A second-order accurate semi-implicit δf method for kinetic MHD simulation

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Outline

1. The model and the tearing mode study
    - The second-order accuracy semi-implicit δf method with kinetic ions and fluid electrons.
    - The simulation of tearing mode.
2. New challenges.
    - To resolve the thin current layer, more k_y modes are needed which leads to 2D domain decomposition of the code (accomplished).
    - In the strong reconnection regime, full-f implementation may be needed.
3. Future works.
    - For larger Δ', the competition between plasmoid formation and island coalescence.
    - Detailed ion diagnostics to show how ions are heated during the reconnection process.

Ion equations of motion and field equations

- Lorentz force ions with collisional drag

$$\frac{d\boldsymbol{v}_i}{dt} = \frac{q_i}{m_i}(\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{B} - \eta(n_i q_i \boldsymbol{v}_i - n_e e \boldsymbol{u}_e))$$

$$\frac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i$$

where $n_e e \boldsymbol{u}_e = \boldsymbol{j}_i - \frac{1}{\mu_0} \bigtriangledown \times \boldsymbol{B}$.

- Ampere's equation

$$\bigtriangledown \times \boldsymbol{B} = \mu_0 \left(n_i q_i \boldsymbol{u}_i - n_e e \boldsymbol{u}_e\right)$$

- Faraday's law

$$\bigtriangledown \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

The generalized Ohm's law and electron modeling

- Using quasi-neutrality $n_i = n_e$, the generalized Ohm's law reads

$$
\begin{aligned}
& en_i(1+\frac{m_e}{m_i}\frac{q_i^2}{e^2})\boldsymbol{E} + \frac{m_e}{\mu_0 e}\,\nabla\times(\nabla\times\boldsymbol{E}) \\
= & -(1+\frac{m_e}{m_i}\frac{q_i}{e})\,\boldsymbol{j}_i\times\boldsymbol{B} + \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} \\
& +\eta\frac{en_i}{\mu_0}(1+\frac{m_e}{m_i}\frac{q_i^2}{e^2})\,\nabla\times\boldsymbol{B} - \nabla\cdot\boldsymbol{\Pi}_e + \frac{m_e}{m_i}\frac{q_i}{e}\,\nabla\cdot\boldsymbol{\Pi}_i,
\end{aligned}
$$

- In general, we need an electron model to calculate Π_e. Here we assume the electrons are isothermal and Π_e reduces to

$$P_e = n_e T_e = n_i T_e$$

More sophisticated models can be employed to include more electron physics.

The second-order semi-implicit δf algorithm

- Given a distribution function $f = f_0 + \delta f$, if we assign a weight of $w_j = \frac{\delta f}{f}|_{x=x_j,v=v_j}$ to particle j, we could then calculate the field quantities by weight δf to the grids. According to Vlasov equation, the particle weight evolves as

$$\frac{d}{dt}w_j = -\frac{d\ln f_0}{dt}$$

- We have implemented a second-order semi-implicit scheme with an adjustable centering parameter which generalizes the previously developed first-order scheme.

- Through direct matrix inversion, the new scheme avoids the Fourier convolutions by solving the fields in real space along the inhomogeneous direction for each Fourier mode in the uniform directions.

Yang Chen, Scott E. Parker, Phys. Plasmas **16**, 052305 (2009)

Normalization and the numerical layout

- The velocity, length and time are normalized to $c_s^2 = T_e/m_i$, $\rho_s = m_i c_s/eB_0$ and $\Omega_{ci}^{-1} = m_i/eB_0$. $\beta_e = \mu_0 n_0 T_e/B_0^2$ is defined upon the uniform background plasma.
- The equations of motion are

$$\frac{\boldsymbol{x}^{n+1} - \boldsymbol{x}^n}{\Delta t} = (1-\theta)\ \boldsymbol{v}^n + \theta\ \boldsymbol{v}^{n+1},$$
$$\frac{\boldsymbol{v}^{n+1} - \boldsymbol{v}^n}{\Delta t} = (1-\theta)\ \boldsymbol{a}^n + \theta\ \boldsymbol{a}^{n+1},$$
$$\frac{w^{n+1} - w^n}{\Delta t} = -(1-\theta)\ (\boldsymbol{v}^n \cdot \bigtriangledown + \boldsymbol{a}^n \cdot \partial_{\boldsymbol{v}}) \ln f_0(\boldsymbol{x}^n, \boldsymbol{v}^n)$$
$$-\theta\ (\boldsymbol{v}^{n+1} \cdot \bigtriangledown + \boldsymbol{a}^{n+1} \cdot \partial_{\boldsymbol{v}}) \ln f_0(\boldsymbol{x}^{n+1}, \boldsymbol{v}^{n+1}),$$

where $\boldsymbol{a} = \frac{q_i}{m_i}[(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) - \eta(\boldsymbol{v} - \boldsymbol{j}_i + \bigtriangledown \times \boldsymbol{B}/\beta_e)]$.

