AN UPGRADE OF THE UNIFIED FLUID/KINETIC MODELS OF MAGNETIZED PLASMAS*

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Abstract

A method for upgrading the three fundamental plasma fluid evolution equations with more physically accurate kinetic closures is presented. The resulting unified fluid/kinetic equations for three perturbed plasma fluid quantities (density, parallel flow and temperature), developed in a sheared slab geometry, are off-shoots of the previous work by Chang and Callen[1]. This hybrid approach to plasma kinetic modeling effectively incorporates kinetic physics into fluid models and provides accurate description of micro-instabilities. The method is more robust, in assessing the behavior of plasma systems of arbitrary collisionality and adiabaticity, than several regime-dedicated models (such as MHD). The generalized kinetic closures (based on the CEL ansatz) are formulated as in the work of Chang and Callen (in which the the plasma drift kinetic equation (DKE) is Fourier-transformed, in time and space). Held's[2] work on generalized closures; and Ji and Held's work on exact linearized Coulomb collision operators[3] also provided the theoretical basis and justification for this work. Two specific aspects of the fluid/kinetic model were overhauled to improve accuracy and physical responsiveness. First, the approximate Lorentz scattering operator, applied in Chang and Callen; and Held et al, is replaced by the exact linearized Coulomb collision operator. Second, the steady state assumption (in the DKE), applied in Held et al, is obviated, and the time-dependent term ($\partial F/\partial t$) is retained. The physical repercussion is that the kinetic distortion, F, solved for in the DKE, is time-dependent, and yields more accurate closure relations for the parallel ion stress and heat flow ($\boldsymbol{\pi}_\parallel$ and $\mathbf{q}_\parallel$), as well as other higher-order moments of F. In effect, the plasma evolution equations (for momentum and energy), closed with such upgraded closures, would capture more accurate kinetic physics in simulations, and also highlight many kinetic-based phenomena which are inaccessible through pure fluid models. The accuracy and efficacy of this upgraded model is tested and validated through the analysis and simulation of ion stress damping of ion acoustic modes. Preliminary results of these tests indicate that the upgraded model captures more kinetic physics, and is robust.

[1]**Z. Chang, J.D. Callen, Phys. Fluids B4 (5), 1167 (1992).**
[2]**E.D. Held, Phys. Plasmas 10, 4708 (2003).**
[3]**J.Y. Ji, E.D. Held, Phys. Plasmas, 13, 102103 (2006)**

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Motivation: Part 1

- **The physical predictions or results from the NIMROD simulation of sound wave damping, based on a steady-state (time-independent) parallel stress, indicate the following:-**

  (1) The damping rate (characterized by $\omega \sim \frac{\partial}{\partial t}$) is equally large compared to the free-streaming term ($k_{\parallel} v_T$) in the collisional limit.

  (2) The damping rate ($\omega \sim \frac{\partial}{\partial t}$) is larger compared to the collision term in the moderately to highly collisional limits.

Motivation (Part 2)

- **The observations, summarized in the table below, justifies the need to formulate a time-dependent parallel ion stress.**

NUMERICAL COMPARISON OF DKE TIME-SCALES (Using Dominant Frequencies)

T_i (eV)	Re(Omega)	$\gamma \sim$ Im(Omega)	$\nu_{free\_streaming}$	$\nu_{collision}$
1	5655.4	1.1	9796.4	38388284.2
10	17883.8	321.1	30978.8	1328832.1
30	30975.6	3425.3	53656.9	266283.1
50	39989.3	7848.9	69270.7	126036.9
100	56553.5	19726.6	97963.5	45654.4
248	89418.9	47644.4	154893.9	11915.5
500	126457.4	68900.1	219053.2	4310.6
1000	178837.7	97525.1	309787.9	1558.6

Objectives

- Re-examine the nature and applicability of the steady-state, generalized parallel ion stress tensor, $\pi_{\parallel}$, within the context of the simulation of basic plasma physics phenomena.

- Upgrade the steady-state, generalized parallel ion stress tensor, by formulating time-dependent closures (ion stress tensor and heat flow) that also incorporate a more physically accurate linearized Coulomb collision operator.

- Analytically study the structure and capabilities of the time-dependent closures (in terms of the transport properties and dissipative effects they generate in plasma models), in various regimes of collisionality or adiabaticity.

- Use NIMROD simulations of ion acoustic waves to compare the physical predictions from the time-dependent $\pi_{\parallel}$, steady-state generalized $\pi_{\parallel}$, and Braginskii-type $\pi_{\parallel}$.

Hybrid Kinetic/Fluid Plasma Models Rely on Kinetic Closures

- **Apply three fundamental conservation equations to the study of momentum diffusion and transport in plasmas.**

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot \left(n\vec{V} \right) = 0 \qquad (1)$$

$$mn\left(\frac{\partial \vec{V}}{\partial t} + \vec{V}.\vec{\nabla}\vec{V} \right) = en\left(E + \frac{1}{c}\vec{V} \times B \right) - \vec{\nabla}p - \vec{\nabla} \cdot \mathbf{\Pi} + \mathbf{R} \qquad (2)$$

$$\frac{3}{2}n\left(\frac{\partial T}{\partial t} + \vec{V}.\vec{\nabla}T \right) = -p\vec{\nabla} \cdot \vec{V} - \mathbf{\Pi} : \vec{\nabla}\vec{V} - \vec{\nabla} \cdot q + Q \qquad (3)$$

- **There is the need to provide kinetic closures for π and q. The ion stress ($\pi = \pi_{\|} + \pi_{gv}$) is emphasized in this work.**

Viscous dissipation is studied using various forms of π closure.

- **Three Braginskii[1] forms of π tensor (parallel, isotropic and kinematic) are investigated or considered in the studies:**

$$\Pi_{\parallel} = -\boldsymbol{mn}\,\nu_{\parallel}(\hat{b}.W.\hat{b})(\hat{b}\hat{b} - \frac{I}{3}) \qquad (4)$$

- where $W = W_{jk} = \frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} - \frac{2}{3}\delta_{jk}\vec{\nabla}\cdot\vec{v}$ is the rate of strain tensor.

$$\mathbf{\Pi}_{iso} = -mn\nu_{iso}(\vec{\nabla}\vec{V} + (\vec{\nabla}\vec{V})^T - \frac{2}{3}\vec{\nabla}\cdot\vec{V}) \qquad (5)$$

$$\mathbf{\Pi}_{kin} = -mn\nu_{kin}\left(\vec{\nabla}\vec{V}\right) \qquad (6)$$

- [1]**Braginskii, S.I., Transport Processes in a Plasma, (Consultants Bureau, New York, edited by** M.A. Leontovich, 1, 1965).

