High-beta analytic equilibria in circular, elliptical and D-shaped large aspect ratio axisymmetric configurations with poloidal and toroidal flows

O. E. López[1] and L. Guazzotto[1]

[1]Physics Department, Auburn University, Auburn, AL, 36849

Motivation

- Macroscopic flows are routinely present in tokamak experimental devices. They can be induced by plasma heating and have also been measured in the absence of a net momentum input (e.g., *[Rice 2008]*).
- There are few analytic solutions of equilibria with poloidal and toroidal flows.
- We introduce a new family of analytic solutions for tokamak equilibrium with flow in arbitrary direction, of sizes relevant to experiments and which allows finite plasma shaping.

Outline

- MHD equilibrium equations.
- Scaling and Lagrangian formulation.
- Asymptotic expansion.
- Circular and Elliptical solutions.
- D-Shaped solution.
- Conclusions.

Stationary MHD equilibria

Equilibrium equations:

$$\mathbf{J}\times\mathbf{B} = \nabla p + \rho\mathbf{V}\cdot\nabla\mathbf{V}$$

$$\nabla\times\mathbf{B} = \mathbf{J}$$

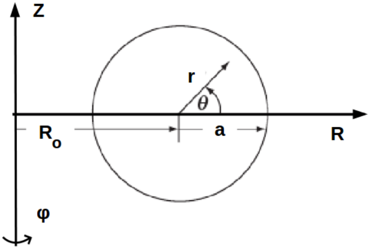
$$\nabla\cdot\mathbf{B} = 0$$

$$\nabla\times(\mathbf{V}\times\mathbf{B}) = 0$$

$$\nabla\cdot(\rho\mathbf{V}) = 0$$

$$p = S\rho^{\gamma}$$

$$\mathbf{V}\cdot\nabla S = 0$$



V = 0 → Grad-Shafranov Eq.

$$\nabla\cdot\left(\frac{\nabla\Psi}{R^2}\right) = -\frac{dp}{d\Psi} - \frac{1}{2R^2}\frac{dF^2}{d\Psi}$$

$\Psi \rightarrow B_p$ flux/radian

$F \rightarrow J_p$ flux/radian

- What is the generalization of the G-S equation?

Stationary MHD equilibria with axisymmetry

The equilibrium problem is reduced to a pair of coupled equations for the magnetic poloidal flux (Ψ) and plasma density (ρ). In cylindrical coordinates (R, φ, Z) these are:

$$\nabla \cdot \left[\left(1 - \frac{\Phi(\Psi)^2}{\rho} \right) \left(\frac{\nabla \Psi}{R^2} \right) \right] = -\frac{B_\varphi}{R} \frac{dF(\Psi)}{d\Psi} - (\mathbf{V} \cdot \mathbf{B}) \frac{d\Phi(\Psi)}{d\Psi}$$
$$-R\rho V_\varphi \frac{d\Omega(\Psi)}{d\Psi} - \rho \frac{dH(\Psi)}{d\Psi} + \frac{\rho^\gamma}{\gamma - 1} \frac{dS(\Psi)}{d\psi} \quad \textbf{(Grad-Shafranov)}$$

$$\frac{1}{2} \left[\frac{\Phi(\Psi) B}{\rho} \right]^2 - \frac{1}{2} [R\Omega(\Psi)]^2 + \frac{\gamma}{\gamma - 1} S(\Psi) \rho^{\gamma - 1} = H(\Psi) \quad \textbf{(Bernoulli)}$$

which depend on the free functions: $\Phi(\Psi)$, $\Omega(\Psi)$, $S(\Psi)$, $H(\Psi)$ and $F(\Psi)$.

Magnetic field, velocity and closure equations

The magnetic and velocity fields are:

$$\mathbf{B}_p = \nabla\Psi \times \frac{1}{R}\hat{\mathbf{e}}_\varphi, \qquad \text{(Poloidal Mag. Field)}$$

$$B_\varphi = \frac{1}{1-\Phi(\Psi)^2/\rho}\left(\frac{F(\Psi)}{R} + R\Phi(\Psi)\Omega(\Psi)\right), \qquad \text{(Toroidal Mag. Field)}$$

$$\mathbf{V}_p = \frac{\Phi(\Psi)}{\rho}\mathbf{B}_p, \qquad \text{(Poloidal Velocity)}$$

$$V_\varphi = \left(\frac{\Phi(\Psi)}{\rho}B_\varphi + R\Omega(\Psi)\right). \qquad \text{(Toroidal Velocity)}$$