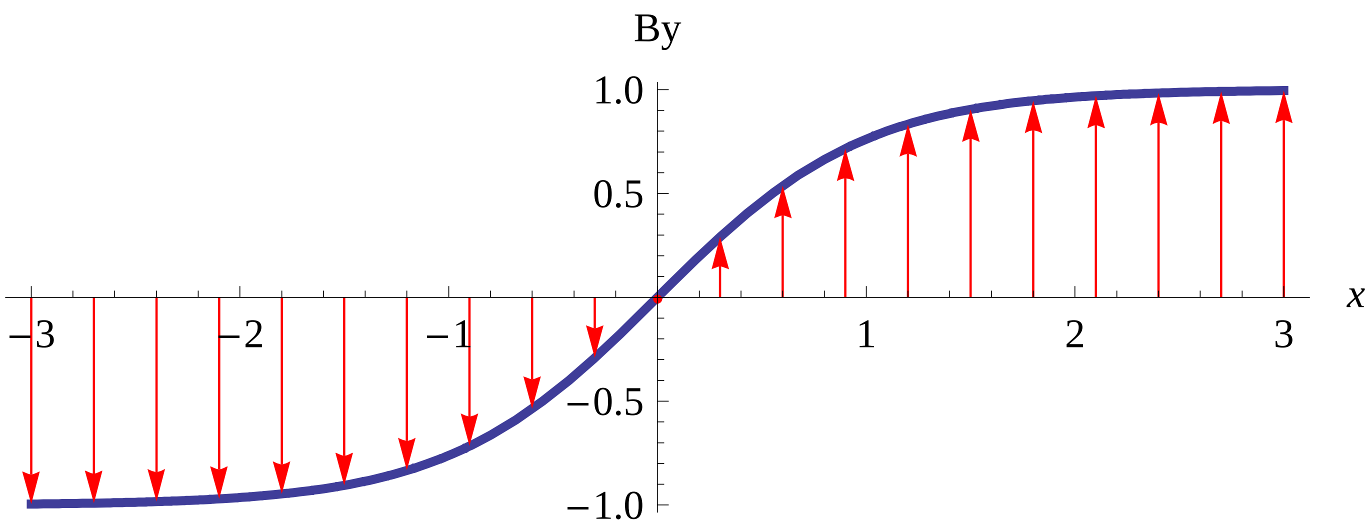
- Generalized Ohm's law:

$$(n_{i0} + \delta n_i^{n+1})(1 + \frac{m_e}{m_i} q_i^2)\boldsymbol{E}^{n+1} + \frac{m_e}{m_i}\frac{1}{\beta_e} \bigtriangledown \times (\bigtriangledown \times \boldsymbol{E}^{n+1})$$
$$= -(1 + \frac{m_e}{m_i} q_i)\delta \boldsymbol{j}_i^{n+1} \times (\boldsymbol{B}_0 + \delta \boldsymbol{B}^{n+1}) + \frac{1}{\beta_e}(\bigtriangledown \times \delta \boldsymbol{B}^{n+1}) \times \boldsymbol{B}_0$$
$$+ \frac{1}{\beta_e}(\bigtriangledown \times (\boldsymbol{B}_0 + \delta \boldsymbol{B}^{n+1})) \times \delta \boldsymbol{B}^{n+1} + \frac{\eta}{\beta_e}(1 + \frac{m_e}{m_i} q_i^2)(n_{i0} + \delta n_i^{n+1}) \bigtriangledown \times \delta \boldsymbol{B}^{n+1}$$
$$- \nabla \delta n_i^{n+1} + \frac{m_e}{m_i} q_i \ \bigtriangledown \cdot P_i^{n+1},$$

Harris sheet equilibrium

- Zero-order $\boldsymbol{B}$

$$\boldsymbol{B}_0(\boldsymbol{x}) = B_{y0} \tanh(\frac{x}{a})\ \hat{\boldsymbol{y}} + B_G\ \hat{\boldsymbol{z}}$$



- The equilibrium distribution function is

$$f_{0s} = n_{h0}\ \mathrm{sech}^2(\frac{x}{a}) \left(\frac{2\pi T_s}{m_s}\right)^{-\frac{3}{2}} \exp\left[-\frac{m(v_x^2 + v_y^2 + (v_z - v_{ds})^2)}{2T_s}\right]$$
$$+n_b \left(\frac{2\pi T_s}{m_s}\right)^{-\frac{3}{2}} \exp\left(-\frac{mv^2}{2T_s}\right),$$

Boundary conditions

- Perfect conducting wall boundary condition is employed in x while periodic boundary conditions in y and z direction.

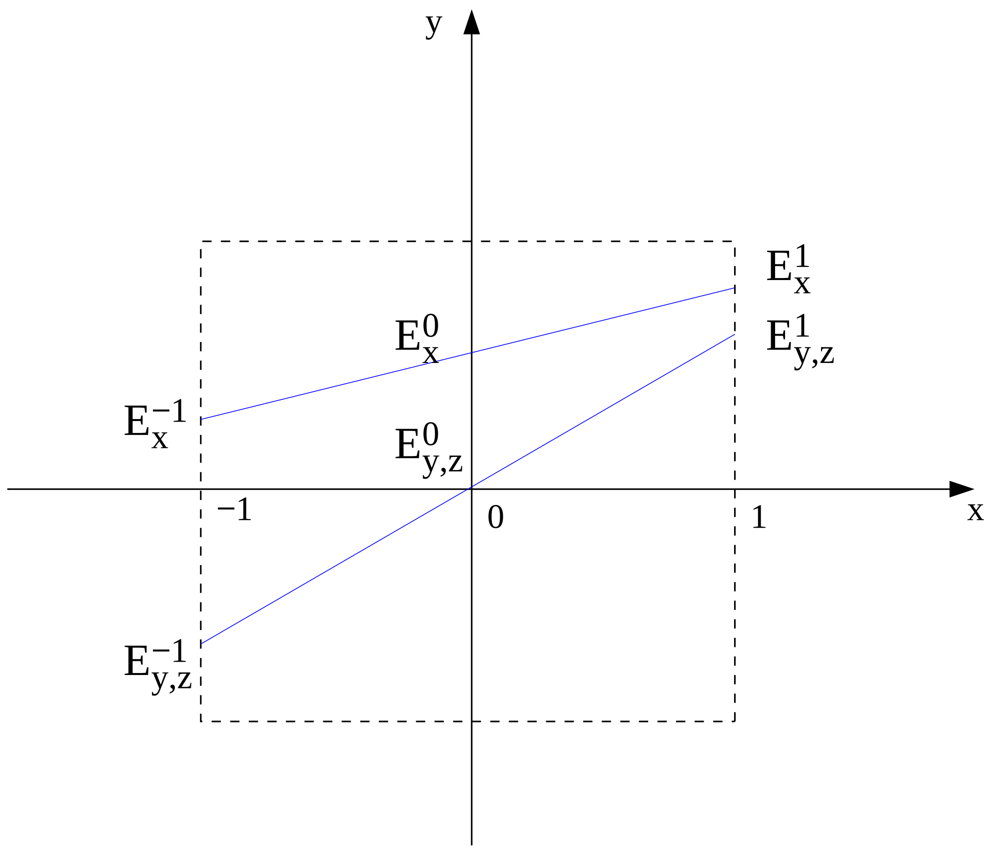
$$\boldsymbol{E}_{y,z}\big|_{x=\pm l_x/2} = 0$$
$$\delta \boldsymbol{B}_x\big|_{x=\pm l_x/2} = 0$$