Momentum diffusion is anisotropic in magnetized plasmas.

- **Consider $\hat{b}\hat{b}$ (zz), xx and yy components of Braginskii stress:**

$$P_{zz} = -\frac{\eta_0}{3}\left(2\frac{\partial V_z}{\partial z} - \frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y}\right),$$

and

$$P_{xx} = P_{yy} = -\frac{\eta_0}{3}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} - 2\frac{\partial V_z}{\partial z}\right),$$

where $\eta_0 = 0.96 n T_i \tau_i$.

- **Here, the magnetic field is assumed to be pointing in the Z direction.**

The steady-state $\pi_{\parallel}$ is derived from the steady-state DKE.

- **The Chapman-Enskog-like (CEL) Drift Kinetic Equation (DKE), is the basis for the derivation of the generalized $\pi_{\parallel}$:-**

$$\frac{\partial F}{\partial t}+\vec{V}_{\parallel}.\nabla F-C(F+f_m)=\frac{-m}{t_0}(\hat{b}\hat{b}-\frac{\bar{\bar{I}}}{3}):\nabla_{\parallel}\vec{u}_{\parallel}v^2P_2(\xi)f_m^{eq}$$

$$+vP_1(\xi)(\hat{b}.\nabla.\bar{\bar{\pi}}_{\parallel}-R_{\parallel})\frac{f_m^{eq}}{p_0}$$

- **Apply few assumptions to the DKE; write F as a series expansion in Legendre polynomials; and apply orthogonality to obtain a system of linear equations to be solved for F.**

$$\underline{I}\vec{F}+\underline{A}\partial_L\vec{F}=\vec{g}$$

- **The solutions to the linear system is of the form:-**

$$F_j=w_{ji}\int^{L^1}dL\frac{(\underline{w}^{-1}\vec{g})_i}{|(\frac{v\gamma_i}{\bar{v}})|}e^{-\frac{\bar{v}}{v\gamma_i}(L^1-L)}$$

- **The generalized $\pi_{\parallel}$ is then defined as:-**

$$\pi_{\parallel}=m\int d^3v\,v^2P_2(\xi)F=m\int d^3v\,v^2P_2^2F_2;$$ **or**

$$\pi_{\parallel}=m\int d^3v\,v^2P_2^2\left(w_{2i}\int^{L^1}dL\frac{(\underline{w}^{-1}\vec{g})_i}{|(k_{\parallel i}^{-1})|}e^{-k_{\parallel i}(L^1-L)}\right).$$

Generalized parallel stress is embedded in an integral equation.

- The implicit form of the integral (nonlocal) generalized $\pi_{\|}$ is:-

$$\begin{aligned}
\mathbf{K}_{11}(U_{\|}) &+ \mathbf{K}_{12}(\pi_{\|}) = \\
\int_0^{\infty} & d\bar{L}\left(u_{\|}(L+\bar{L}) + u_{\|}(L-\bar{L})\right)\frac{\partial K_1(\bar{L})}{\partial \bar{L}} + B_1 u_{\|}(L), \\
\mathbf{K}_{21}(U_{\|}) &+ (1+B_2)\pi_{\|} + \mathbf{K}_{22}(\pi_{\|}) = \\
\int_0^{\infty} & d\bar{L}\left(u_{\|}(L+\bar{L}) - u_{\|}(L-\bar{L})\right)\frac{\partial K_2(\bar{L})}{\partial \bar{L}},
\end{aligned}$$

- $\pi_{\|}$ appears on right side of the flow evolution equation:

$$\rho\frac{d\vec{V}}{dt} = \vec{J}\times\vec{B} - \vec{\nabla}p - \vec{\nabla}\cdot(\mathbf{\Pi}_{\|} + \mathbf{\Pi}_{gv}),$$

where $\mathbf{\Pi}_{\|} = (\hat{\mathbf{b}}\hat{\mathbf{b}} - \mathbf{I}/3)\pi_{\|}$

Detailed form of the integral equation embedding the $\pi_{\parallel}$.

- The integrals for the different limits of integration, imposed for positive and negative values of $k_{\parallel_i}$, are as follows:-

$$\pi_{\parallel}=\int_0^{\infty} ds\, s^4\left\{\int_{-\infty}^{L} dL'\left\{c_i^+(3)\left(\frac{16}{15\pi^{1/2}}\partial_L \pi_{\parallel}-\frac{3\,G(s)}{5a\,s^3}\bar{U}_{\parallel}\right)+c_i^+(4)\frac{16}{15\pi^{1/2}}s\,\partial_L \bar{u}_{\parallel}\right\} f_m^{eq}\, e^{-k_{\parallel_i}(L^1-L)}\right\}$$

$$+\int_0^{\infty} ds\, s^4\left\{\int_{L}^{\infty} dL'\left\{c_i^-(3)\left(\frac{16}{15\pi^{1/2}}\partial_L \pi_{\parallel}-\frac{3\,G(s)}{5a\,s^3}\bar{U}_{\parallel}\right)+c_i^-(4)\frac{16}{15\pi^{1/2}}s\,\partial_L \bar{u}_{\parallel}\right\} f_m^{eq}\, e^{-k_{\parallel_i}(L^1-L)}\right\}$$

and

$$U_{\parallel}=0=\int_0^{\infty} ds\, s^3\left\{\int_{-\infty}^{L} dL'\left\{c_i^+(1)\left(\frac{8}{9\pi^{1/2}}\partial_L \pi_{\parallel}-\frac{1}{2a}\frac{G(s)}{s^3}\bar{U}_{\parallel}\right)+c_i^+(2)\frac{8}{9\pi^{1/2}}s\,\partial_L \bar{u}_{\parallel}\right\} f_m^{eq}\, e^{-k_{\parallel_i}(L^1-L)}\right\}$$

$$+\int_0^{\infty} ds\, s^3\left\{\int_{L}^{\infty} dL'\left\{c_i^-(1)\left(\frac{16}{15\pi^{1/2}}\partial_L \pi_{\parallel}-\frac{1}{2a}\frac{G(s)}{s^3}\bar{U}_{\parallel}\right)+c_i^-(2)\frac{8}{9\pi^{1/2}}s\,\partial_L \bar{u}_{\parallel}\right\} f_m^{eq}\, e^{-k_{\parallel_i}(L^1-L)}\right\}$$

Where $\bar{u}_{\parallel}=n\,m\,v_{th}u_{\parallel}$; $\bar{U}_{\parallel}=n\,m\,v_{th}U_{\parallel}$ and $K_{\parallel_i}=\dfrac{\bar{\nu}}{v\gamma_i}$.

