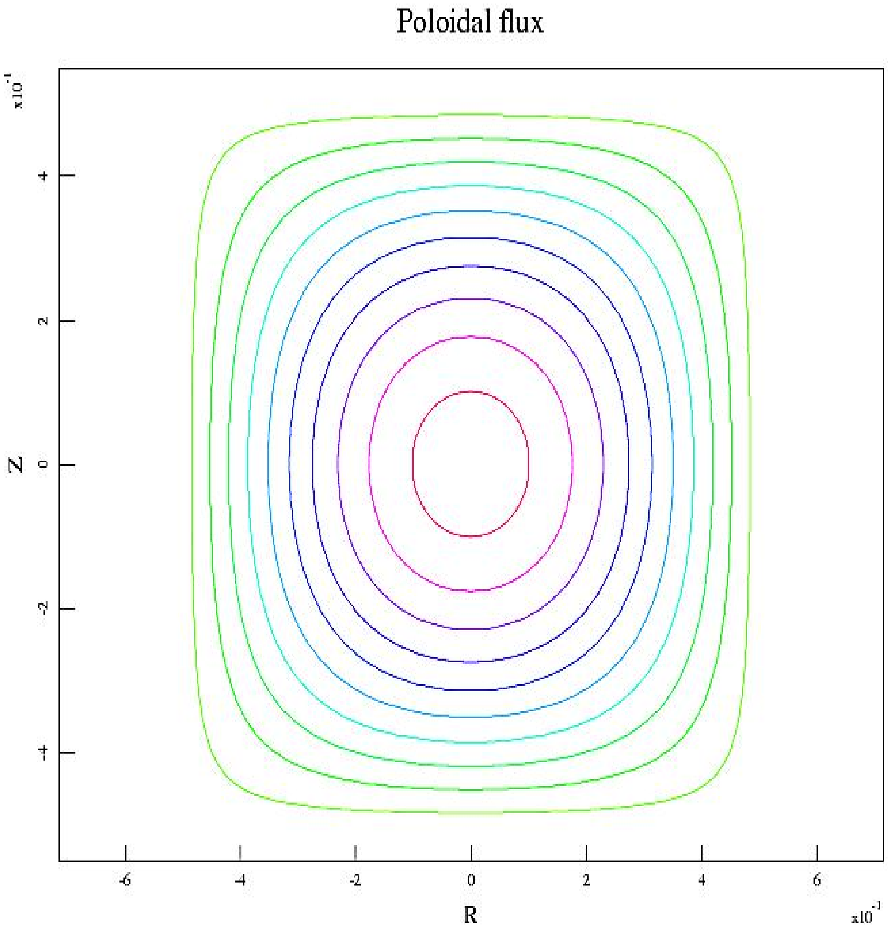
A Mixed Finite-Element Method for Parallel Heat Conduction