- Numerically, the boundary condition for $\boldsymbol{E}$ is treated using linear interpolation

$$\frac{\boldsymbol{E}_{y,z}^{-1} + \boldsymbol{E}_{y,z}^{1}}{2} = 0$$
$$\frac{\boldsymbol{E}_{x}^{-1} + \boldsymbol{E}_{x}^{1}}{2} = \boldsymbol{E}_x^0$$

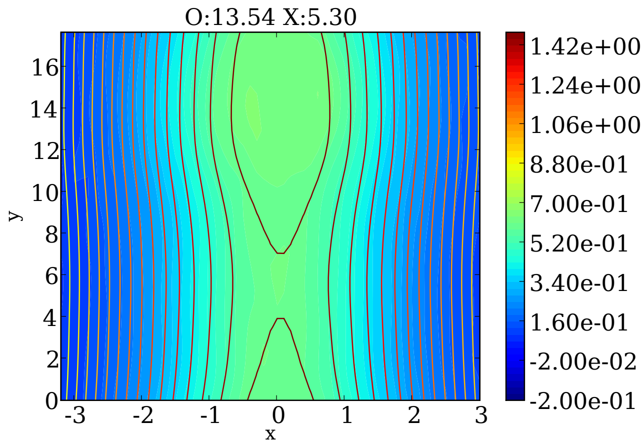
at $x = \pm l_x/2$.

- Boundary condition for $\delta \boldsymbol{B}$ is assumed in Faraday's equation.

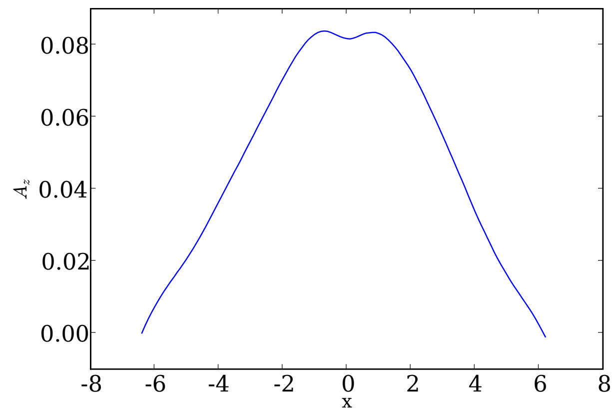


First grid on the left boundary of x direction

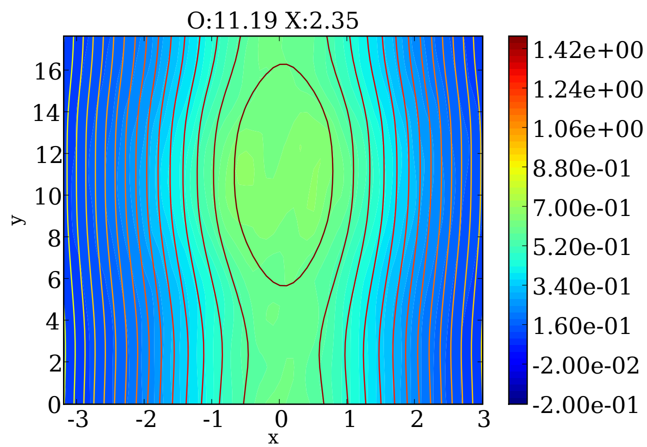
Island and eigenmode structure



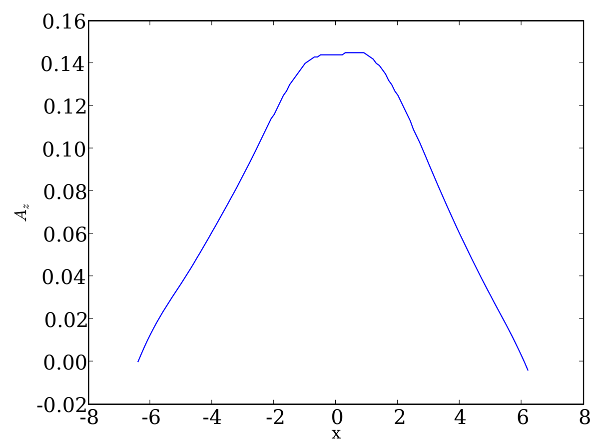
(a)



(b)



(c)