Basic Physics of Stress-induced Sound Wave Damping.

- Ignore advection and friction in momentum evolution:

$$\hat{\mathbf{b}}. \left(\rho \frac{\partial \vec{V}}{\partial t} = \left(\frac{1}{c} \vec{J} \times B \right) - \vec{\nabla} p - \vec{\nabla} \cdot \mathbf{\Pi} \right) \qquad (8)$$

- Write adiabatic equation and derive pressure evolution from it:

$$\frac{\partial p}{\partial t} = -\gamma \rho^{\mathbf{0}} \vec{\nabla} \cdot \vec{V}_{\|} \hat{\mathbf{b}} \qquad (9)$$

- Combine equations to obtain an approximation of damped sound waves:

$$\frac{\partial^2 V_{\|}}{\partial t^2} - \frac{(\gamma - 1)\, p_0}{m n_0} \nabla^2{}_{\|} V_{\|} = -\hat{b} . \nabla . \frac{\partial \pi_{\|}}{\partial t} \qquad (10)$$

In NIMROD: Stabilization of the integral stress term.

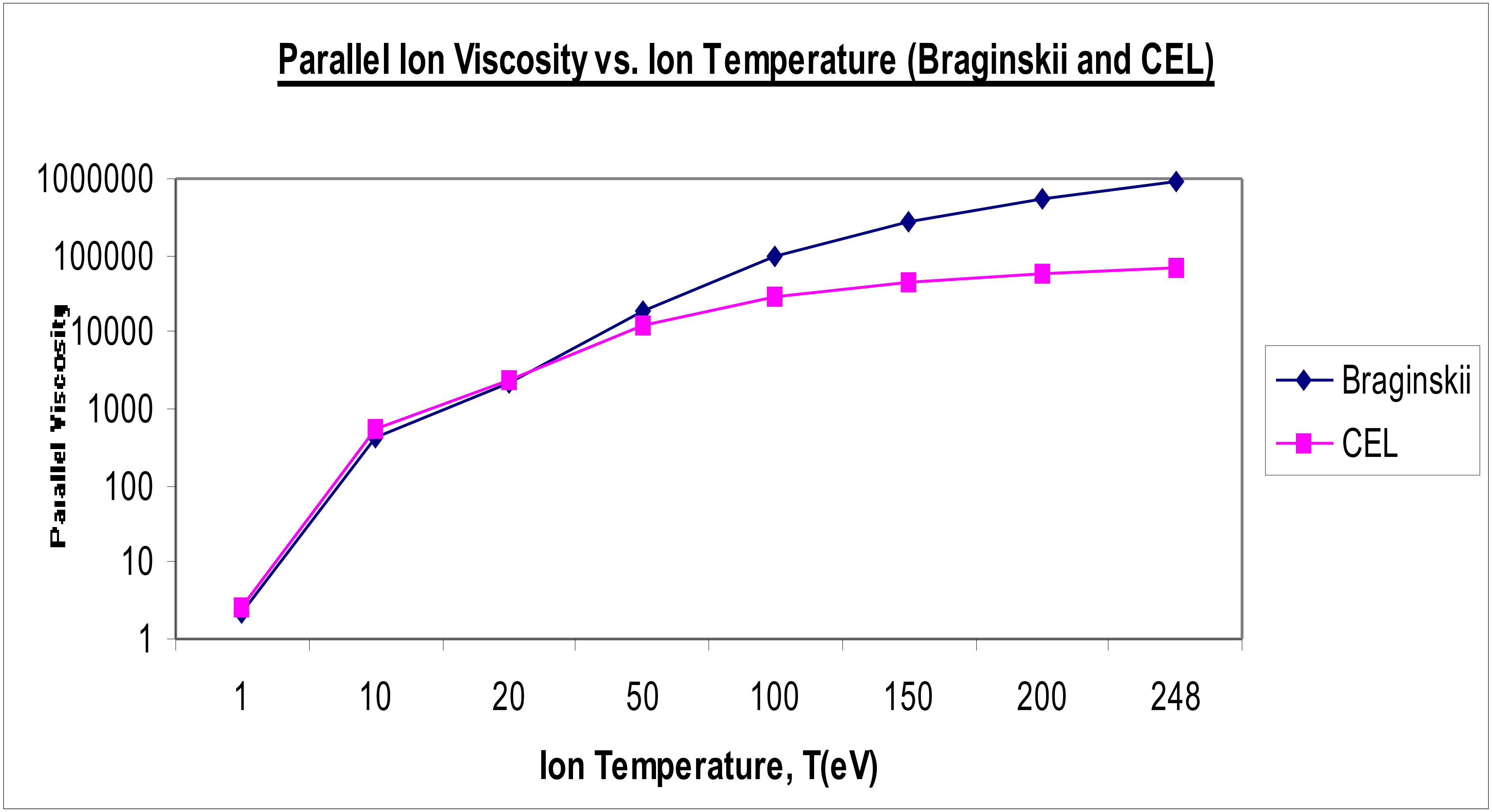
- NIMROD has used $\mathbf{\Pi} = -\rho\nu\vec{\nabla}\vec{V}$ in the past to introduce viscous dissipation.
- We now use isotropic form (ignoring gyroviscosity for now): $\mathbf{\Pi} = -\rho\nu(\vec{\nabla}\vec{V} + (\vec{\nabla}\vec{V})^T - \vec{\nabla}\cdot\vec{V}/3)$. .
- Semi-implicit operator is used to stabilize the evolution of the explicit stress term:

$$\left(\rho^n - \Delta t f \vec{\nabla}\cdot(\rho^n\nu\vec{\nabla} - \mathbf{\Pi}_{si})\right)\Delta\vec{V} = \Delta t\vec{\nabla}\cdot\left(\rho^n\nu\vec{\nabla}\vec{V}^n - \mathbf{\Pi}_{\|}\right)$$

where $\mathbf{\Pi}_{\|} = -\rho\nu_{\|}(\hat{\mathbf{b}}\hat{\mathbf{b}} - \mathbf{I}/3)\hat{\mathbf{b}}\hat{\mathbf{b}} : \vec{\nabla}\vec{V}$ and $\mathbf{\Pi}_{si}$ is analogous but acts on $\Delta\vec{V}$.