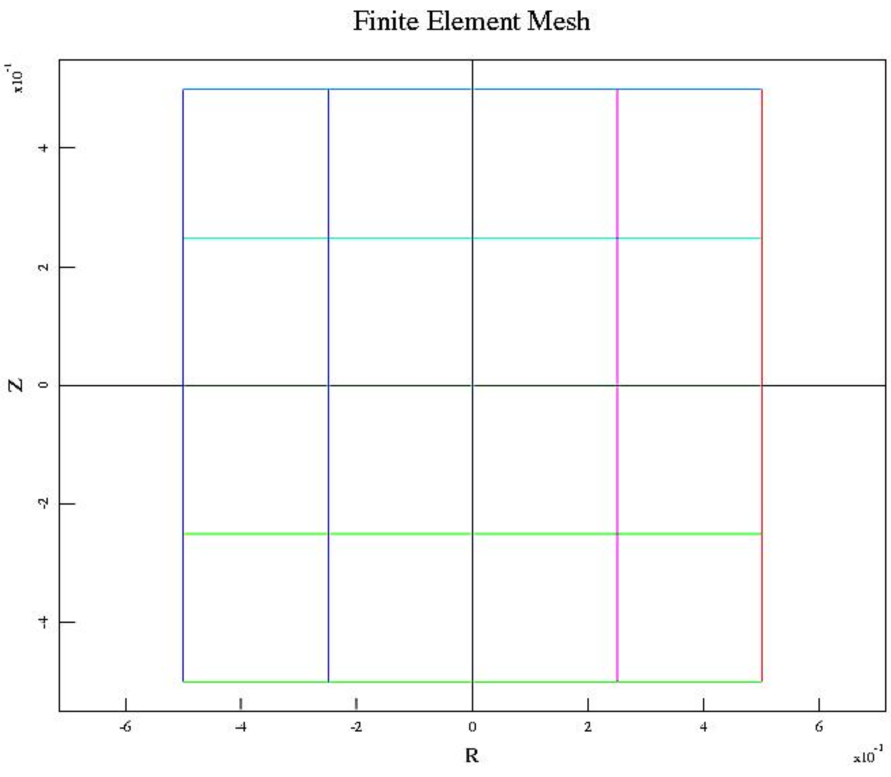
with Carl Sovinec and Jeong-Young Ji

NIMROD Team Meeting
3/29/08
Boulder, CO

Higher-order finite elements have had success treating highly anisotropic thermal conduction.

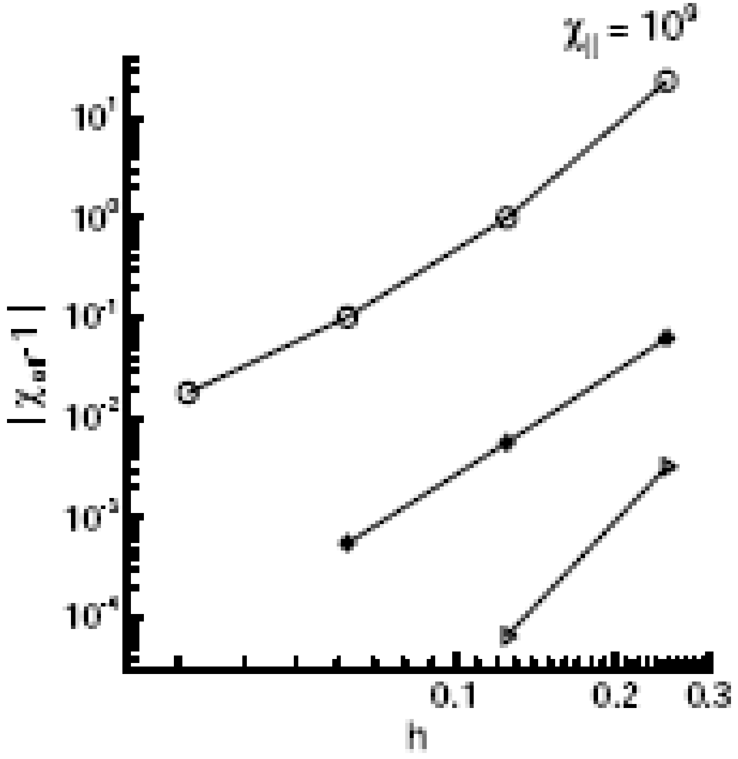
- Magnetic field and grid need not be aligned.
- Accuracy tested with problem that has flux-function heat source in rectangular geometry.



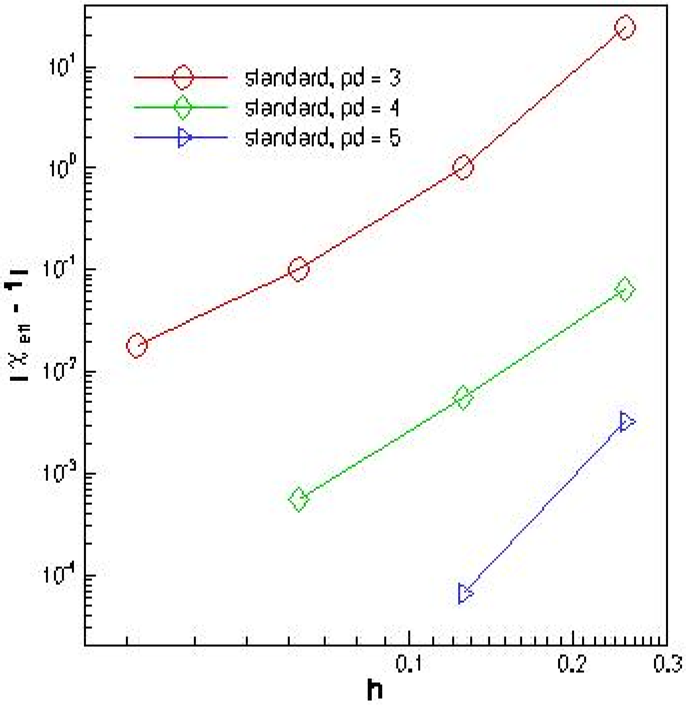


Spatial convergence with increasing polynomial degree at $\kappa_{||} = 10^9$.