The MHD-closure is:

$$p = S(\Psi)\rho^\gamma. \qquad \text{(Kinetic Pressure)}$$

Scaling factors and ordering

Ordering in terms of $\epsilon << 1$:

$$\epsilon = \frac{a}{R_o} \qquad \text{(Inverse-aspect-ratio)}$$

$$\beta_\varphi \sim \frac{p}{B_{\varphi o}^2} \sim O(\epsilon) \qquad \text{(High Beta)}$$

$$\frac{B_p}{B_{\varphi o}} \sim O(\epsilon) \rightarrow q \sim \frac{aB_{\varphi o}}{R_o B_p} \sim O(\epsilon^0)$$

q is the safety factor at edge.

$$\frac{V_p}{V_{Ao}} \sim O(\epsilon^2) \quad \text{and} \quad \frac{V_\varphi}{V_{Ao}} \sim O(\epsilon)$$

$$\frac{C_s}{V_{Ao}} = \frac{\sqrt{\gamma p/\rho}}{V_{Ao}} \sim O(\epsilon^{1/2})$$

$V_{Ao}/B_{\varphi o} \rightarrow$ Alfvén velocity / toroidal magnetic field on axis. Minor radius (a), geometric center at R_o.

Table : Scaled variables $\sim O(\epsilon^0)$.

Scaled	Original	Factor
ψ	Ψ	$a^2 B_{\varphi o}$
$\overline{\rho}$	ρ	$V_{Ao}^{-2} B_{\varphi o}^2$
$\overline{S}(\psi)$	$S(\Psi)$	$\epsilon\, V_{Ao}^{2\gamma} B_{\varphi o}^{2(1-\gamma)}$
$\overline{F}(\psi)$	$F(\Psi)$	$\epsilon^{-1}\, aB_{\varphi o}$
$\overline{\Omega}(\psi)$	$\Omega(\Psi)$	$\epsilon^2\, a^{-1} V_{Ao}$
$\overline{\Phi}(\psi)$	$\Phi(\Psi)$	$\epsilon\, V_{Ao}^{-1} B_{\varphi o}$
$\overline{H}(\psi)$	$H(\Psi)$	$\epsilon\, V_{Ao}^2$
$\overline{z}$	Z	a
$\overline{R}$	R	R_o
$\overline{x}$	$R - R_o$	a
$\overline{L}$	L	$a^2 R_o B_{\varphi o}^2$

Asymptotic expansion using a variational formulation of equilibria

Grad-Shafranov-Bernoulli system $\longleftarrow \delta L(\Psi,\rho)=0$

$$L(\Psi,\rho,\nabla\Psi)=\int\left[\frac{1}{2R^2}\left(1-\frac{\Phi(\Psi)^2}{\rho}\right)|\nabla\Psi|^2-\frac{1}{2}\frac{(F(\Psi)/R+R\Phi(\Psi)\Omega(\Psi))^2}{1-\Phi(\Psi)^2/\rho}\right.$$
$$\left.+\frac{S(\Psi)\rho^\gamma}{\gamma-1}-H(\Psi)\rho-\frac{1}{2}\rho R^2\Omega(\Psi)^2\right]RdRdZ$$

- Rewrite the functional in the scaled form: $\overline{L}(\psi,\overline{\rho},\overline{\nabla}\psi)$.
- Expand each quantity (Q) in $\overline{L}$ in the usual way: $Q=Q_0+\epsilon\ Q_1+\epsilon^2\ Q_2+\ldots$ where $Q_k\sim O(\epsilon^0)$ for k =1, 2, 3....
- Retrieve the Grad-Shafranov-Bernoulli expansion applying an order-by-order variational process *[Hameiri 2013]*.

Retrieving the Bernoulli expansion

$\delta\overline{L}/\delta\overline{\rho}_0 = 0 \longrightarrow$ Bernoulli equation.

For each order the Bernoulli equation is an algebraic equation in which the successive orderings of density appear as an unknown one at a time.