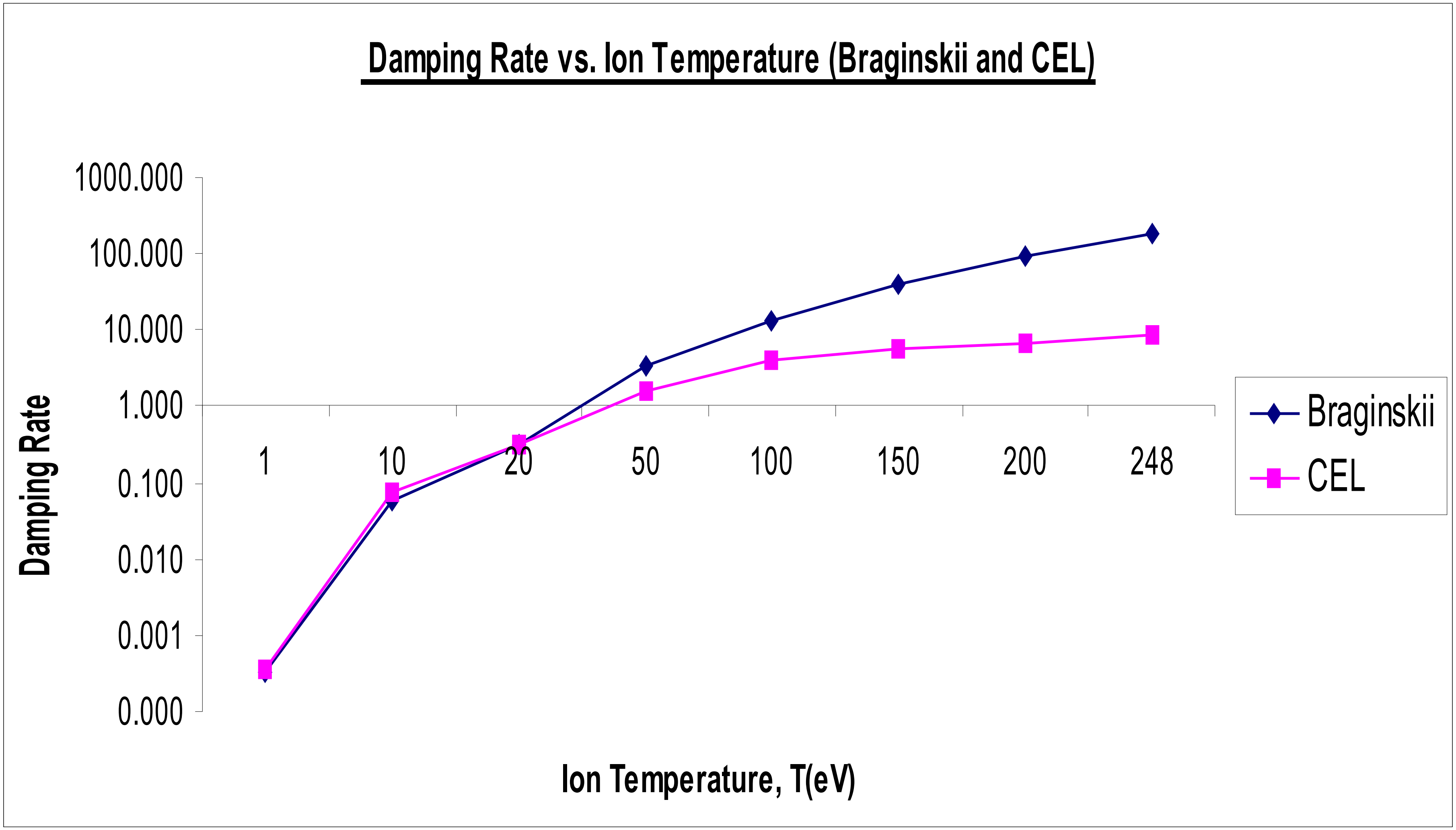
- For centering coefficient $f = 1$, this represents a fully implicit advance for anisotropic momentum diffusion.

Key results from the steady-state parallel stress simulations. (Part 1)



- **Generalized $\pi_{\|}$ reproduces parallel ion viscosity predicted by Braginskii-type $\pi_{\|}$ at low temperatures. It also produces well-behaved results at higher temperatures.**

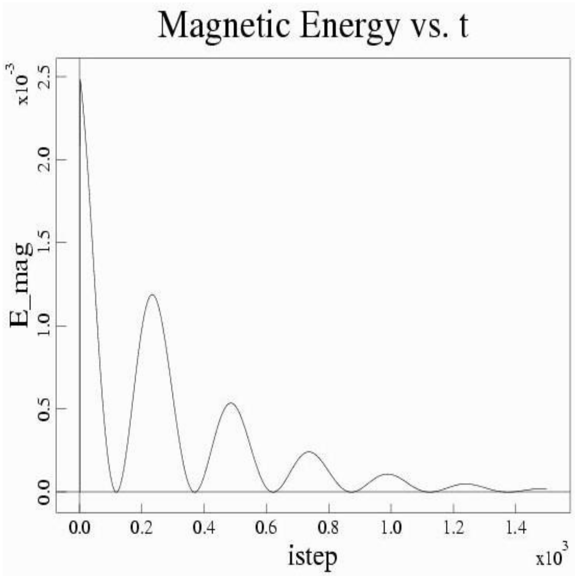
Key results from the steady-state parallel stress simulations. (Part 2)

