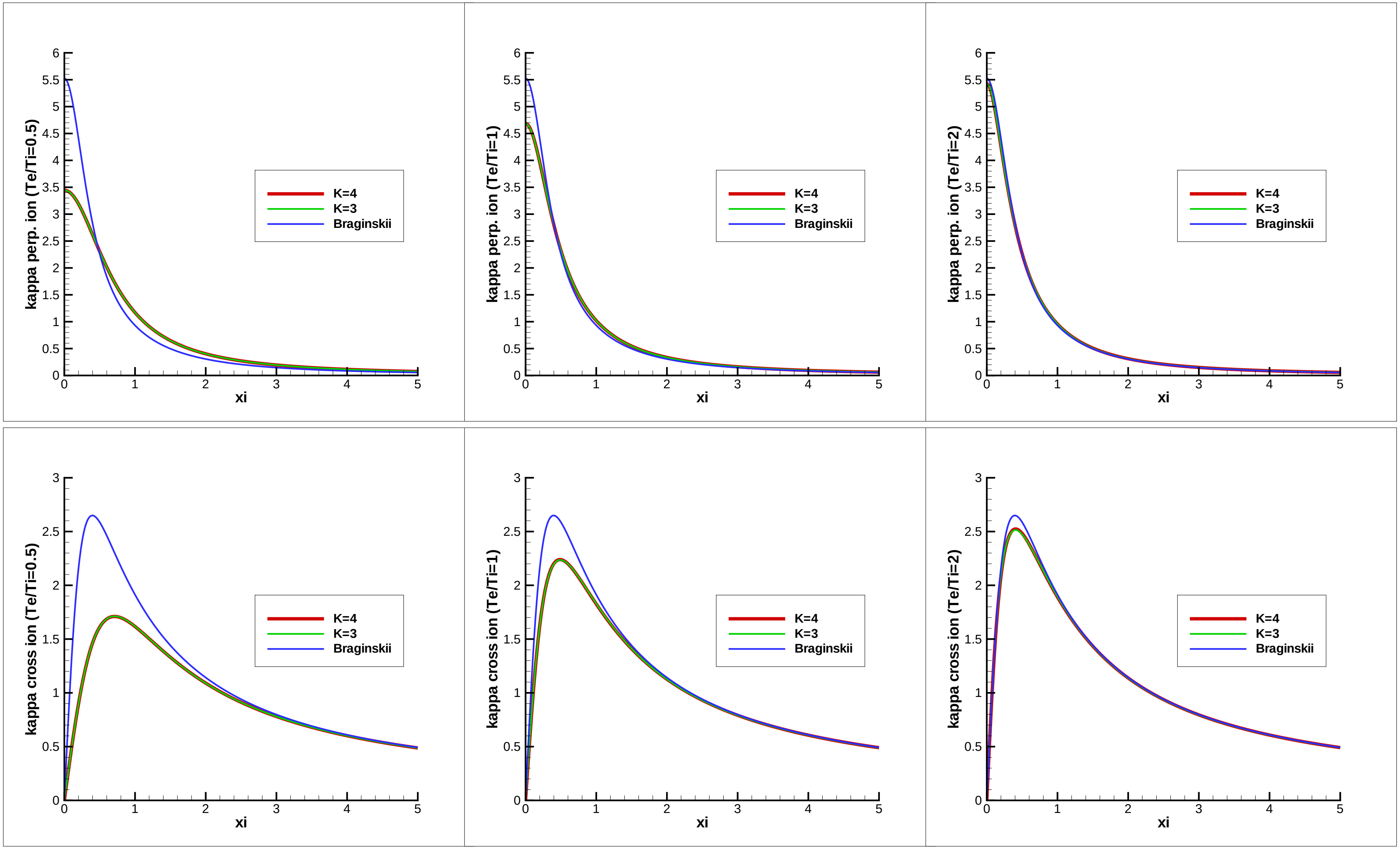
$$\mathbf{q}_\mathrm{i} = -\kappa_{\mathrm{i}\|}\nabla_\| T_\mathrm{i} - \kappa_{\mathrm{i}\perp}\nabla_{\mathrm{i}\perp} T + \kappa_{\mathrm{i}\times}\mathbf{b}\times\nabla T_\mathrm{i}$$

$$\kappa_{\mathrm{i}A} = \hat{\kappa}_{\mathrm{i}A}\frac{n_\mathrm{i} T_\mathrm{i} \tau_\mathrm{i}}{m_\mathrm{i}} \quad (A = \|, \perp, \times)$$