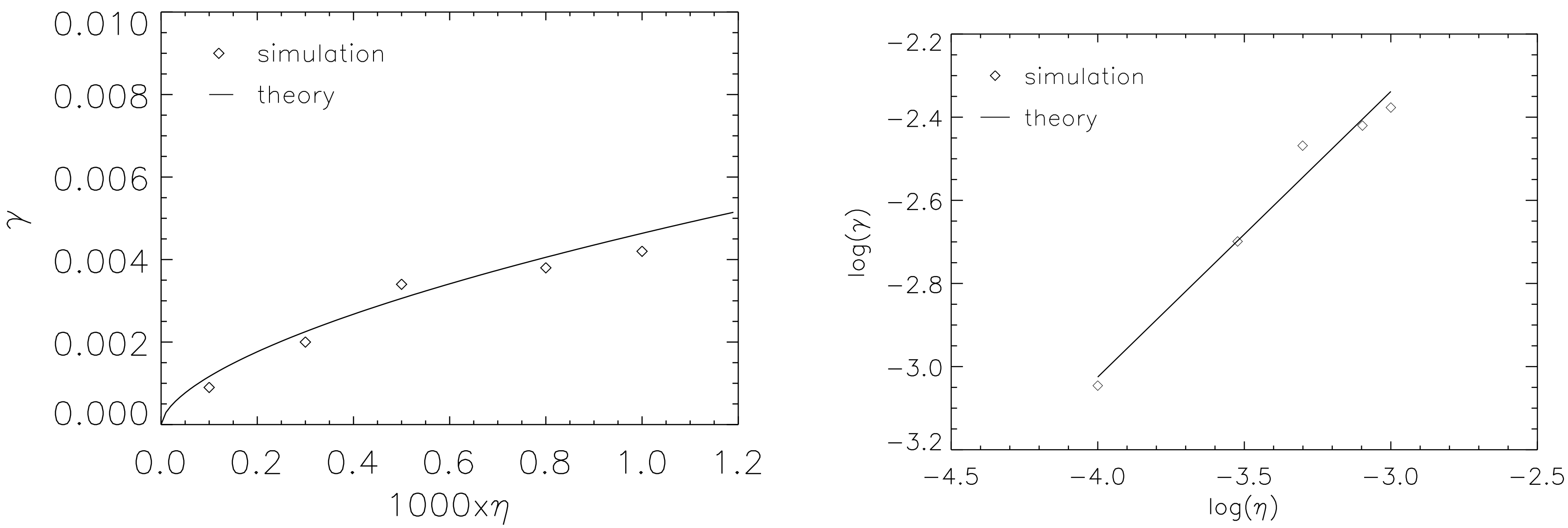
(d)

(a)(b) $t = 1640\Omega_i^{-1}$, (c)(d) $t = 3260\Omega_i^{-1}$ $128 \times 32 \times 64$ grids, 8388608 particles. $\frac{a}{\rho_i} = 2.0, \beta_e = \frac{\mu_0 n_0 T_e}{B_0^2} = 0.5,$
$\eta \frac{e n_0}{B_0} = 5 \times 10^{-4}$, $\frac{\boldsymbol{B}_G}{B_0} = 0$, $\frac{T_i}{T_e} = 1$, $\frac{l_x}{\rho_i} = 12.8$, $\frac{l_y}{\rho_i} = 18.84$

The linear growth rate

- Linear Tearing mode theory shows that the growth rate is (scaled)

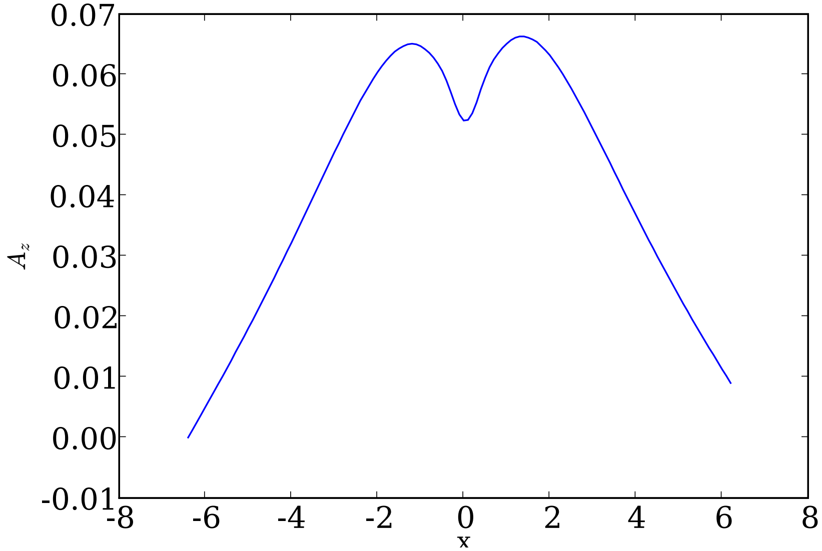
$$\gamma = 0.55\ (\frac{1}{\beta})^{1/5}\ \Delta'^{4/5}\eta^{3/5}\ (k\ B'_{y0})^{2/5}.$$



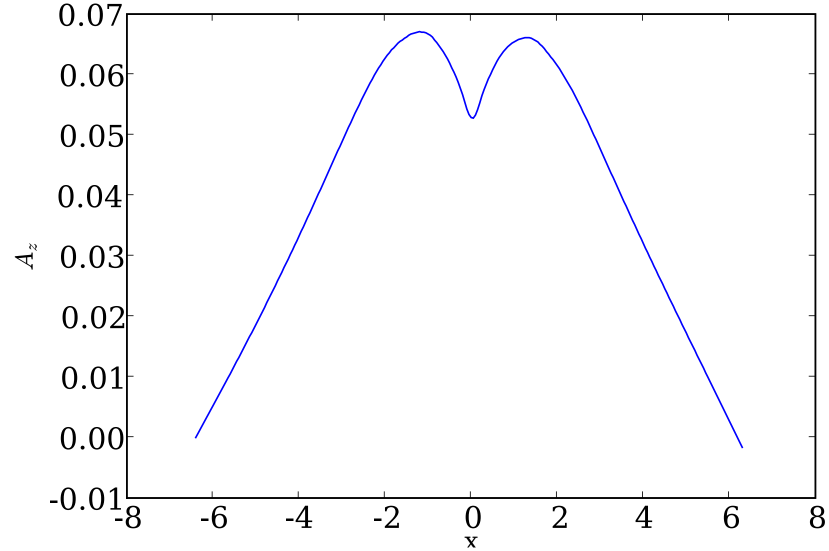
(a) (b)

$128 \times 32 \times 64$ grids, 8388608 particles. $\frac{a}{\rho_i} = 2.0$, $\beta_e = \frac{\mu_0 n_0 T_e}{B_0^2} = 0.5$, $\frac{\boldsymbol{B}_G}{B_0} = 0$, $\frac{T_i}{T_e} = 1$, $\frac{l_x}{\rho_i} = 12.8$, $\frac{l_y}{\rho_i} = 18.84$

Convergence of mesh grids



(a)