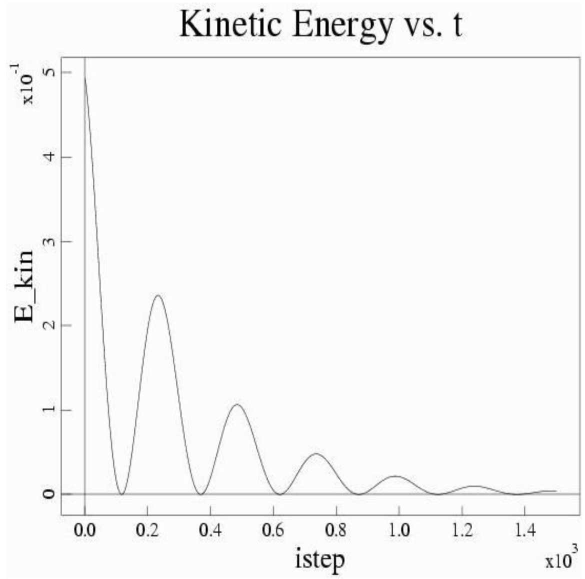


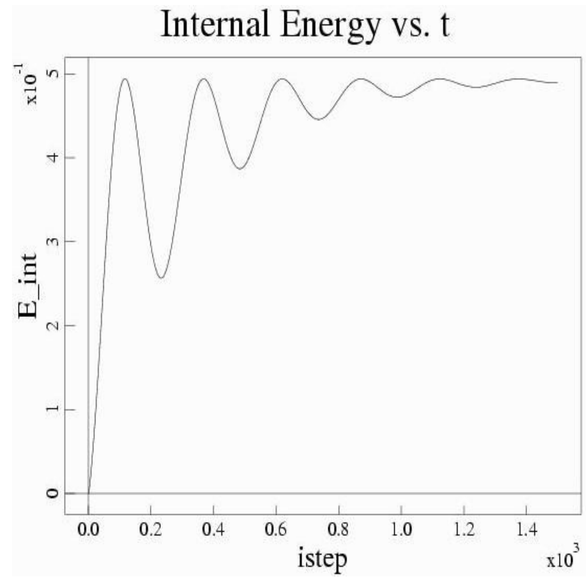
- **Generalized $\pi_{\|}$ reproduces plasma wave damping rates predicted by Braginskii-type $\pi_{\|}$ at low temperatures. It also produces well-behaved results at higher temperatures.**

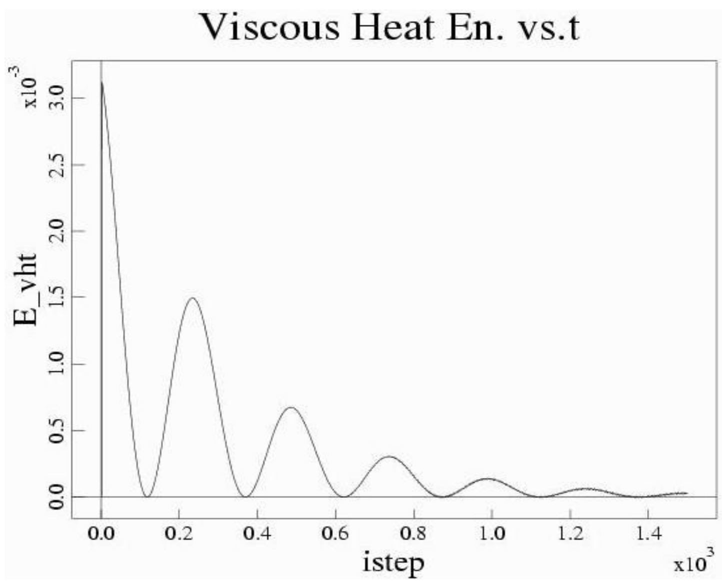
Key results from the steady-state parallel stress simulations. (Part 3)

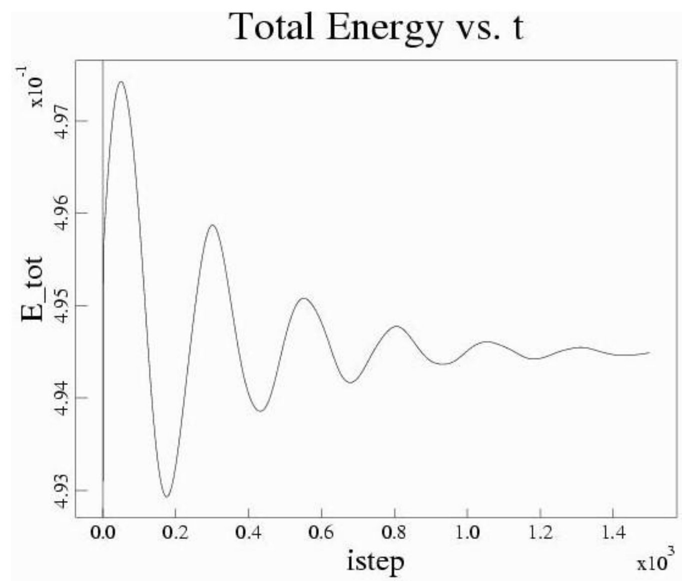
Plots of Energies Indicate that Total Energy is Conserved: There is Convergence as Time-Step Decreases.











- **Simulations, based on the generalized $\pi_{\parallel}$, correctly produce the associated stress-related (viscous) heating, for which energies satisfy the Energy Conservation Law.**

Previous Work of Interest: Time-Dependent Closures

- Recall Chang and Callen's[1] formulation of time-dependent closure relations for the perturbed $\pi_{\parallel}$ and $q_{\parallel}$.

- The CEL Drift Kinetic Equation, which was solved for the kinetic distortion (F), was Fourier-transformed in both space and time.

- To simplify the problem or analysis in the formulation, model collisional terms were introduced.

- Based on these closure relations, the real and imaginary parts of several dynamic pseudo-transport coefficients were calculated correctly.

- These pseudo-transport coefficients compare favorably with values from known models due to Waltz; Hammett & Perkins; and Lee & Diamond, in specified regimes of collisionality.

This Work: Time-dependent parallel ion stress tensor.

- **Three main feature upgrades are simultaneously applied to the physical model (DKE) used in deriving the steady-state, generalized $\pi_{\|}$:-**

  (1) the term embodying the time-dependent evolution of F in the DKE $(\partial F_1/\partial t)$ is retained, as done in Chang and Callen[1].

  (2) the simplified collision operator, consisting of model collisional terms introduced into the DKE in Chang and Callen, is replaced by the exact Linearized Coulomb Collision Operator (LCCO) in the moment expansion, formulated by Ji and Held[3].

  (3) heat flow and temperature gradient drive terms, ignored in the previous derivation of the stress tensor, are retained in the DKE.