- Reproduce results of Sovinec, *et al.*, JCP, 2004



JCP result

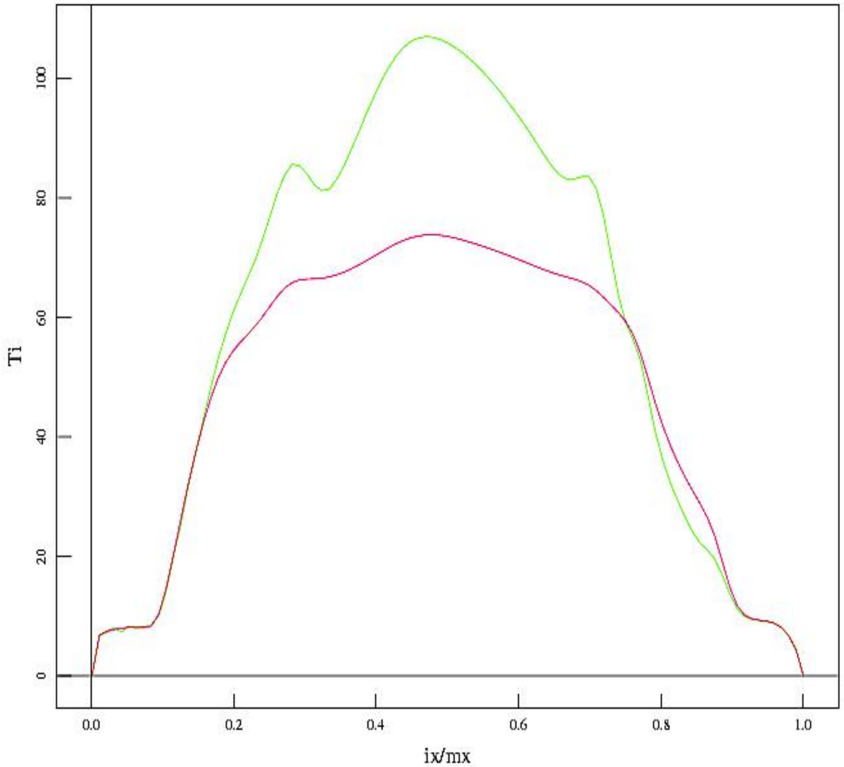


recent result with nimdevel

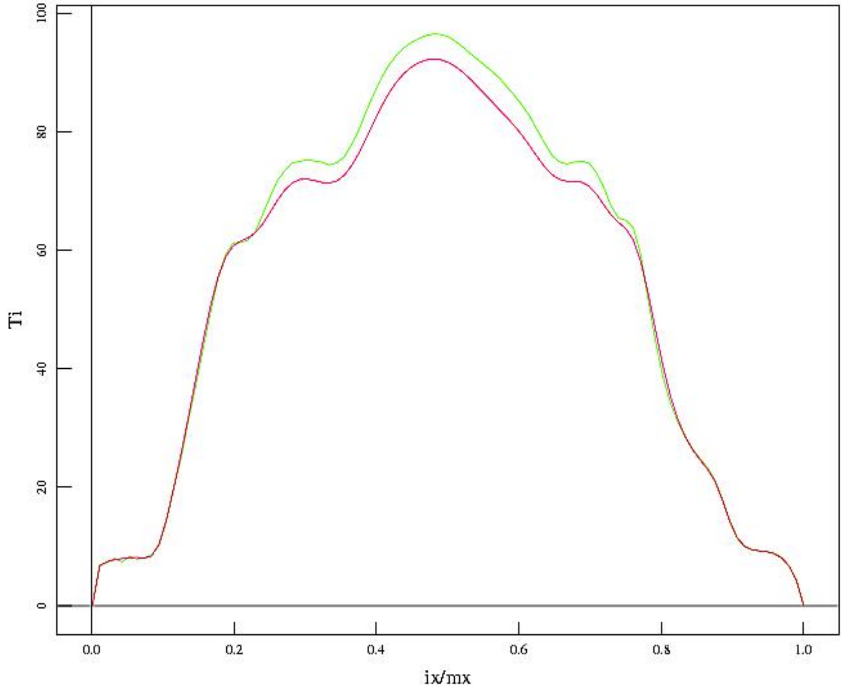
Simulations with stochastic fields may lack spatial resolution, however.