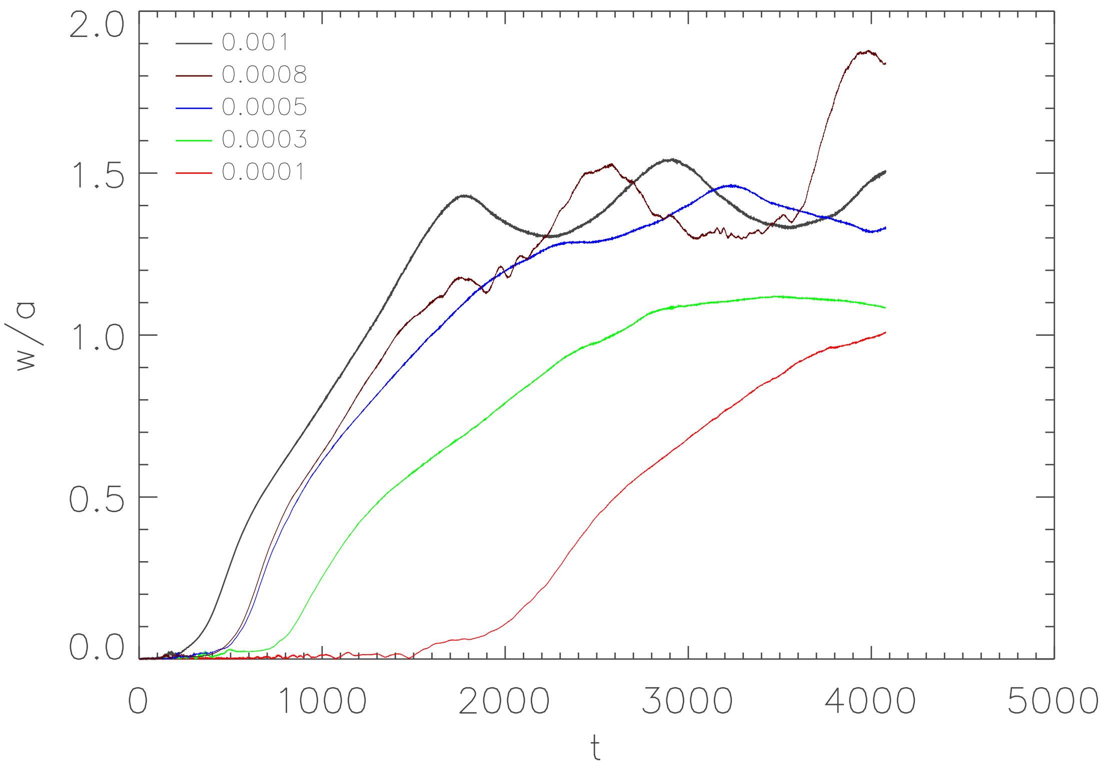
(b)

(a) $128 \times 16 \times 32$ (b) $256 \times 16 \times 32$.

$$\frac{a}{\rho_i} = 2.0, \beta_e = \frac{\mu_0 n_0 T_e}{B_0^2} = 0.5, \eta \frac{e n_0}{B_0} = 8 \times 10^{-4}, T_i/T_e = 1.0, \frac{l_x}{\rho_i} = 12.8, \frac{l_y}{\rho_i} = 25.12.$$

Full evolution with different η

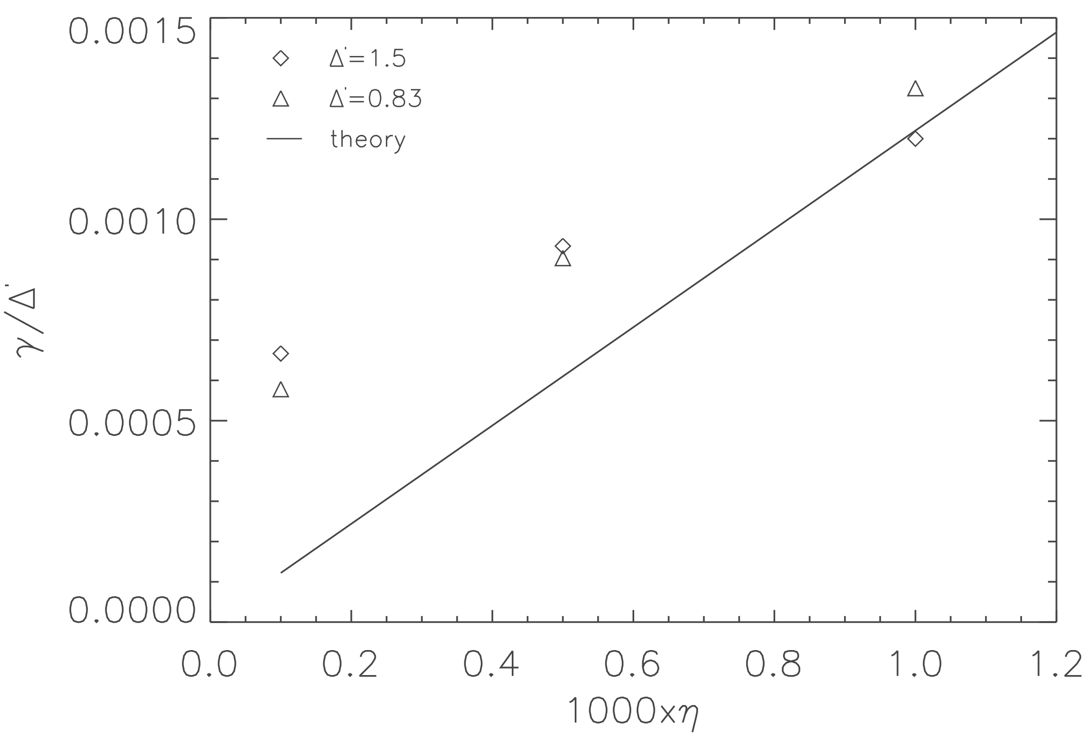
- Resistivity η scan,



$128 \times 32 \times 64$ grids, 8388608 particles. $\frac{a}{\rho_i} = 2.0, \beta_e = \frac{\mu_0 n_0 T_e}{B_0^2} = 0.5,$
$\frac{\boldsymbol{B}_G}{B_0} = 0, \frac{T_i}{T_e} = 1, \frac{l_x}{\rho_i} = 12.8, \frac{l_y}{\rho_i} = 18.84$