- **The analytical methodology has two main features:-**

  (1) Similar to Chang and Callen, but time is Laplace-transformed.

  (2) Finite Method for integration involving the Landau propagator $\left(\frac{1}{\omega - k_{\|} v_{\|}}\right)$

Incorporation of Time-Dependence and the LCCO into the DKE

- The retention of the explicit, partial derivative of time in the DKE effects the time-dependence of the model. The complete form of the DKE must be solved:-

$$\frac{\partial \tilde{F}_1}{\partial t} + \vec{V}_{\parallel} . \nabla \tilde{F}_1 + \langle \frac{e}{m} \vec{E} . \nabla_{\vec{v}} \tilde{F}_1 \rangle = \langle C(f_{M_1} + \tilde{F}_1) \rangle + \langle CEL\, drives \rangle .$$

- The third term on the LHS of the DKE above is ordered small compared to all the other terms.

- The substitution below is made in the DKE in order to incorporate the (LCCO) into the physical model.

$$\langle C(f_{M_1} + \tilde{F}_1) \rangle_{\gamma} = \langle C(f_i^{lk}, f_i^{(0)}) \rangle_{\gamma} + \langle C(f_i^{(0)}, f_i^{lk}) \rangle_{\gamma} .$$

Brief overview of the Exact Linearized Coulomb Collision Operator (LCCO) : Part 1

- **In the exact LCCO line of thought, both the plasma particle distribution function and the particle collision operator (for any species, a) are cast into series expansion of Sonine polynomials, with dimensionless fluid moments as the coefficients of the expansion. Specifically, we have:-**

$$f_a = \sum_{lk} \frac{1}{\sigma_k^l} f_a^{lk}$$

And

$$C_{ab}^{(1)} = \sum_{lk} \frac{1}{\sigma_k^l} \left[\left\{ C_{ab}^{(lk,0)} + C_{ab}^{(0,lk)} \right\} \right]$$

where

$$f_a^{lk} = \frac{1}{\sigma_k^l} f_a^{(0)} P_a^{lk} . \hat{M}_a^{lk}$$

$$f_a^{(0)} = \frac{n}{\pi^{3/2} v_{Ta}^3} e^{-s_a^2}$$

$$\hat{M}_a^{lk} = v_{Ta}^{-(l+2\mathrm{k})} \left\{ M_a^{lk}(t, \boldsymbol{x}) = \frac{1}{n_a} \int d\vec{v}\, P^l(\boldsymbol{v})\, v_{Ta}^{2\mathrm{k}} L_k^{(l+1/2)} \right\}.$$

Brief overview of the Exact Linearized Coulomb Collision Operator (LCCO) : Part 2

- **The test particle operator and the field particle operator in the LCCO are, respectively:-**

$$C_{ab}^{(lk,0)} = f_a^{(0)} P^l(\hat{\boldsymbol{v}}) . \hat{M}_a^{lk} \nu_{ab}^{(lk,0)}$$

And

$$C_{ab}^{(0,lk)} = f_a^{(0)} P^l(\hat{\boldsymbol{v}}) . \hat{M}_b^{lk} \nu_{ab}^{(0,lk)} .$$

The definitions of the normalization constants $(\sigma_k^l = \sigma_l \lambda_k^l)$ **and the speed functions,** $\nu_{ab}^{(lk,0)}$ **and** $\nu_{ab}^{(0,lk)}$**, associated with** $P^l L_k^{(l+1/2)}$ **, are all provided in reference Ji and Held's paper.**

Details of the Time-Dependent $\pi_\parallel$ formulation: Part 1

The Fourier-transformed (in time) DKE is:-

$$(w-k_\parallel v_\parallel)\tilde{F}=-\frac{2}{3}k_\parallel v_\parallel f_m\frac{\tilde{\pi}_\parallel}{p}-[\frac{2}{3}(s^2-\frac{3}{2}k_\parallel v_t)]f_m\frac{\tilde{q}_\parallel}{p v_t}$$
$$+\frac{4}{3}s^2 P_l(\xi)k_\parallel u_\parallel f_m-i(s^2-\frac{5}{2})v_\parallel f_m\frac{(\nabla_\parallel T)_1}{T}$$
$$+i v_\parallel \tilde{R}_\parallel\frac{\tilde{f}_m}{p}+i\frac{2}{3}(s^2-\frac{3}{2})\tilde{Q}+\langle C(f_i)\rangle.$$

This equation is summarized as:-

$$(\omega-k_\parallel v_\parallel)\tilde{F}=i\overline{\langle C(f_{M_1}+\tilde{F}_1)\rangle}+i\overline{\langle CEL\,drives\rangle};$$

It is solved to obtain the kinetic distortion:-

$$\tilde{F}=\frac{1}{(w-k_\parallel v_\parallel)}\left\{-\frac{2}{3}k_\parallel v_\parallel f_m\frac{\tilde{\pi}_\parallel}{p}-[\frac{2}{3}(s^2-\frac{3}{2}k_\parallel v_t)]f_m\frac{\tilde{q}_\parallel}{p v_t}\right\}$$
$$+\frac{1}{(w-k_\parallel v_\parallel)}\left\{\frac{4}{3}s^2 P_l(\xi)k_\parallel u_\parallel f_m-i(s^2-\frac{5}{2})v_\parallel f_m\frac{(\nabla_\parallel T)_1}{T}\right\}$$
$$+\frac{1}{(w-k_\parallel v_\parallel)}\left\{\sum_{lk}\frac{1}{\sigma_k^l}\left[\left\{f_i^{(0)}P_l(\xi)\boldsymbol{P}^{\boldsymbol{l}}(\hat{\boldsymbol{b}}).(\hat{M}_i^{lk}\nu_{ii}^{(lk,0)}+\hat{M}_i^{lk}\nu_{ii}^{(0,lk)})\right\}\right]\right\}$$

Details of the Time-Dependent $\pi_\parallel$ formulation: Part 2

Two closely-related operators for generating the generalized moments of the total particle distribution function, f_a (and its component terms or functionals, such as F) are defined:-

$$\Omega_a^{q,r} = \int d^3 v\, v^q P_q(\xi) L_r^{q+1/2}$$

and

$$\Omega_{a(L)}^{q,r} = \int d^3 v \frac{v^q P_q(\xi) L_r^{q+1/2}}{\omega - k_\parallel v_\parallel}$$