- SSPX heat confinement studies need ~ 20 Fourier modes and polynomial degree ~ 4 or 5. This is prohibitive for integral closures.

6 Fourier modes, polynomial degree = 4



22 Fourier modes, polynomial degree = 4



For ease, consider temperature evolution due to anisotropic conduction only.

- Simplified temperature equation is:

$$\frac{3}{2} n \frac{\partial T}{\partial t} = -\vec{\nabla} \cdot \vec{q}$$

$$= \vec{\nabla} \cdot \left[\kappa_{\perp} \vec{\nabla} T + \left(\kappa_{\parallel} - \kappa_{\perp} \right) \hat{b} \hat{b} \cdot \vec{\nabla} T \right].$$

- In NIMROD, T is expanded using C^0 finite-element basis functions, which are appropriate when solving weak form of problem.

To convert to weak formulation, apply finite-element test function and integrate parts.

- Expand $T = \Sigma_i T_i \alpha_i$, multiply by test function, α_j, and integrate by parts:

$$-\int_R dV \vec{\nabla}\alpha_j \cdot \left[\kappa_\perp \vec{\nabla}\alpha_i + \left(\kappa_\| - \kappa_\perp\right)\hat{b}\hat{b}\cdot\vec{\nabla}\alpha_i\right]$$
$$+\int_{\partial R} d\vec{S}\cdot\alpha_j\left[\kappa_\perp \vec{\nabla}\alpha_i + \left(\kappa_\| - \kappa_\perp\right)\hat{b}\hat{b}\cdot\vec{\nabla}\alpha_i\right]$$

- Formulation allows for C^0 finite-elements and is amenable to robust conjugate gradient solvers.

Mixed finite-element method (MFEM) treats $\mathbf{q}_{||}$ like fundamental variables (n, **V**, T and **B**).

- Define parallel conduction type term in T evolution:

$$\frac{3}{2} n \frac{\partial T}{\partial t} = \vec{\nabla} \cdot \left[\kappa_{\perp} \vec{\nabla} T - q_{||} \hat{b} \right],$$

$$q_{||} = -\left(\kappa_{||} - \kappa_{\perp} \right) \hat{b} \cdot \vec{\nabla} T$$

- Expand $q_{||} = \Sigma_i q_{||i} \alpha_i$ (like T) and solve expanded system for T and $q_{||}$ simultaneously.
- Leads to non-Hermitian matrices which require non-symmetric solvers.

Scaling variables helps solver.

- Re-define parallel conduction term:

$$\frac{3}{2} n \frac{\partial T}{\partial t} = \vec{\nabla} \cdot \left[\kappa_{\perp} \vec{\nabla} T - n_0 \kappa_0^{1/4} q_{\parallel} \hat{b} \right],$$

$$\frac{n_0^2}{\kappa_0^{1/2} \langle \kappa_{\parallel} - \kappa_{\perp} \rangle} q_{\parallel} + n_0 \kappa_0^{1/4} \hat{b} \cdot \vec{\nabla} T = 0,$$

where n_0 and κ_0 are suitable averages and $\langle \kappa_{\parallel} - \kappa_{\perp} \rangle = (\kappa_{\parallel} - \kappa_{\perp}) / \kappa_0$.

Integrate conduction terms in T equation by parts but leave $q_{\|}$ equation in strong form.

- Ignoring surface terms yields coupled system:

$$\int dV \left[\alpha_j \frac{3}{2} n \frac{\partial T}{\partial t} + \vec{\nabla} \alpha_j \cdot \left(\kappa_{\perp} \vec{\nabla} T - n_0 \kappa_0^{1/4} q_{\|} \hat{b} \right) \right] = 0,$$

$$\int dV \, \alpha_j [\{ \frac{n_0^2}{\kappa_0^{1/2} \langle \kappa_{\|} - \kappa_{\perp} \rangle} q_{\|} + n_0 \kappa_0^{1/4} \hat{b} \cdot \vec{\nabla} T \}] = 0,$$