Rutherford stage

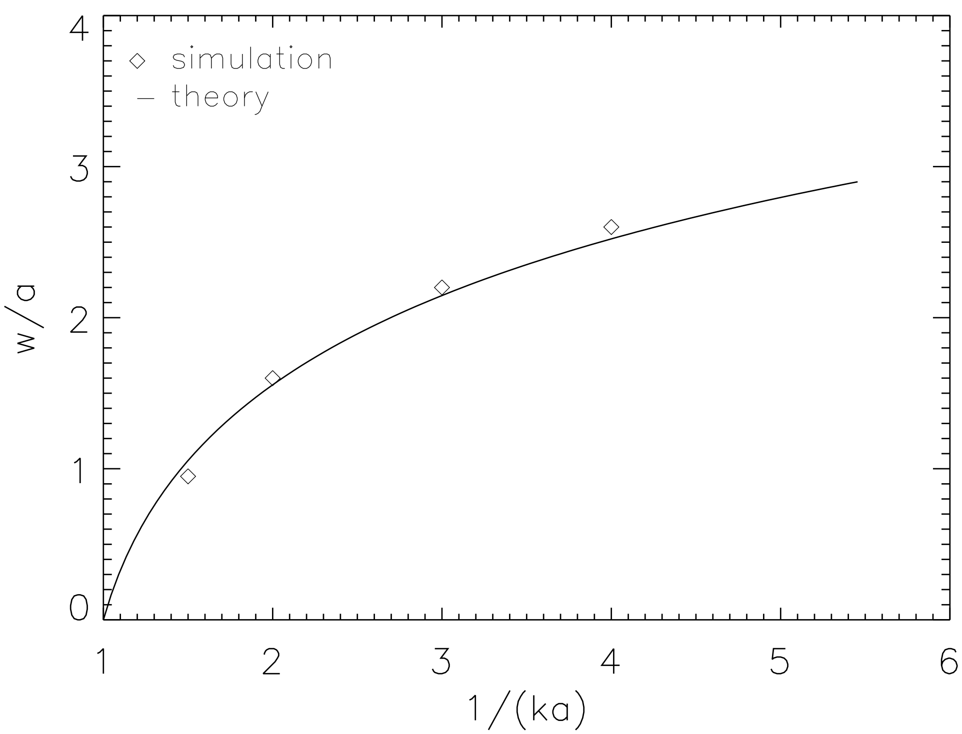
- The growth rate of island width calculated by Rutherford is $\frac{dw}{dt} = 1.22\eta\Delta'$,



$128 \times 32 \times 64$ grids, 8388608 particles. $\frac{a}{\rho_i} = 2.0, \beta_e = \frac{\mu_0 n_0 T_e}{B_0^2} = 0.5,$
$\frac{\boldsymbol{B}_G}{B_0} = 0, \frac{T_i}{T_e} = 1, \frac{l_x}{\rho_i} = 12.8, \frac{l_y}{\rho_i} = 25.12(18.84)$