Thus, we have the generalized moment:-

$$\begin{aligned}\Omega_{a(L)}^{q,r}(\tilde{F}) &= \int d^3 v \frac{v^q P_q(\xi) L_r^{q+1/2}}{\omega - k_\parallel v_\parallel}\left\{-\frac{2}{3} k_\parallel v_\parallel f_m \frac{\tilde{\pi}_\parallel}{p} - [\frac{2}{3}(s^2 - \frac{3}{2} k_\parallel v_t)] f_m \frac{\tilde{q}_\parallel}{p v_t}\right\} \\ &+ \int d^3 v \frac{v^q P_q(\xi) L_r^{q+1/2}}{\omega - k_\parallel v_\parallel}\left\{\frac{4}{3} s^2 P_l(\xi) k_\parallel u_\parallel f_m - i(s^2 - \frac{5}{2}) v_\parallel f_m \frac{(\nabla_\parallel T)_1}{T}\right\} \\ &+ \int d^3 v \frac{v^q P_q(\xi) L_r^{q+1/2}}{\omega - k_\parallel v_\parallel}\left\{\sum_{lk} \frac{1}{\sigma_k^l}\left[\left\{f_i^{(0)} P_l(\xi) \boldsymbol{P}^{\boldsymbol{l}}(\hat{\boldsymbol{b}}).(\hat{M}_i^{lk} \nu_{ii}^{(lk,0)} + \hat{M}_i^{lk} \nu_{ii}^{(0,lk)})\right\}\right]\right\}.\end{aligned}$$

It is observed immediately that:-

$$\Omega_a^{2,0} = \int d^3 v\, v^2 P_2(\xi) L_0^{2+1/2} \quad \Rightarrow \quad \Omega_a^{2,0}(\tilde{F}) = M_\pi(\tilde{F}) = \tilde{\pi}_\parallel$$

$$\Omega_a^{1,1} = \int d^3 v\, v^1 P_1(\xi) L_1^{1+1/2} \quad \Rightarrow \quad \Omega_a^{1,1}(\tilde{F}) = M_q(\tilde{F}) = \tilde{q}_\parallel$$

Details of the Time-Dependent $\pi_\parallel$ formulation: Part 3

For "classical reasons" we convert to Laplace-transformation of the DKE (in time), with the understanding that the continuous Fourier transform is equivalent to evaluating the bilateral Laplace transform with complex argument, $s = i\omega$. So we have:-

$$-i\,(ip+k_\parallel v_\parallel)\,\tilde{F}=\tilde{F}_0+\overline{\langle C(f_{M_1}+\tilde{F}_1)\rangle}+\overline{\langle CEL\,drives\rangle};$$

On solving for F, we obtain:-

$$-i\,\tilde{F}=\frac{1}{(ip+k_\parallel v_\parallel)}\left\{\tilde{F}_0+\overline{\langle C(f_{M_1}+\tilde{F}_1)\rangle}+\overline{\langle CEL\,drives\rangle}\right\};$$

Thus, we have the generalized moment of F below:-

$$\begin{aligned}\Omega^{q,r}_{a(L)}(\tilde{F})=&\int d^3v\,\frac{v^q P_q(\xi)\,L_r^{q+1/2}}{ip+k_\parallel v_\parallel}\left\{-\frac{2}{3}k_\parallel v_\parallel f_m\frac{\tilde{\pi}_\parallel}{p}-[\frac{2}{3}(s^2-\frac{3}{2}k_\parallel v_t)]\,f_m\frac{\tilde{q}_\parallel}{p\,v_t}\right\}\\&+\int d^3v\,\frac{v^q P_q(\xi)\,L_r^{q+1/2}}{ip+_\parallel v_\parallel}\left\{\frac{4}{3}s^2 P_l(\xi)\,k_\parallel u_\parallel f_m-i\,(s^2-\frac{5}{2})\,v_\parallel f_m\frac{(\nabla_\parallel T)_1}{T}\right\}\\&+\int d^3v\,\frac{v^q P_q(\xi)\,L_r^{q+1/2}}{ip+k_\parallel v_\parallel}\left\{\sum_{lk}\frac{1}{\sigma_k^l}\Big[\Big\{f_i^{(0)}P_l(\xi)\,\boldsymbol{P}^{\boldsymbol{l}}(\hat{\boldsymbol{b}}).(\hat{M}_i^{lk}\nu_{ii}^{(lk,0)}+\hat{M}_i^{lk}\nu_{ii}^{(0,lk)})\Big\}\Big]\right\}.\end{aligned}$$

Details of the Time-Dependent $\pi_\parallel$ formulation: Part 4

By writing the velocity-space integral in a magnetic slab coordinate system, the generalized moment of F becomes:-

$$\Omega^{q,r}_{a(L)}(\tilde{F}) = 2\pi\int_0^\infty d^3v\, v^2 \int_{-1}^{+1} d\xi \frac{v^q P_q(\xi) L_r^{q+1/2}}{ip+k_\parallel v_\parallel}\left\{-\frac{2}{3}k_\parallel v_\parallel f_m \frac{\tilde{\pi}_\parallel}{p} - [\frac{2}{3}(s^2-\frac{3}{2}k_\parallel v_t)]f_m \frac{\tilde{q}_\parallel}{p v_t}\right\}$$

$$+2\pi\int_0^\infty d^3v\, v^2 \int_{-1}^{+1} d\xi \frac{v^q P_q(\xi) L_r^{q+1/2}}{ip+_\parallel v_\parallel}\left\{\frac{4}{3}s^2 P_l(\xi) k_\parallel u_\parallel f_m - i(s^2-\frac{5}{2})v_\parallel f_m \frac{(\nabla_\parallel T)_1}{T}\right\}$$

$$+2\pi\int_0^\infty d^3v\, v^2 \int_{-1}^{+1} d\xi \frac{v^q P_q(\xi) L_r^{q+1/2}}{ip+k_\parallel v_\parallel}\left\{\sum_{lk}\frac{1}{\sigma_k^l}\left[\left\{f_i^{(0)} P_l(\xi)\boldsymbol{P}^{\boldsymbol{l}}(\hat{\boldsymbol{b}}).(\hat{M}_i^{lk}\nu_{ii}^{(lk,0)}+\hat{M}_i^{lk}\nu_{ii}^{(0,lk)})\right\}\right]\right\}.$$