	Braginskii	$C_\mathrm{ii} + C_\mathrm{ie}$
$\hat{\kappa}_\|$	3.906	3.302 (18%)
$\hat{\kappa}_\perp$	$(2x^2 + 2.64)/\Delta$	$(2.25x^2 + 3.98)/\Delta$
$\hat{\kappa}_\times$	$x(\frac{5}{2}x^2 + 4.65)/\Delta$	$x(\frac{5}{2}x^2 + 2.31)/\Delta$
Δ	$x^4 + 2.70x^2 + 0.677$	$x^4 + 3.32x^2 + 1.21$

Heat flux for ions

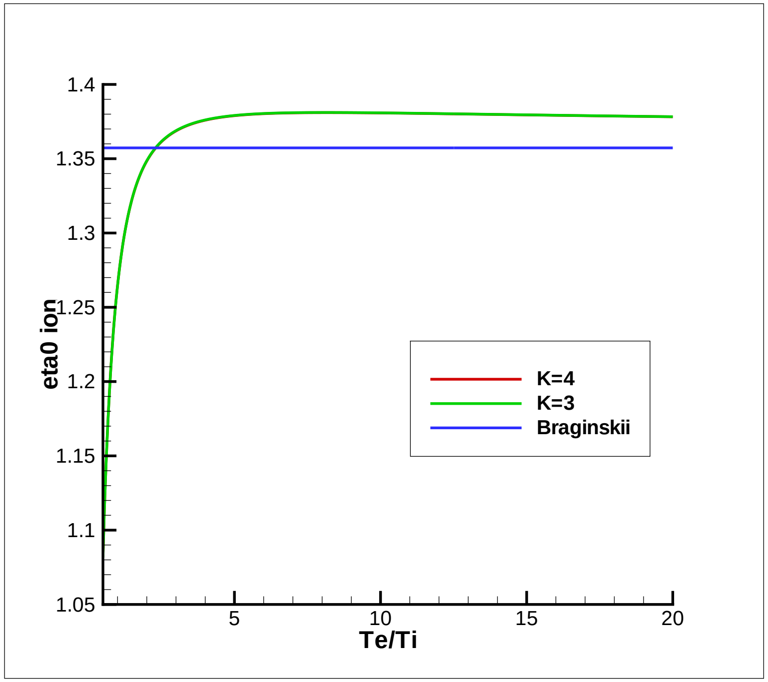


Viscous stress tensor for ions

2×2 calculation

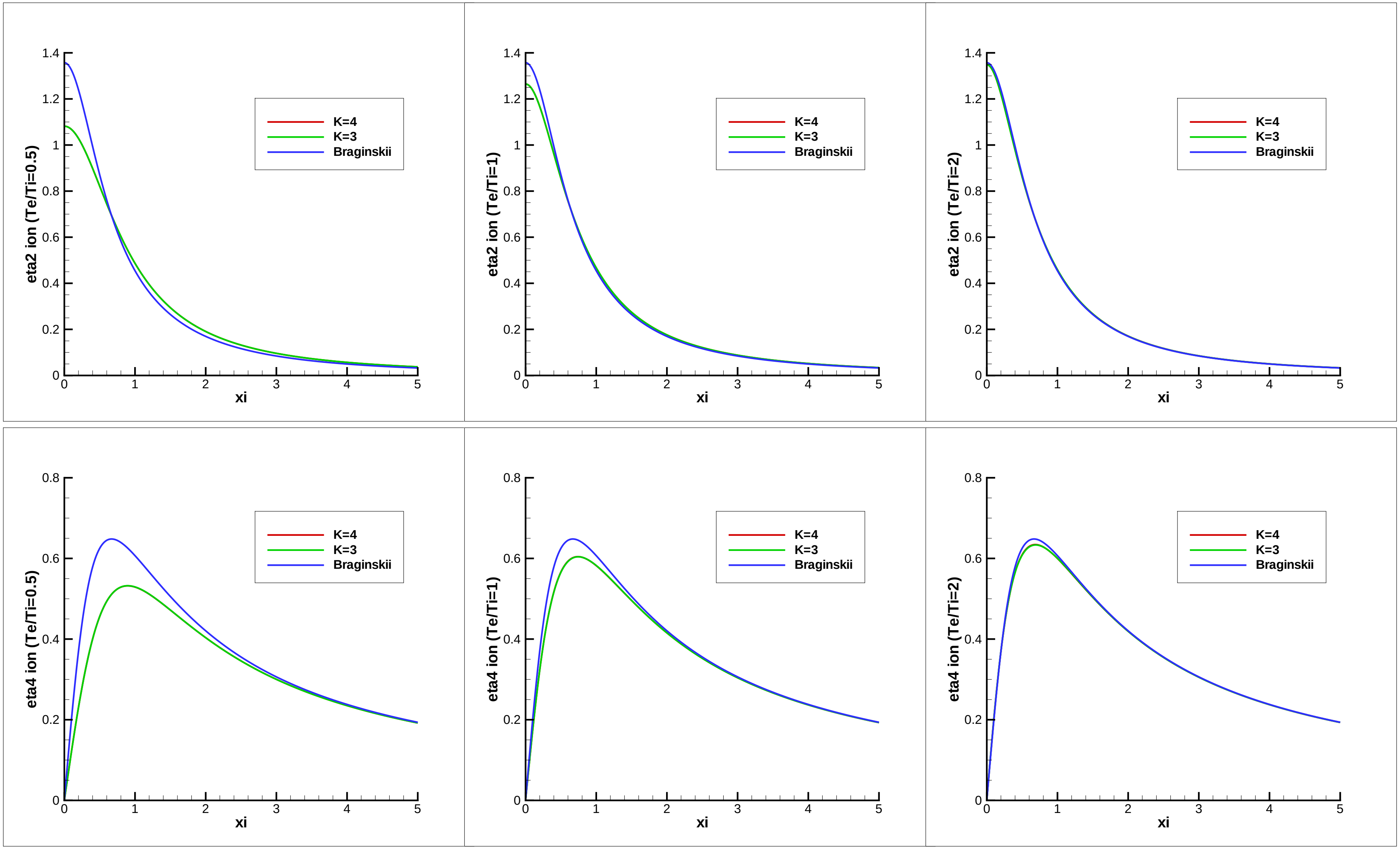
$$\boldsymbol{\pi}_{\mathrm{i}} = \eta_0 \mathsf{W}_0 - \eta_1 \mathsf{W}_1 - \eta_2 \mathsf{W}_2 + \eta_3 \mathsf{W}_3 + \eta_4 \mathsf{W}_4$$