- 0th-order Bernoulli $\rightarrow$ trivial.
- 1st-order Bernoulli $\rightarrow$ analytic $\overline{\rho}_0 = \overline{\rho}_0(\psi_0)$.
- 2nd-order Bernoulli $\rightarrow$ analytic $\overline{\rho}_1 = \overline{\rho}_1(\psi_0, \psi_1)$.

Retrieving the Grad-Shafranov expansion

$\delta\overline{L}/\delta\psi_0 = 0 \longrightarrow$ Grad-Shafranov equation.

The PDE Grad-Shafranov equation is solved for ψ_0. We have to substitute here the previously found expression for $\overline{\rho}_0(\psi_0)$.

- 0th-order G-S $\rightarrow \overline{F}_0 = constant$.
- 1st-order G-S $\rightarrow \overline{F}'_1(\psi_0)$ as a function of $\overline{S}_0(\psi_0)$, $\overline{H}_0(\psi_0)$ and their derivatives.
- 2nd-order G-S $\rightarrow$ equation in which ψ_0 and the leading order of the toroidal velocity appear. In general, this is a non-linear equation.

Assume that the free functions are polynomials

We choose a polynomial dependence on ψ for the free functions:

$$\overline{F}(\psi) = f_0 + \epsilon\,(f_1\,\psi) + \epsilon^2\left(f_2\,\psi + \frac{1}{2}f_{22}\,\psi^2\right),$$

$$\overline{S}(\psi) = s_0\,\psi + \epsilon\left(s_1\,\psi + \frac{1}{2}s_{11}\,\psi^2\right),$$

$$\overline{H}(\psi) = h_0\,\psi + \epsilon\left(h_1\,\psi + \frac{1}{2}h_{11}\,\psi^2\right),$$

$$\overline{\Phi}(\psi) = \phi_0\,\psi,$$

$$\overline{\Omega}(\psi) = \omega_0\,\psi,$$

where the "*free coefficients*": f_0, f_1, f_2, f_{22}, s_0, s_1, s_{11}, h_0, h_1, h_{11}, ϕ_0 and ω_0 are dimensionless constants $\sim O(\epsilon^0)$.

The plasma density is a flux function up to first order in the asymptotic expansion

The 0th-order plasma density is:

$$\bar{\rho}_0 = \left(\frac{(\gamma - 1) h_0}{\gamma s_0} \right)^{1/(\gamma - 1)} = constant.$$

With this choice of free functions the 1st-order density is decoupled from ψ_1, it is a linear function of ψ_0:

$$\bar{\rho}_1(\psi_0) = \left[\frac{s_1 \bar{\rho}_0}{(1-\gamma) s_0} + \frac{h_1 \bar{\rho}_0^{2-\gamma}}{\gamma s_0} \right] + \frac{\psi_0}{2 s_0} \left[\frac{1}{\gamma} \left(\bar{\rho}_0^{2-\gamma} (h_{11} + {\omega_0}^2) - \bar{\rho}_0^{-\gamma} {f_0}^2 {\omega_0}^2 \right) + \frac{s_{11} \bar{\rho}_0}{1 - \gamma} \right].$$

Helmholtz equation for the poloidal flux

2nd-order G-S Eq. $\rightarrow$ Inhomogeneous Helmholtz PDE:

$$(\overline{\nabla}^2 + \lambda)\psi_0 = A + C\,\overline{x},$$

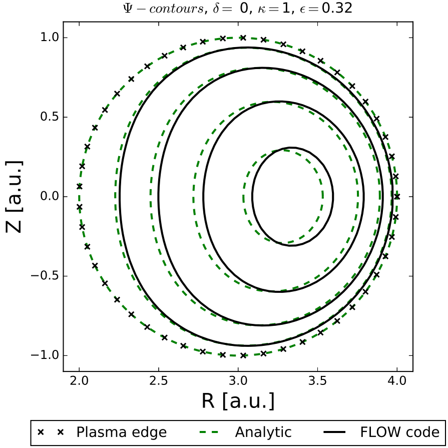
where the constants λ, A and C are given by expressions that depend on the free coefficients:

- $A = A(f_0, f_2, h_0, h_1, s_0, s_1)$
- $C = C(f_0, f_1)$
- $\lambda = \lambda(f_0, f_1, f_{22}, h_0, h_{11}, s_0, s_{11}, \omega_0, \phi_0)$

Closed form solution for a circular boundary

Circular solution:

- $\overline{x} = r\cos(\theta)$, $\overline{z} = r\sin(\theta)$.
- $\psi_0(1,\theta) = 0$, $\theta \in [0, 2\pi)$.
- Construct Green's functions using an expansion in a complete set of functions for θ and a piecewise function for r.
- A closed form solution for the poloidal flux which involves Bessel's functions of the 1st- and 2nd-kind and the Meijer-G-function is found.