Specifically, the generalized, parallel ion stress closure moment calculation involves:-

$$\Omega^{2,0}_{i(L)}(\tilde{F}) = 2\pi\int_0^\infty d^3v\, v^2 \int_{-1}^{+1} d\xi \frac{v^2 P_2(\xi) L_0^{2+1/2}}{ip+k_\parallel v_\parallel}\left\{-\frac{2}{3}k_\parallel v_\parallel f_m \frac{\tilde{\pi}_\parallel}{p} - [\frac{2}{3}(s^2-\frac{3}{2}k_\parallel v_t)]f_m \frac{\tilde{q}_\parallel}{p v_t}\right\}$$

$$+2\pi\int_0^\infty d^3v\, v^2 \int_{-1}^{+1} d\xi \frac{v^2 P_2(\xi) L_0^{2+1/2}}{ip+k_\parallel v_\parallel}\left\{\frac{4}{3}s^2 P_l(\xi) k_\parallel u_\parallel f_m - i(s^2-\frac{5}{2})v_\parallel f_m \frac{(\nabla_\parallel T)_1}{T}\right\}$$

$$+2\pi\int_0^\infty d^3v\, v^2 \int_{-1}^{+1} d\xi \frac{v^2 P_2(\xi) L_0^{2+1/2}}{ip+k_\parallel v_\parallel}\left\{\left\{f_i^{(0)} P_2(\xi)\boldsymbol{P}^{\boldsymbol{2}}(\hat{\boldsymbol{b}}).(\hat{M}_i^{20}\nu_{ii}^{20})\right\}+\frac{1}{\sigma_1^1}\left\{f_i^{(0)} P_1(\xi)\boldsymbol{P}^1(\hat{\boldsymbol{b}}).(\hat{M}_i^{11}\nu_{ii}^{11})\right\}\right\}.$$

The most difficult task in the moment calculations is the integration of terms involving the Landau propagator, Maxwellians and error functions. Especially, the integral terms that also involve the free-streaming parameter, ξ, are very hard to integrate. That is:-

$$\int_{-1}^{+1} d\xi \frac{\xi^q P_q(\xi)\xi^{q'} P_{q'}(\xi)}{ip+k_\parallel v_\parallel} \quad \textbf{or} \quad \int_{-1}^{+1} d\xi \frac{\xi^q P_q(\xi)\xi^{q'} P_{q'}(\xi)}{\omega-k_\parallel v_\parallel}$$

Integration of the integrals involving ξ

Since the Legendre functions of ξ are just polynomials involving powers of ξ, the relevant ξ integrals required are, effectively, of the form:-

$$\int_{-1}^{+1} d\xi \frac{\xi^n}{ip+k_\parallel v_\parallel}$$

This is re-written as:-

$$\frac{1}{v k_\parallel}\int_{-1}^{+1} d\xi \frac{\xi^n}{i\zeta+\xi}; \qquad \text{where} \qquad \zeta=\frac{p}{v k_\parallel}.$$

We write ζ as a complex variable, split into real and complex parts (x and y, respectively):-

$$\zeta = x+iy$$

Substitution of this complex form of ζ into the integral leads to:-

$$\int_{-1}^{+1} d\xi \frac{\xi^n}{ip+k_\parallel v_\parallel}=\frac{1}{v k_\parallel}\int_{-(1+1)}^{(1-y)} dz \frac{(z+y)^n (z-ix)}{z^2+x^2}$$

This leads to the result:-

$$\int_{-1}^{+1} d\xi \frac{\xi^n}{ip+k_\parallel v_\parallel} = \frac{1}{s^n}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left(\frac{k}{n+1-k}-\frac{ix}{y}\right)\left(\frac{\bar{x}}{\bar{y}}\right)^{(k-1)}\bar{y}^n\int_{-\theta}^{+\theta} d\theta \tan^n\theta+\bar{x}^n\int_{-\theta}^{+\theta} d\theta \tan^k\theta$$

where:-

$$-\theta=1\frac{(1+y)}{x}; \quad +\theta=\frac{(1-y)}{x}; \qquad \bar{x}=x\,s \quad \text{and} \quad \bar{y}=y\,s\,.$$

This is a finite integral, and it obviates the use of limit approximations and dispersion functions!

Work-In-Progress

Detailed methods of integration have been devised to resolve all the complicated integrals involved in generalized kinetic closure formulations. The most troublesome part is that involving free-streaming coordinate, ξ.

$$\int d\xi \frac{\xi^{q} P_{q}(\xi)\xi^{q'} P_{q'}(\xi)}{ip + k_{\|} v_{\|}}$$

or

$$\int d\xi \frac{\xi^{q} P_{q}(\xi)\xi^{q'} P_{q'}(\xi)}{\omega - k_{\|} v_{\|}}$$

Various fortran code modules have been developed to calculate these integrals computationally.

Comprehensive strategies have been mapped out for implementing the time-dependent parallel ion stress tensor in NIMROD.

Conclusion.

- The properties of the steady state $\pi_{\|}$ closure, in both analytical and numerical (NIMROD) modeling of plasmas, are well-behaved and understood.

- The time-dependent $\pi_{\|}$ closure would be used in all simulations, both analytical and numerical (NIMROD), in which the steady state version was previously used.

- Real and imaginary components of dynamic pseudo-transport coefficients would be calculated and plotted in a manner similar to Chang and Callen's, for comparison.

- The time-dependent $\pi_{\|}$ closure would be used in various applications in the future, including non-linear studies of the the growth rate of single modes of ELMs.

Future Research Work.

- The kinetic-based generalized $\pi_{\|}$ produces results that are consistent with plasma physics at both high and low temperatures. It would therefore be used extensively in studying various kinetic effects in high-temperature plasmas.

- Upgrade kinetic model of the generalized $\pi_{\|}$ to include explicit time-dependence, acceleration and a more accurate collision operator.

- Extend the scope of studies to the analysis of $\pi_{\|}$ effects in parallel ion flow damping across stochastic magnetic islands, reconnection, anomalous ion heating and ELMs.

Key References

- [1]Z. Chang, J.D. Callen, Phys. Fluids B4 (5), 1167 (1992).
- [2]E.D. Held, Phys. Plasmas 10, 4708 (2003).
- [3]J.Y. Ji, E.D. Held, Phys. Plasmas, 13, 102103 (2006).