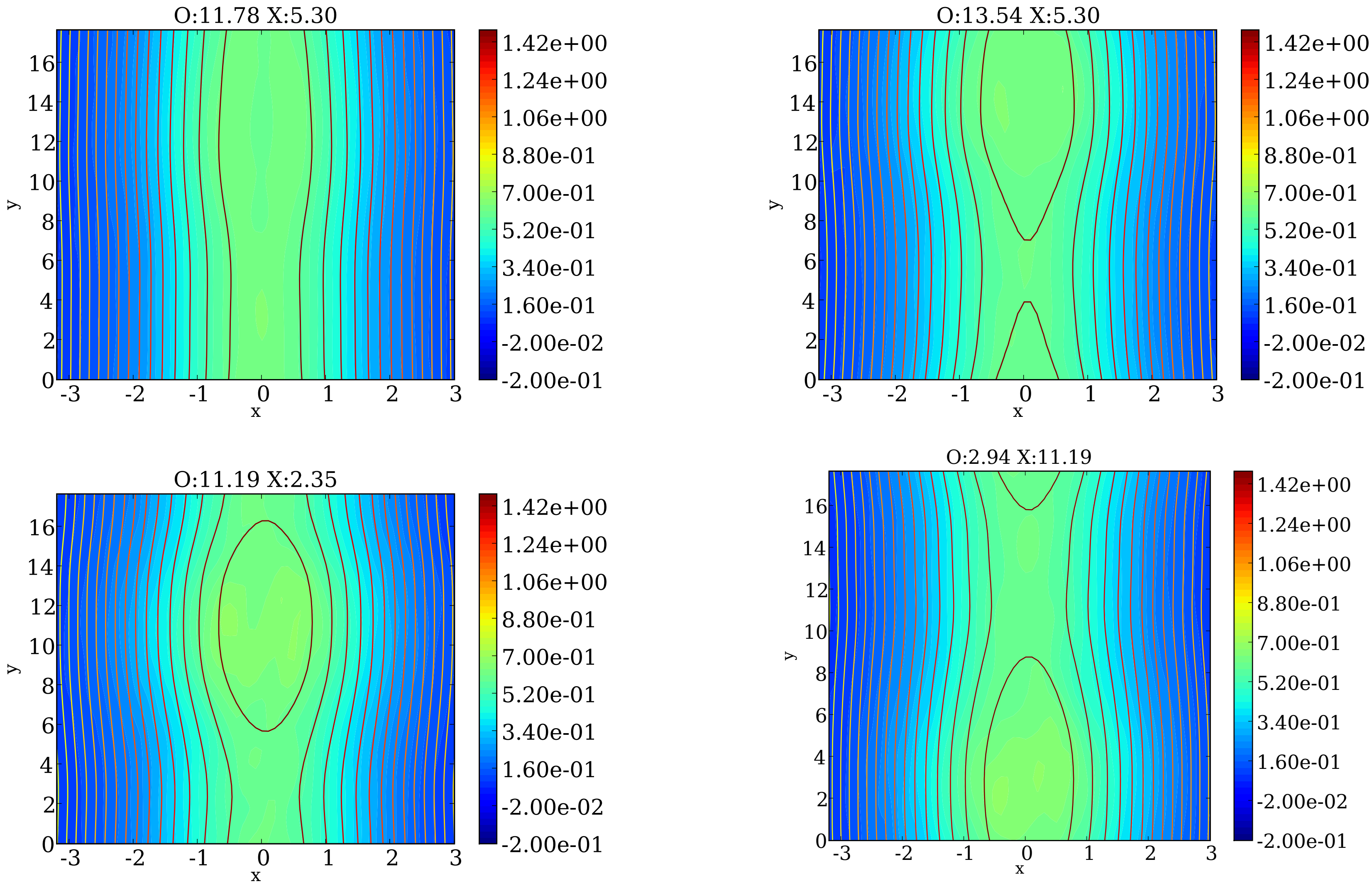
Saturation

- The generalized $\Delta'(w) = \frac{1}{\psi_1(x_s)}(\frac{d\psi_1}{dx}|_{x_+} - \frac{d\psi_1}{dx}|_{x_-})$, where $x_\pm = x_s \pm w/2$. One interpretation of saturation is $\Delta'(w_s) = 0$,



128 × 32 × 64 grids, 8388608 particles. $\frac{a}{\rho_i} = 2.0, \beta_e = \frac{\mu_0 n_0 T_e}{B_0^2} = 0.5,\ \frac{\boldsymbol{B}_G}{B_0} = 0, \frac{T_i}{T_e} = 1, \frac{l_x}{\rho_i} = 12.8, \eta\frac{en_0}{B_0} = 0.0005$

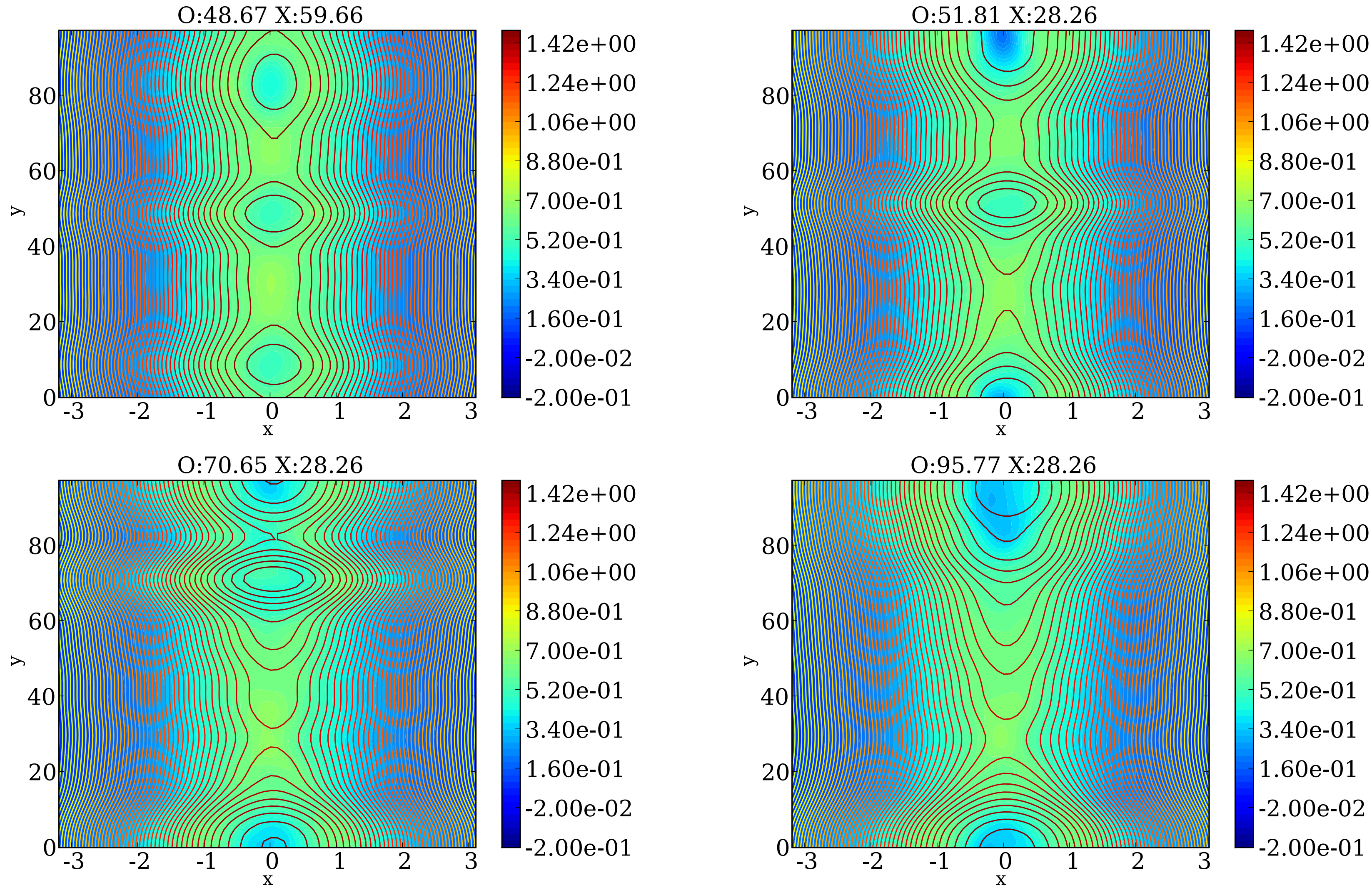
Island evolution—$\Delta' = 0.833$



$128 \times 32 \times 64$, 8388608 particles. $\frac{a}{\rho_i} = 2.0, \beta_e = \frac{\mu_0 n_0 T_e}{B_0^2} = 0.5,\ \eta \frac{e n_0}{B_0} = 0.0005,\ \frac{\boldsymbol{B}_G}{B_0} = 0,\ \frac{T_i}{T_e} = 1,\ \frac{l_x}{\rho_i} = 12.8,\ \frac{l_y}{\rho_i} = 18.84$

