Update on Verification of delta-F Kinetic Implementation

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Thesis

- Poloidal flow damping serves as a test case to benchmark NIMROD kinetic formulations.
- A delta-F kinetic formulation has been implemented in the code.
- Numerical results disagree slightly (12-15% error) with analytic predictions for the steady-state ion poloidal flow.
- A higher order analytic approach for the steady-state poloidal flow does not solve this discrepancy.
- Potential reasons for this discrepancy are discussed herein.
We are solving self-consistently the drift kinetic equation (DKE) in NIMROD

- Drift kinetic equation, as constructed from Hazeltine [1]:

\[
\frac{\partial f}{\partial t} + (\mathbf{v}_\parallel + \mathbf{v}_D) \cdot \nabla f + \left[ \mu \frac{\partial B}{\partial t} + e (\mathbf{v}_\parallel + \mathbf{v}_D) \cdot \mathbf{E} \right] \frac{\partial f}{\partial w} = C(f)
\]

- \( w = \text{kinetic energy, and } \mu = \text{magnetic moment} \)

- \( \mathbf{v}_\parallel = \sigma \sqrt{\frac{2}{m}} (w - \mu B) \)

- \( \mathbf{v}_D = \frac{E \times B}{B^2} + \frac{1}{\Omega} \mathbf{b} \times \left( \mu \nabla B + \mathbf{v}_\parallel \frac{\partial \mathbf{b}}{\partial t} + \mathbf{v}_\parallel^2 \mathbf{b} \cdot \nabla \mathbf{b} \right) \)

- In NIMROD the \( \dot{\mu} \) terms are also added in for numerical regularity purposes.
We expand $f$ in a perturbation expansion

- We use the small parameter $\delta \overset{\text{def}}{=} \frac{\rho}{L}$, where $\rho$ = the Larmor radius, and $L$ = a typical macroscopic length scale.

- In the “delta-F” approach, $f_0 = n_0 \left( \frac{m}{2\pi T_0} \right)^{3/2} e^{-\frac{mv^2}{2T_0}}$

  where $n_0$ and $T_0$ are the equilibrium number density and temperature (Grad-Shafranov solutions).

- Then, $f = f_0 + f_1 + \ldots$, and we solve the first-order DKE in NIMROD for $f_1$. 
Some additional assumptions are made for the delta-F approach

- delta-F assumptions:
  \[ \frac{\partial B}{\partial t} = 0 \] and \( B \) axisymmetric.
  \[ m_i n_i \frac{\partial \langle Ru_{i1\zeta} \rangle}{\partial t} \approx 0 \]

- This last assumption leads to a closure of the form:
  \[ \phi'_0 = \frac{\langle \frac{u_{i1\parallel}}{B} \rangle - \langle \frac{u_{i1\parallel}}{B} \rangle_{t=0}}{\langle \frac{B^2}{\theta} \rangle} - \frac{p'_0}{e n_0} \]

- In a Chapman-Enskog-like approach \( \phi'_0 \) doesn’t directly enter the calculation.
To check the validity of the delta-F implementation, we look at the steady-state poloidal ion flow

- Analytic results can be obtained [2,3].
- A traditional approach keeps 2 vector moments (the velocity and heat flux of each species), and two tensor moments (stress tensor and heat flux stress tensor).
- In the banana regime, this leads to an inversion of a 2x2 matrix which leads to the following formula for the steady-state ion poloidal flow:

\[
\frac{u_\theta}{B_\theta} = C(f_t) \frac{I}{e \langle B^2 \rangle} \frac{dT_i}{d\psi}, \quad \text{where} \quad C(f_t) = \left( \frac{1.173}{1 + 0.462 \frac{f_t}{1-f_t}} \right), \quad \text{and}
\]

\[
f_t = 1 - \frac{3}{4} \langle B^2 \rangle \int_0^{\lambda_c(\psi)} d\lambda \frac{\lambda}{\langle \sqrt{1 - \lambda B} \rangle}.
\]
This analytic procedure can be extended to arbitrarily higher order

- One more vector moment and tensor moment can be retained, leading to an inversion of a 3x3 matrix [3].
- The steady state poloidal flow has the same form as with the 2x2 matrix inversion, but has a different form for $C \left( f_t \right)$. 
- Although the form for $C \left( f_t \right)$ is different, it’s value doesn’t differ much from that of the 2x2 matrix approach for representative values of $f_t$. 
Formula for $C(f_t)$ differs when using 3x3 matrix approach, but small quantitative difference

$$C(f_t) = \left( \frac{1.173 \left( 1.003 + 2.929 \frac{f_t}{1-f_t} + 2.666 \left( \frac{f_t}{1-f_t} \right)^2 + 0.904 \left( \frac{f_t}{1-f_t} \right)^3 + 0.101 \left( \frac{f_t}{1-f_t} \right)^4 \right)}{1 + 3.408 \frac{f_t}{1-f_t} + 4.071 \left( \frac{f_t}{1-f_t} \right)^2 + 2.162 \left( \frac{f_t}{1-f_t} \right)^3 + 0.512 \left( \frac{f_t}{1-f_t} \right)^4 + 0.043 \left( \frac{f_t}{1-f_t} \right)^5} \right)$$

Value of $C(f_t)$ for different $f_t$:

<table>
<thead>
<tr>
<th>$f_t$</th>
<th>2x2</th>
<th>3x3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.091</td>
<td>1.1212</td>
<td>1.1184</td>
</tr>
<tr>
<td>0.474</td>
<td>0.8285</td>
<td>0.8197</td>
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<tr>
<td>0.697</td>
<td>0.5687</td>
<td>0.5651</td>
</tr>
<tr>
<td>0.800</td>
<td>0.4119</td>
<td>0.4133</td>
</tr>
</tbody>
</table>
For an NSTX Equilibrium, we use five synthetic probes located at the following locations.
We use a Gaussian initial condition for $f_1$.
For the NSTX equilibrium, $\nu_\star \approx 10^{-4}$ is found to be sufficient for convergence of $C(f_t)$

$C(f_t)$ vs. $\log_{10}(\nu_\star)$

Max Difference between Prediction and Result: 15.8%
Min Difference between Prediction and Result: 12.8%

Polynomial Degree in $\chi_1$ is 25 here.
For an NSTX equilibrium, convergence in pitch angle for $C(f_t)$ occurs at high polynomial degree.

- “even” poly degree results still increasing, “odd” poly degree results have reached stationary value.
Potential reasons for discrepancy between analytic steady-state poloidal flow predictions and results

- Perhaps the incorporating of cubic splines in the evaluation of the flux-surface averaged quantities in the DKE prevents complete convergence of results.
- Maybe there is a boundary layer near the trapped-passing boundary that is not being resolved (something a trapped-passing grid could help with).
- Andy and Eric are working on calculating the Rosenbluth potentials in field particle collision term more accurately in NIMROD. This could potentially effect error in $C(f_t)$. (This would also explain why error in “even” poly degree in $\xi$ is greater).
Summary

• A delta-F kinetic formulation has been implemented in the code.

• Numerical results differ from predictions by a max difference of about 16%.

• There are potential explanations for these discrepancies, which need to be tested.

Future Work

• Refine errors in delta-F implementation

• Test the Chapman-Enskog-Like formulation in NIMROD to benchmark against delta-F formulation and analytics.
References