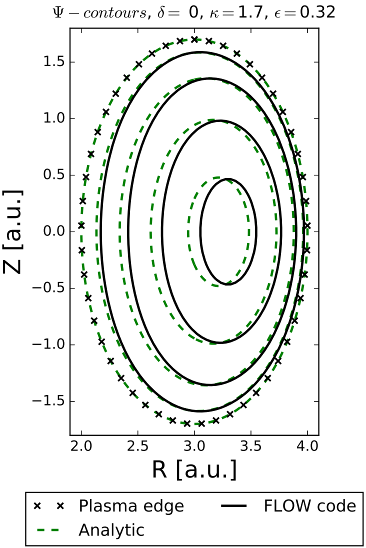


Contour plot of magnetic surfaces in a circular configuration.

Series solution for an elliptical boundary

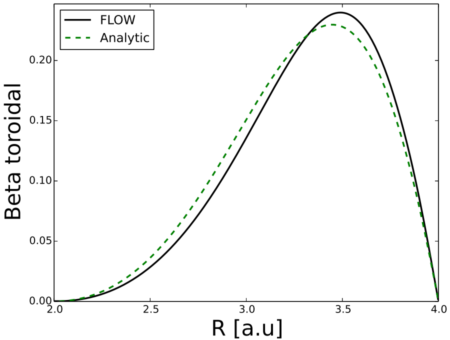
Elliptical solution:

- $\bar{x} = \bar{f}\sinh(\zeta)\sin(\eta)$,
  $\bar{z} = \bar{f}\cosh(\zeta)\cos(\eta)$.
- $\bar{f} := \kappa\sqrt{1-\kappa^{-2}}$ where κ is the elongation.
- $\psi_0(\zeta_o, \eta) = 0$ for $\eta \in [0, 2\pi)$ where $\zeta_o = \text{arctanh}(\kappa^{-1})$.
- Construct Green's functions using an expansion in a complete set of functions for η and a piecewise function for ζ.
- A series solution in terms of Mathieu functions is found.

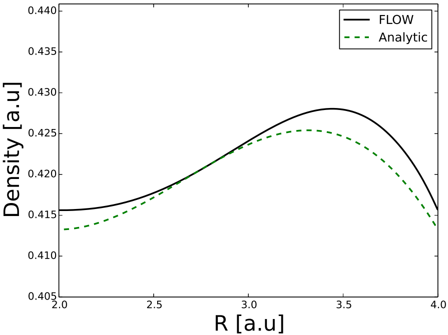


Contour plot of magnetic surfaces in an elliptical configuration.

Plasma beta and density: FLOW comparison



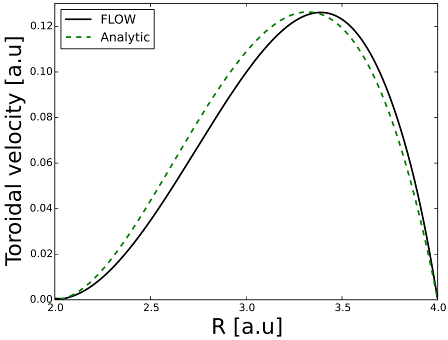
a) Beta Poloidal.



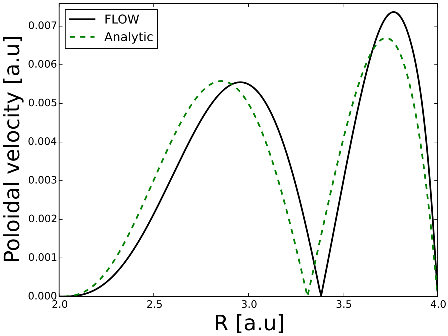
b) Plasma density.

- Typical mid-plane equilibrium quantities for an elliptically shaped plasma ($\kappa = 1.7$ and $\epsilon = 0.32$).
- Observe $\beta_\varphi \sim O(\epsilon)$.
- $\rho = \rho_0 + \epsilon\, \rho_1(\Psi_0)$.