From left to right, top to bottom: $t = 880, 1620, 3260, 4600\ \Omega_i^{-1}$

Island evolution—$\Delta' = 7.875$



$128 \times 32 \times 64$, 8388608 particles. $\frac{a}{\rho_i} = 2.0, \beta_e = \frac{\mu_0 n_0 T_e}{B_0^2} = 0.5$, $\eta \frac{e n_0}{B_0} = 0.0005$, $\frac{\boldsymbol{B}_G}{B_0} = 0$, $\frac{T_i}{T_e} = 1$, $\frac{l_x}{\rho_i} = 12.8$, $\frac{l_y}{\rho_i} = 100.48$

From left to right, top to bottom: $t = 744, 1064, 1532, 1776\ \Omega_i^{-1}$

Technical issues and next step work

- Technical issues and possible future work.
  1. In order to resolve the thin current layer, more k_y modes are needed. 2D domain decomposition has been completed.
  2. In the strong reconnection regime, we may need to implement full f method for particle ions. A full-f scheme coupled with the current field solver has been derived.
  3. In the large Δ' regime, Loureiro's resistive MHD simulation suggests Sweet-Parker scenario with a shrinked elongated current layer. Another interesting problem is the formation of secondary islands and their coalescence. We expect to investigate this in more detail and check whether additional physics beyond resistive MHD is involved.
  4. We can study how ions are heated during reconnection by more detailed ion diagnostics.

N. F. Loureiro et al, Phys. Rev. Lett. **95**, 235003 (2005)

Full f scheme

- Ion current

$$\begin{aligned}
\boldsymbol{J}_i^{n+1} &= \sum_j q_i \ \boldsymbol{v}_j^{n+1} \\
&= \sum_j q_i \boldsymbol{v}^\star + \theta \Delta t \ \frac{q_i^2}{m} \ n^{n+1} \ \left(\boldsymbol{E}^{n+1} + \boldsymbol{J}_i^{n+1} \times \boldsymbol{B}^{n+1}/q_i - \frac{\eta}{\mu_0} \bigtriangledown \times \boldsymbol{B}^{n+1} \right) \\
&\approx \boldsymbol{J}_i^\star + \theta \Delta t \ \frac{q_i^2}{m} \ (n_0 + \delta n^{n+1}) \ \left(\boldsymbol{E}^{n+1} + \boldsymbol{J}_i^\star \times \boldsymbol{B}^{n+1}/q_i - \frac{\eta}{\mu_0} \bigtriangledown \times \boldsymbol{B}^{n+1} \right)
\end{aligned}$$

- Continuity equation

$$\frac{\delta n^{n+1} - \delta n^n}{\Delta t} = -(1-\theta) \ \bigtriangledown \cdot \boldsymbol{J}_i^n - \theta \ \bigtriangledown \cdot \boldsymbol{J}_i^{n+1}$$

- Ion density

$$\begin{aligned}
\delta n^{n+1} &= \delta n^\star - \theta \Delta t \ \bigtriangledown \cdot \boldsymbol{J}_i^{n+1} \\
&\approx \delta n^\star - \theta \Delta t \ \bigtriangledown \cdot \left(\boldsymbol{J}_i^\star + \theta \Delta t \ \frac{q_i^2}{m} \ (n_0 + \delta n^{n+1}) \ \left(\boldsymbol{E}^{n+1} + \boldsymbol{J}_i^\star \times \boldsymbol{B}^{n+1}/q_i - \frac{\eta}{\mu_0} \bigtriangledown \times \boldsymbol{B}^{n+1} \right) \right)
\end{aligned}$$

The Lorentz ion/Drift kinetic electron model

Lorentz ions:

$$\frac{d\mathbf{v}_i}{dt} = \frac{q}{m_i}(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}), \qquad \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i$$

Drift kinetic electrons: $\varepsilon = \frac{1}{2} m_e v^2$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}_G \equiv v_\parallel \left(\mathbf{b} + \frac{\delta \mathbf{B}_\perp}{B_0}\right) + \mathbf{v}_D + \mathbf{v}_E$$

$$\frac{d\varepsilon}{dt} = -e\mathbf{v}_G \cdot \mathbf{E} + \mu \frac{\partial B}{\partial t}, \qquad \frac{d\mu}{dt} = 0$$

Ampere's equation

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J}_i - en_e(\mathbf{V}_{\mathrm{e}\perp} + u_{\parallel\mathrm{e}}\mathbf{b}))$$

$$\mathbf{V}_{\mathrm{e}\perp} = \frac{1}{B}\mathbf{E} \times \mathbf{b} - \frac{1}{enB}\mathbf{b} \times \nabla P_{\perp\mathrm{e}}$$

$$\mathbf{J}_i = \int f_i \mathbf{v}\, d\mathbf{v}, \quad u_{\parallel\mathrm{e}} = \int f_e\, v_\parallel\, d\mathbf{v}, \qquad P_{\perp\mathrm{e}} = \int f_e \frac{1}{2} m_e v^2\, d\mathbf{v}$$

Faraday's equation,

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

Summary

1. We implemented an second-order accurate semi-implicit algorithm with Lorentz force ions and isothermal fluid electrons which is
   - Quasi-neutral and fully electromagnetic.
   - Suitable for MHD scale plasmas.

2. Demonstrated 3-D slab simulation for Alfvèn waves, whistler wave, and the ion acoustic wave. Showed that the time-centered second order scheme brings no numerical damping through whistler waves studies.

3. The full evolution of resistive tearing mode is investigated with Harris sheet equilibrium.

4. Future work includes: larger Δ' case study, extension to full f scheme and the drift-kinetic electrons.