$$\eta_A = \hat{\eta}_A n_{\mathrm{i}} T_{\mathrm{i}} \tau_{\mathrm{i}} \quad (A = 0, 1, 2, 3, 4)$$



	Braginskii	$C_{\mathrm{ii}} + C_{\mathrm{ie}}$
$\hat{\eta}_0$	0.96	0.89 (7.9%)
$\hat{\eta}_2$	$(\frac{6}{5}x^2 + 2.23)/\Delta$	$(1.27x^2 + 2.86)/\Delta$
$\hat{\eta}_4$	$x(x^2 + 2.38)/\Delta$	$x(x^2 + 2.78)/\Delta$
Δ	$x^4 + 4.03x^2 + 2.33$	$x^4 + 4.61x^2 + 3.20$

Viscous stress tensor for ions



Solving moment equations for general collisionality in a strong magnetic field

- $\delta \equiv 1/x = \tau^{-1}/\Omega \ll 1$

$$\mathsf{n} = \mathsf{n}^{(0)} + \delta \mathsf{n}^{(1)} + \cdots$$

- δ^{-1} order

$$\mathbf{b} \wedge \mathsf{n}^{(0)} = 0$$

- δ^{0} order

$$D\mathsf{n}^{(0)} + \mathbf{b} \wedge \mathsf{n}^{(1)} = C\mathsf{n}^{(0)} + \mathsf{g}$$

  - Parallel components: integral closures

$$\partial_L \mathsf{n}_{\parallel}^{(0)} = C\mathsf{n}_{\parallel}^{(0)} + \mathsf{g}_{\parallel}$$

  - Perpendicular components: find $\mathsf{n}^{(1)}$ using geometric method

$$\mathbf{b} \wedge \mathsf{n}^{(1)} = -D\mathsf{n}^{(0)} + C\mathsf{n}^{(0)} + \mathsf{g}$$

Summary

- High collisionality in a general magnetic field
  - Electron vector transport coefficients
    - 3×3 calculation is good enough for $\kappa_{\mathrm{e}A}^{T}$
    - $\kappa_{\mathrm{e}A}^{V}$ and collisional friction : $\Omega\tau$-dependent : 8×8 calculation
  - Ion vector and tensor transport coefficients
    - C_{ie} effect
    - Temperature dependent: $T_{\mathrm{e}}/T_{\mathrm{i}} \sim .5 - 1$ $\Rightarrow$significant correction

- General collisionality in a strong magnetic field
  - Parallel moments $\Leftarrow$ integral closures
  - Perpendicular moments $\Leftarrow$ geometric method
  - Include neoclassical closures