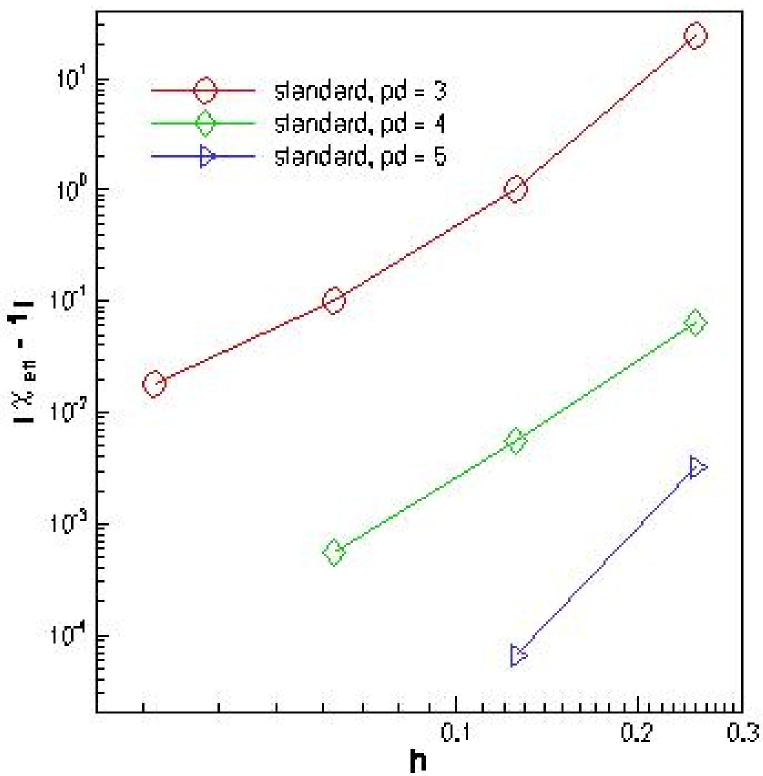
- Note $q_{\|}$ is undifferentiated

In lieu of iterative solver with 3-D preconditioning, compute full anisotropic conduction matrix and send to SuperLU.

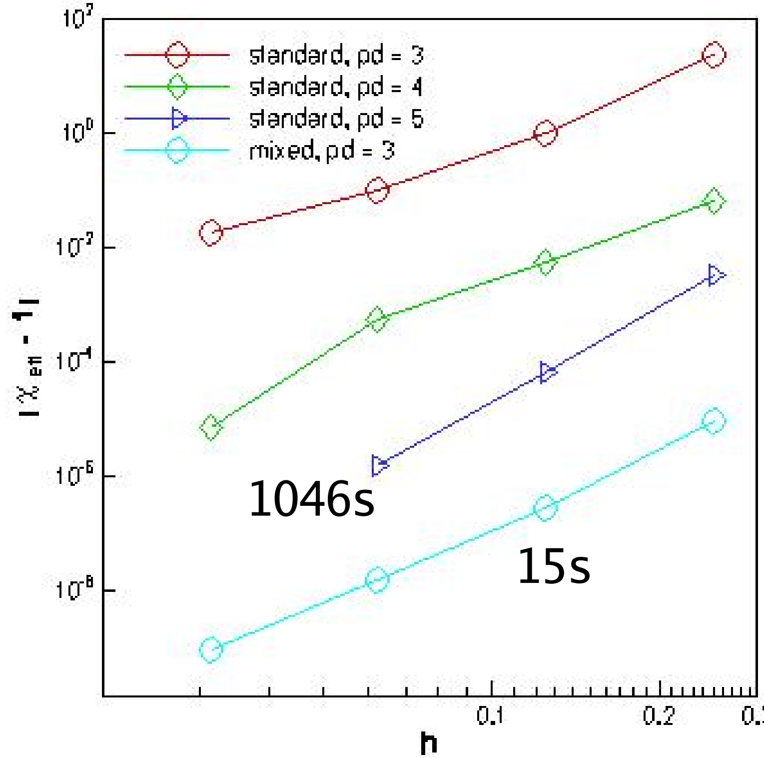
- Anisotropic conduction split from remainder of T advance.
- First advance T without conduction then perform split step with conduction terms only.
- t_aniso_op_full routine written to compute full anisotropic conduction matrix.
- outer loop of routine cycles over Fourier index of test function
- decent performance on smaller problems but not a viable route to improved scaling efficiency

Apply to JCP anisotropic conduction test problem.

- Error reduced considerably with MFEM method (bottom curve on right plot).