Toroidal and poloidal velocity: FLOW comparison



a) Toroidal velocity.

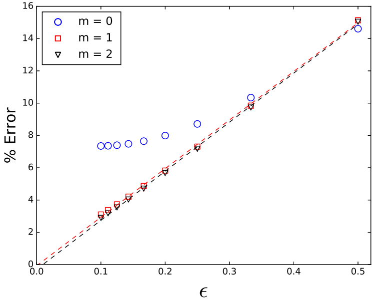


b) Poloidal velocity.

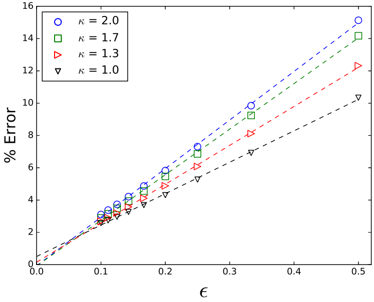
- Typical mid-plane equilibrium quantities for an elliptically shaped plasma ($\kappa = 1.7$ and $\epsilon = 0.32$).
- Observe $V_p/V_\varphi \sim O(\epsilon)$.

Analytical/FLOW solution error determination

- Norm $\rightarrow ||X|| := \sqrt{\sum_{i,j}\left(X_{ij}\right)^2}$, X is a matrix.
- $\mathrm{E}(\Psi_{FLOW}, \Psi_{analytical}) := \frac{||\Psi_{FLOW} - \Psi_{analytical}||}{||\Psi_{FLOW}||} \times 100\%$ is an error function.
- $2m+2 \rightarrow$ number of terms in the series solution in an elliptically shaped plasma.

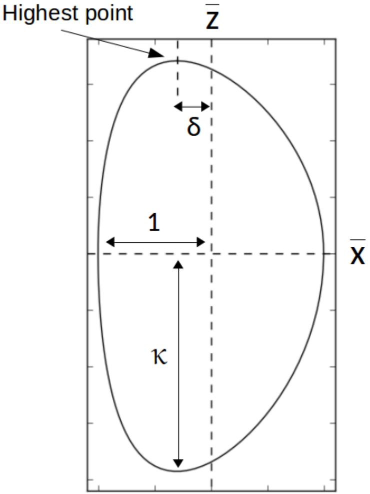


a) $\kappa = 2$.



b) Varying κ.

D-Shape boundary parametrization in terms of elliptical-like coordinates



- D-Shape parametrization:

$$\bar{x} = \cos(\tau + \alpha \sin(\tau)),$$
$$\bar{z} = \kappa \sin(\tau).$$

$\delta = \sin^{-1}(\alpha) \rightarrow$ *triangularity*,

$\kappa \rightarrow$ *elongation*,

$\bar{x} = (R - R_o)/a$,

$\bar{z} = Z/a$.

- Alternative:

$$\bar{x} = \bar{f} \sinh(\zeta_o) \sin(\eta - \alpha \cos \eta)$$
$$\bar{z} = \bar{f} \cosh(\zeta_o) \cos(\eta)$$

Boundary perturbation for a D-shaped plasma

1) Given a triangularity $\delta = \sin(\alpha)$, the D-shaped boundary (∂D) is expressed as a perturbation of the elliptical boundary (∂E):

$$\bar{z} = \bar{f} \cosh(\zeta_o) \cos(\eta),$$

$$\bar{x} = \bar{f} \sinh(\zeta_0) \sin(\eta) + \alpha \; \overline{\delta x}_1 + \alpha^2 \; \overline{\delta x}_2 + \ldots,$$

where $(\overline{\delta x}_1, \overline{\delta x}_2, \ldots)$ are known functions.

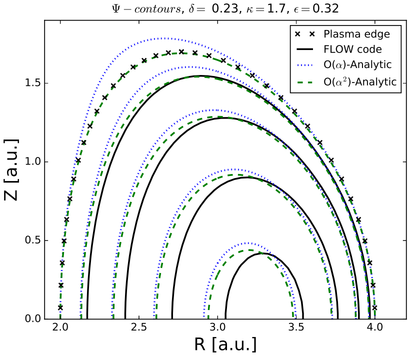
2) Triangularity expansion: $\psi_0 = \psi_0^{(0)} + \alpha \; \psi_0^{(1)} + \ldots$

