# General closure theory for magnetized plasmas

Jeong-Young Ji and Eric D. Held

Department of Physics, Utah State University

MINROD Team Meeting

May 1, 2009

# Closure vs. transport theory

- Closure theory for 5 moment $(n_a, \mathbf{V}_a, T_a)$ equations

$$d_a n_a + n_a \nabla \cdot \mathbf{V}_a = 0 \, (d_a \equiv \partial_t + \mathbf{V}_a \cdot \nabla)$$

$$\tfrac{3}{2} n_a d_a T_a + n_a T_a \nabla \cdot \mathbf{V}_a + \nabla \cdot \mathbf{q}_a + \nabla \mathbf{V}_a : \boldsymbol{\pi}_a = Q_a$$

$$m_a n_a d_a \mathbf{V}_a - n_a e_a (\mathbf{E} + \mathbf{V}_a \times \mathbf{B}) + \nabla p_a + \nabla \cdot \boldsymbol{\pi}_a = \mathbf{R}_a$$

Solve the kinetic equation or equivalently ($\mathsf{N}_a$:non-Maxwellian moments)

$$D\mathsf{N}_a + \Omega_a \mathbf{b} \times \mathsf{N}_a = C\mathsf{N}_a + \mathsf{G}_a$$

Express $\mathbf{q}_a(\mathsf{N}_a^{11})$, $\boldsymbol{\pi}_a(\mathsf{N}_a^{20})$, $Q_a$, $\mathbf{R}_a$ in terms of $n_a, \mathbf{V}_a, T_a$

$$\mathbf{R}_\mathrm{e} = -(\alpha)(\mathbf{V}_\mathrm{ei}) - (\beta)(\nabla T_\mathrm{e}), \;\; \mathbf{q}_\mathrm{e} = (\beta)(\mathbf{V}_\mathrm{ei}) - (\kappa)(\nabla T_\mathrm{e})$$

- Transport theory

Solve momentum balance equation

$$ne\mathbf{E}' + ne\mathbf{V}_\mathrm{ei} \times \mathbf{B} = \mathbf{R}_\mathrm{e} \text{ where } \mathbf{E}' = \mathbf{E} + \mathbf{V}_\mathrm{i} \times \mathbf{B} + (ne)^{-1} \nabla p_\mathrm{e}$$

Express fluxes in terms of thermodynmic drives

$$\mathbf{J} = (\sigma)(\mathbf{E}') - (\alpha')(\nabla T_\mathrm{e}), \;\; \mathbf{q}_\mathrm{e} = (\alpha')(\mathbf{E}') - (\kappa')(\nabla T_\mathrm{e})$$

## Transport/closure theories

| | Braginskii | Neoclassical | transport | Unified closure |
|---|---|---|---|---|
| Collisionality | high | high: PS | low | general$^{\star 1}$ |
| Magnetic field strength | general | strong | strong | strong |
| Magnetic geometry | general | nested | nested | general$^{\star 2}$ |
| Collision operator | Landau | Landau | model | Landau$^{\star 3}$ |

- Solve general moment equations (not the drift kinetic equation)

$\star 1$ No ordering on collisionality (no subsidiary expansion) $\Rightarrow$ Braginskii

$\star 2$ No flux surface average

$\star 3$ Exact full linearized Coulomb collision operators

# Moment expansion of a distribution function: $\mathbf{s}_a = \mathbf{v}_a / v_{Ta}$

- Landau kinetic equation

$$\partial_t f_a + \mathbf{v} \cdot \nabla f_a + \frac{e_a}{m_a}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \partial_{\mathbf{v}} f_a = \sum_b C(f_a, f_b)$$

- Moment expansion: expansion coefficients, $\mathsf{M}^{lk}$'s, are symmetric traceless fluid moments

$$f_a(t, \mathbf{x}, \mathbf{v}) = f_a^{(0)} \sum_{lk} \frac{1}{\sqrt{\sigma_k^l}} \mathsf{M}_a^{lk}(t, \mathbf{x}) \cdot \mathsf{P}^{lk}(\mathbf{s}_a)$$

$$\mathsf{N}_a^{lk} \equiv n_a \mathsf{M}_a^{lk}(t, \mathbf{x}) = \int d\mathbf{v} \frac{1}{\sqrt{\sigma_k^l}} \mathsf{P}_a^{lk} f_a(t, \mathbf{x}, \mathbf{v})$$

- $\mathsf{P}^{lk}$'s are orthogonal, irreducible, tensorial polynomials and form a complete set

$$\int d\mathbf{v} \mathsf{P}^{jp} \mathsf{P}^{lk} \cdot \mathsf{M}^{lk} f^{(0)} = \delta_{jl} \delta_{pk} \sigma_p^j \mathsf{M}^{jp}$$

# Moment equations for electrons and ions ($a$ =e,i)

Ji and Held, PoP (2006, 2008)

$$\hat{D}_a \mathsf{N}_a + \Omega_a \mathbf{b}\breve{\times}\mathsf{N}_a = (\hat{C}_{aa} + \hat{A}_{ab})\mathsf{N}_a + \mathsf{G}_a$$

$$\mathsf{N}_a = \begin{bmatrix} \mathsf{N}_a^0 \\ \mathsf{N}_a^1 \\ \mathsf{N}_a^2 \\ \mathsf{N}_a^3 \\ \vdots \end{bmatrix}, \ \mathsf{N}_a^0 = \begin{pmatrix} \mathsf{N}_a^{02} \\ \mathsf{N}_a^{03} \\ \mathsf{N}_a^{04} \\ \vdots \end{pmatrix}, \ \mathsf{N}_a^1 = \begin{pmatrix} \mathsf{N}_a^{11} \\ \mathsf{N}_a^{12} \\ \mathsf{N}_a^{13} \\ \vdots \end{pmatrix}, \ \mathsf{N}_a^2 = \begin{pmatrix} \mathsf{N}_a^{20} \\ \mathsf{N}_a^{21} \\ \mathsf{N}_a^{22} \\ \vdots \end{pmatrix}, \ \cdots$$