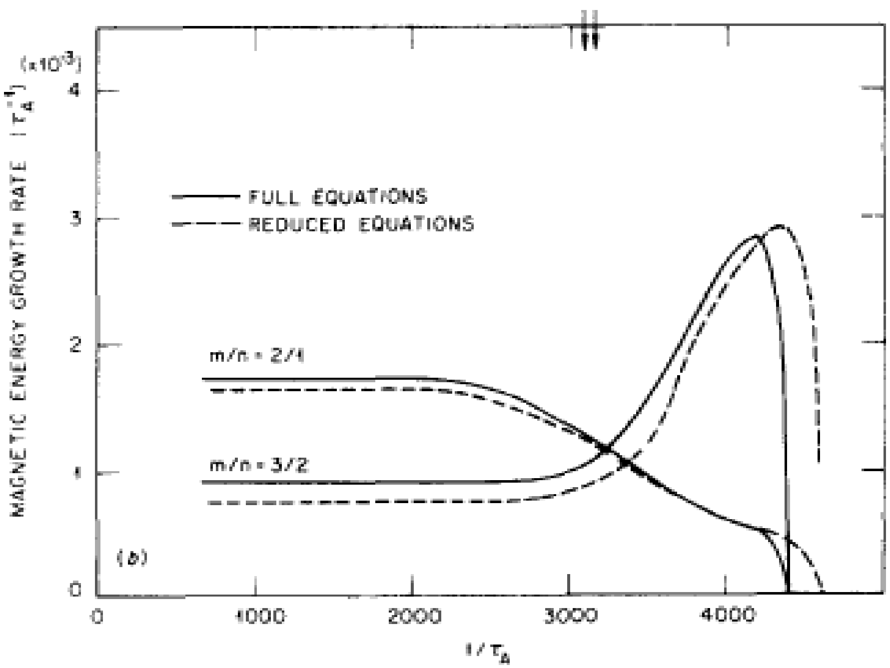
previous result

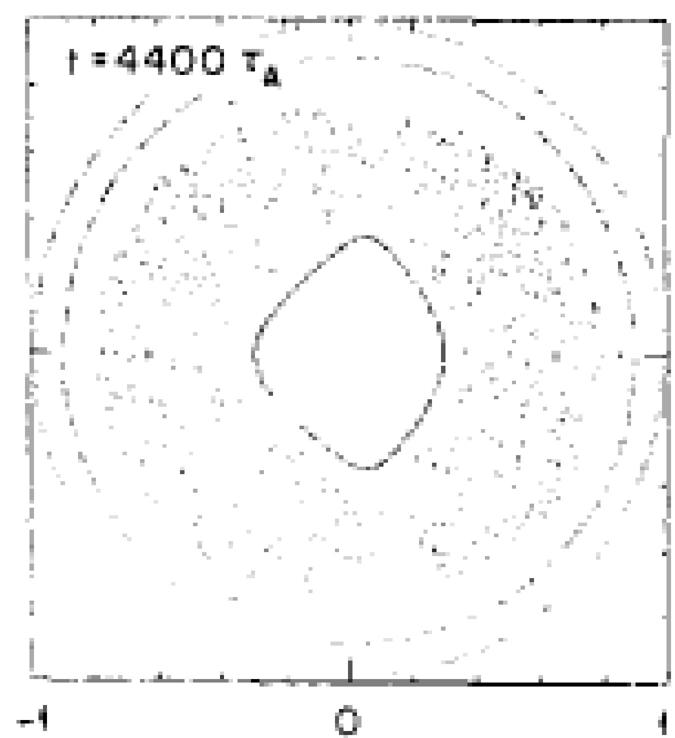


MFEM result (bottom curve)

Apply to cylindrical tearing mode case as additional test of MFEM.

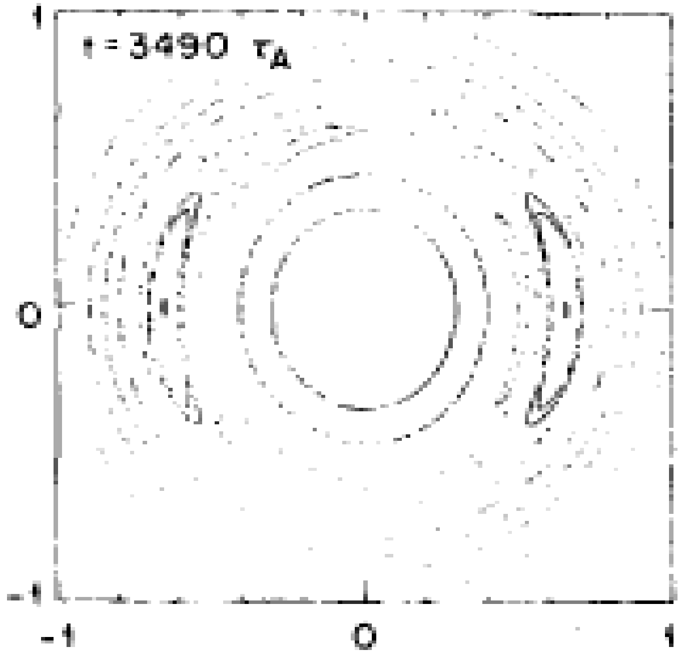
- Coupled 2/1 and 3/2 islands interact leading to stochasticity (Holmes, *et al*. 1983 Phys Fluids).