3) Transfer boundary condition $\partial D \rightarrow \partial E$:

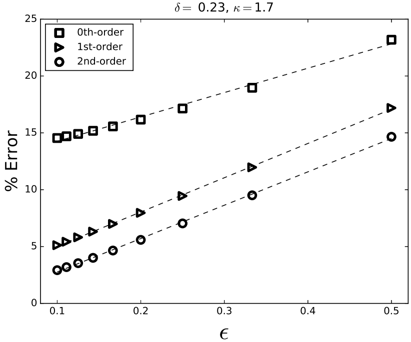
$O(\alpha^0)$: $(\bar{\nabla}^2 + \lambda)\psi_0^{(0)} = A + C\,\bar{x}$, $\psi_0^{(0)}\Big|_{\partial E} = 0 \leftarrow$ Already solved.

$O(\alpha^k)$: $(\bar{\nabla}^2 + \lambda)\psi_0^{(k)} = 0$ + B.C. at $\partial E \leftarrow$ Use elliptical Green's functions again.

D-shaped 1st- and 2nd-order corrections



Contour plot of magnetic surfaces in a D-shaped configuration.



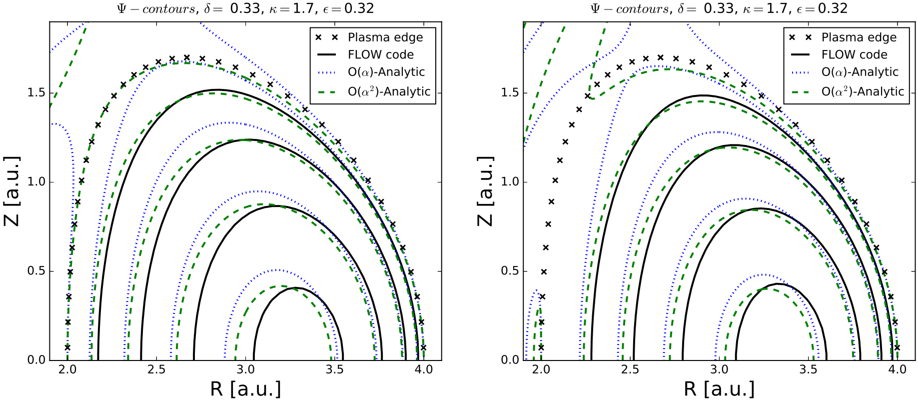
$E(\Psi_{FLOW}, \Psi_{analytical})$ in the $O(\alpha^0)$, $O(\alpha)$ and $O(\alpha^2)$ approximations.

$O(\alpha)$ and $O(\alpha^2)$ boundary conditions:

$$\psi_0^{(1)}\Big|_{\partial E} = -\overline{\delta x}_1 \left.\frac{\partial \psi_0^{(0)}}{\partial \bar{x}}\right|_{\partial E}, \quad \psi_0^{(2)}\Big|_{\partial E} = -\overline{\delta x}_1 \left.\frac{\partial \psi_0^{(1)}}{\partial \bar{x}}\right|_{\partial E} - \overline{\delta x}_2 \left.\frac{\partial \psi_0^{(0)}}{\partial \bar{x}}\right|_{\partial E} - \frac{\overline{\delta x}_1^2}{2} \left.\frac{\partial^2 \psi_0^{(0)}}{\partial \bar{x}^2}\right|_{\partial E}.$$

D-shaped boundary

- The $\Psi_{analytic} = 0$ curve is not guaranteed to be closed.
- The D-shaped solution can describe configurations with relevant elongations and triangularities.



Contour plots of magnetic surfaces in D-shaped configurations with the same geometrical parameters but different values of the free coefficients.

Conclusions

- We have constructed a family of analytic solutions for high-beta equilibria in circular, elliptical and D-shaped axisymmetric configurations with poloidal and toroidal flows.

- The closed form solution for circularly shaped plasmas is given in terms of the Meijer-G-function and Bessel's functions. The series solution for elliptically shaped plasmas of arbitrary elongation is expressed in terms of Mathieu functions. The solution for D-shaped plasmas is obtained from the elliptical one using a boundary perturbative method.

Conclusions

- The plasma density is calculated up to first order in an inverse aspect ratio expansion.
- The relative error between the numerical (from the code FLOW) and analytic solutions converges to zero in the large aspect ratio limit.
- The solution can handle ITER-like elongations, triangularities and values of the inverse aspect ratio.

Thank you for your attention.

Green's functions

The Green's function have the generic form:

$$g\left(u,v|u',v'\right) = G\sum_{m=M}^{\infty}\frac{f_{1m}\left(u_<\right)f_{4m}\left(u_>\right)}{f_{1m}\left(u_o\right)}\,f_{3m}(v,v'),$$

$u \rightarrow$ radial variable, $v \rightarrow$ angular variable, G $\rightarrow$ constant, $\{u_<, u_>\} = min, max\{u', u_o\}$ and

$$f_{4m}\left(u,u_o\right) = f_{1m}(u_o)f_{2m}(u) - f_{1m}(u)f_{2m}(u_o).$$

Type	u_o	G	M	f_1	f_2	f_3
Circle ($\lambda > 0$)	1	$1/4$	$-\infty$	$\mathrm{J}_m\left(\sqrt{\lambda}\ r\right)$	$\mathrm{N}_m\left(\sqrt{\lambda}\ r\right)$	$\cos\left(m(\theta-\theta')\right)$
Circle ($\lambda < 0$)	1	$-1/(2\pi)$	$-\infty$	$\mathrm{I}_m\left(\sqrt{\lvert\lambda\rvert}\ r\right)$	$\mathrm{K}_m\left(\sqrt{\lvert\lambda\rvert}\ r\right)$	$\cos\left(m(\theta-\theta')\right)$
Ellipse	ζ_o	$1/2$	0	$\mathrm{Je}_m(\zeta,\Lambda)$	$\mathrm{Ne}_m(\zeta,\Lambda)$	$\mathrm{ce}_m(\eta,\Lambda)\ \mathrm{ce}_m(\eta',\Lambda)$
($\Lambda > 0$)	ζ_o	$1/2$	1	$\mathrm{Jo}_m(\zeta,\Lambda)$	$\mathrm{No}_m(\zeta,\Lambda)$	$\mathrm{se}_m(\eta,\Lambda)\ \mathrm{se}_m(\eta',\Lambda)$
Ellipse	ζ_o	$-1/\pi$	0	$\mathrm{Ie}_m(\zeta,\Lambda)$	$\mathrm{Ke}_m(\zeta,\Lambda)$	$\mathrm{ce}_m(\eta,\Lambda)\ \mathrm{ce}_m(\eta',\Lambda)$
($\Lambda < 0$)	ζ_o	$-1/\pi$	1	$\mathrm{Io}_m(\zeta,\Lambda)$	$\mathrm{Ko}_m(\zeta,\Lambda)$	$\mathrm{se}_m(\eta,\Lambda)\ \mathrm{se}_m(\eta',\Lambda)$

Radial Mathieu functions $\rightarrow$ Jo_m, Je_m, Ne_m, No_m, Ie_m, Io_m, Ke_m, Ko_m.
Angular Mathieu functions $\rightarrow$ ce_m, se_m ; $\Lambda := \bar{f}^2\lambda/4$.

Analytical solutions I

The circular solution is given by:

$$\psi_0^{\text{circular}}(r,\theta) = Re\left\{\frac{A}{\lambda}\left[1-\frac{J_0\left(\sqrt{\lambda}r\right)}{J_0\left(\sqrt{\lambda}\right)}\right]+\frac{C\pi\cos(\theta)}{2\sqrt{\lambda}}\left[r^2J_2\left(\sqrt{\lambda}r\right)N_1\left(\sqrt{\lambda}r\right)-\frac{J_1\left(\sqrt{\lambda}r\right)J_2\left(\sqrt{\lambda}\right)N_1\left(\sqrt{\lambda}\right)}{J_1\left(\sqrt{\lambda}\right)}\right.\right.$$

$$\left.\left.+J_1\left(r\sqrt{\lambda}\right)\left[G^{2,1}_{2,4}\left(\frac{\sqrt{\lambda}}{2},\frac{1}{2}\middle| \begin{matrix}0,-\frac{1}{2}\\ 0,1,-1,-\frac{1}{2}\end{matrix}\right)-\frac{\sqrt{\lambda}r^3}{2}G^{2,1}_{2,4}\left(\frac{r\sqrt{\lambda}}{2},\frac{1}{2}\middle| \begin{matrix}-\frac{1}{2},-1\\ -\frac{1}{2},\frac{1}{2},-\frac{3}{2},-1\end{matrix}\right)\right]\right]\right\}.$$