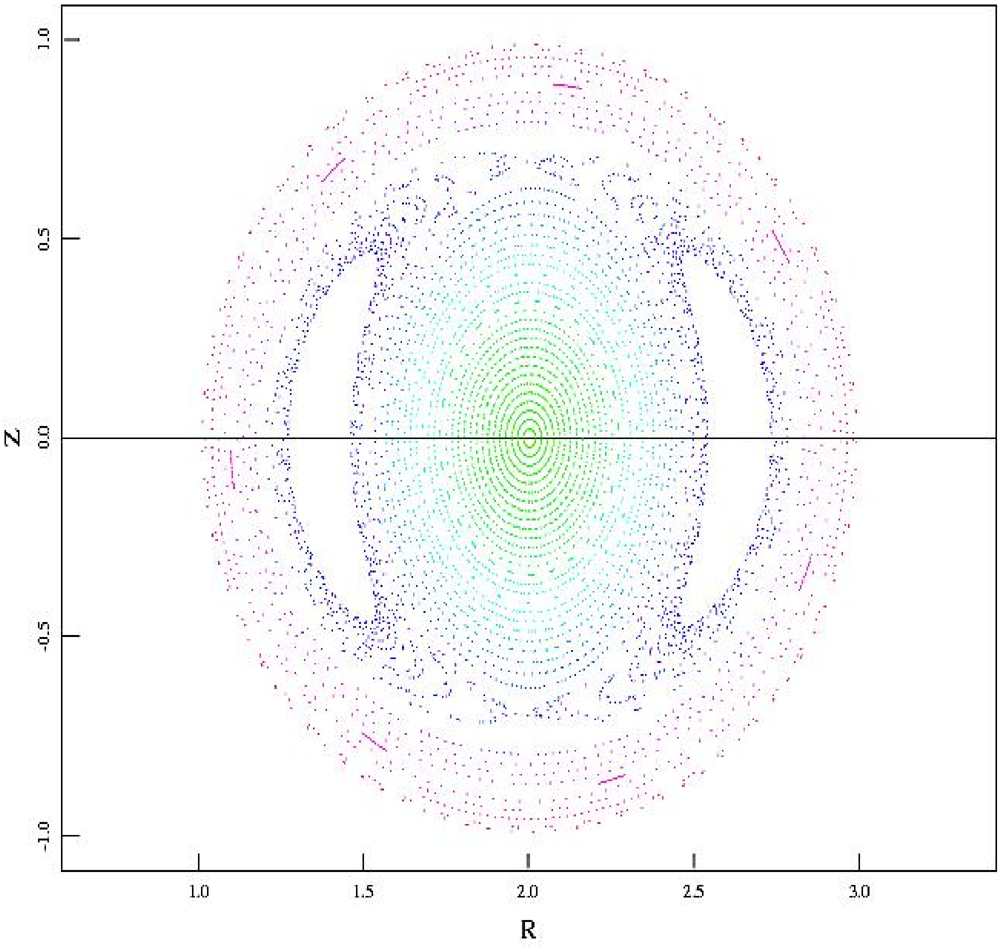




Preliminary results on cylindrical tearing mode.

- Reproduce linear growth rates of 2/1 and 3/2.
- Nonlinear results differ.





Future work.

- Develop effective preconditioning in Fourier direction.
  - compute diagonal in finite-element basis index matrices and apply in preconditioning step.
- Revisit SSPX calculations with MFEM.
- Employ MFEM in conjunction with integral heat flow closures to provide more accuracy in semi-implicit operator for anisotropic conduction.
- Apply 3-D iterative solves in continuum solution of CEL-DKE and/or higher order moment equations.