For $\Lambda > 0$ the elliptical solution is given by:

$$\psi_0^{\text{elliptical}}(\zeta,\eta) = \sum_{m=0}^{\infty}\left[A_{2m}(\zeta,\Lambda)\mathrm{ce}_{2m}(\eta,\Lambda)+C_{2m+1}(\zeta,\Lambda)\mathrm{se}_{2m+1}(\eta,\Lambda)\right],$$

$$A_m(\zeta,\Lambda) := \frac{A\bar{f}^2\pi}{4\mathrm{Je}_m(\zeta_o,\Lambda)}\int_0^{\zeta_o}\mathrm{Je}_m(\zeta_<,\Lambda)\mathrm{JNe}_m(\zeta_>,\Lambda)\mathrm{KA}_m(\zeta',\Lambda)d\zeta',$$

$$C_m(\zeta,\Lambda) := \frac{C\bar{f}^3\pi}{4\mathrm{Jo}_m(\zeta_o,\Lambda)}\int_0^{\zeta_o}\mathrm{Jo}_m(\zeta_<,\Lambda)\mathrm{JNo}_m(\zeta_>,\Lambda)\mathrm{KC}_m(\zeta',\Lambda)d\zeta'.$$

Analytical solutions II

$$\mathrm{JNe}_m(\zeta,\Lambda) := \mathrm{Je}_m(\zeta_o,\Lambda)\mathrm{Ne}_m(\zeta,\Lambda) - \mathrm{Je}_m(\zeta,\Lambda)\mathrm{Ne}_m(\zeta_o,\Lambda),$$
$$\mathrm{JNo}_m(\zeta,\Lambda) := \mathrm{Jo}_m(\zeta_o,\Lambda)\mathrm{No}_m(\zeta,\Lambda) - \mathrm{Jo}_m(\zeta,\Lambda)\mathrm{No}_m(\zeta_o,\Lambda).$$
$$\mathrm{KA}_m(\zeta,\Lambda) := 2D_0^m(\Lambda)\cosh(2\zeta) - D_2^m(\Lambda),$$
$$\mathrm{KC}_m(\zeta,\Lambda) := \left[B_1^m(\Lambda)\left[\frac{1}{2}+\cosh(2\zeta)\right] - \frac{1}{2}B_3^m(\Lambda)\right]\sinh(\zeta).$$

$D_0^m(\Lambda)$, $D_2^m(\Lambda)$, $B_1^m(\Lambda)$ and $B_3^m(\Lambda)$ are Fourier components of the angular Mathieu functions.
For $\Lambda > 0$ the D-shaped solution is given by:

$$\psi_0^{(i)}{}^{\text{D-shape}}(\zeta,\eta) = \sum_{n=0}^{\infty} C_n^{(i)}\mathrm{Je}_n(\zeta,\Lambda)\mathrm{ce}_n(\eta,\Lambda) + \sum_{n=1}^{\infty} A_n^{(i)}\mathrm{Jo}_n(\zeta,\Lambda)\mathrm{se}_n(\eta,\Lambda).$$

$C_n^{(i)}$ and $A_n^{(i)}$ are determined with the aid of the elliptical Green's function by a contour integral of the appropriate inhomogeneity over the elliptical boundary. The $\Lambda < 0$ case is solved in a similar way.

Bibliography and Acknowledgments

- J. E. Rice, A. C. Ince-Cushman, M. L. Reinke, Y. Podpaly, M. J. Greenwald, B. LaBombard and E. S. Marmar, *Plasma Phys. Contr.Fusion*, **50**, 124042 (2008).
- E. Hameiri, *Physics of Plasmas*, **20**, 024504 (2013).
- L. Guazzotto, R. Betti, J. Manickam, and S. Kaye *Phys. Plasmas*, **11**, 604 (2004).
- N. W. McLachlan, *Theory and Application of Mathieu Functions* (Claredon Press, Oxford England, 1951).
- E. J. Hinch, *Perturbation Methods* (Cambridge University Press, Oxford England, 1991).
- J. P. Freidberg, *Ideal MHD*, (Cambridge University Press, 2014).

This work has been supported by the U.S. Department of Energy under Contract No. DE-FG02-93ER54215.