$$\mathsf{G}_a = \begin{bmatrix} 0 \\ \mathsf{G}_a^1 \\ \mathsf{G}_a^2 \\ 0 \\ \vdots \end{bmatrix}, \ \mathsf{G}_a^1 = \begin{pmatrix} \mathsf{G}_{Ta}^{11} + \delta_{a\mathrm{e}}\hat{A}_{\mathrm{ei}}^{11,00} n_\mathrm{e} \\ \delta_{a\mathrm{e}}\hat{A}_{\mathrm{ei}}^{12,00} n_\mathrm{e} \\ \delta_{a\mathrm{e}}\hat{A}_{\mathrm{ei}}^{13,00} n_\mathrm{e} \\ \vdots \end{pmatrix}, \ \mathsf{G}_a^2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} n_a \mathsf{W}_a \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$\mathsf{G}_{Ta}^{11} = \sqrt{\sigma_1^1} n_a v_{Ta} \nabla \ln T_a, \ \hat{A}_{\mathrm{ei}}^{1k,00} = \tau_{\mathrm{ei}}^{-1}\sqrt{2} a_{\mathrm{ei}}^{10k} \frac{\mathbf{V}_{\mathrm{ei}}}{v_{T\mathrm{e}}}$$

$$\mathsf{W}_a = \nabla \mathbf{V}_a + \widetilde{\nabla \mathbf{V}}_a - \frac{2}{3}\mathsf{I}\nabla \cdot \mathbf{V}_a$$

# Small gyroradius ordering $\delta = \rho/L \ll 1$

$$\hat{D}_a \mathsf{N}_a + \underbrace{\Omega_a}_{\delta^{-1}} \mathbf{b}\breve{\times}\mathsf{N}_a = \hat{C}_a \mathsf{N}_a + \mathsf{G}_a$$

$$\mathsf{N}_a = \mathsf{N}_a^{(0)} + \mathsf{N}_a^{(1)} + \mathsf{N}_a^{(2)} + \cdots\,, \quad \mathsf{G}_a = \mathsf{G}_a^{(0)} + \mathsf{G}_a^{(1)}$$

$$\begin{aligned}(\hat{D}_a \mathsf{N}_a)^{lp} \quad = \quad & \hat{\Psi}^{l+}_{pk} \nabla \cdot \mathsf{N}_a^{l+1,k} + \hat{\Psi}^{l-} \nabla \mathsf{N}_a^{l-1,k} \\ & + \hat{\Phi}^{l+}_{pk} \nabla \ln T \cdot \mathsf{N}_a^{l+1,k} + \hat{\Phi}^{l-} \nabla \ln T \mathsf{N}_a^{l-1,k} \\ & + \hat{\Theta}^{l+}_{pk} \mathbf{E} \cdot \mathsf{N}_a^{l+1,k} + \hat{\Theta}^{l-} \mathbf{E} \mathsf{N}_a^{l-1,k}\end{aligned}$$

- $\mathsf{G}^{(1)} \neq 0$ only when $\mathsf{G}^{(0)} = 0$
  - Flux surfaces: $\nabla_\| T^{(0)} = 0 \Rightarrow \nabla_\| T = \nabla_\| T^{(1)}$
  - Neoclassical ordering: $\mathbf{V} = \delta^1 \mathbf{V}_1$
- $\delta^0$: $\Omega \mathbf{b}\breve{\times}\mathsf{N}^{(0)} = 0$
- $\delta^1$: $\hat{D}\mathsf{N}^{(0)} + \Omega \mathbf{b}\breve{\times}\mathsf{N}^{(1)} = \hat{C}\mathsf{N}^{(0)} + \mathsf{G}^{(0)}$
- $\delta^2$: $\hat{D}\mathsf{N}^{(1)} + \Omega \mathbf{b}\breve{\times}\mathsf{N}^{(2)} = \hat{C}\mathsf{N}^{(1)} + \mathsf{G}^{(1)}$

# Solutions of $\mathrm{b}\breve{\times}\mathsf{N}^{lk(0)} = 0$

$$\mathsf{b}_{\parallel}\mathbf{V} = \mathbf{bb}\cdot\mathbf{V} \equiv \mathbf{V}_{\parallel},\ \mathsf{b}_{\times}\mathbf{V} \equiv \mathbf{b}\times\mathbf{V} = \mathbf{V}_{\times},\ \mathsf{b}_{\perp}\mathbf{V} \equiv (\mathsf{I}-\mathbf{bb})\cdot\mathbf{V} = \mathbf{V}_{\perp}$$

$$\mathsf{b}_{\times\perp}\mathsf{W} \equiv \mathsf{W}_{\times\perp},\ \mathsf{K}^{-1} \equiv \tfrac{1}{2}\mathsf{b}_{\times\perp} + 2\mathsf{b}_{\parallel\times}$$

- Vector moments: $\mathbf{b}\times\mathsf{N}^{1k(0)} = 0 \ \Rightarrow\ \mathsf{N}_{\perp}^{1k(0)} = 0$
- Rank 2 tensor moments: $\mathbf{b}\breve{\times}\mathsf{N}^{2k(0)} = \mathbf{b}\times\mathsf{N}^{2k(0)} - \mathsf{N}^{2k(0)}\times\mathbf{b} = 0$

$$\mathsf{K}^{-1}(\mathbf{b}\times\mathsf{N}^{2k(0)} - \mathsf{N}^{2k(0)}\times\mathbf{b}) = \mathsf{N}_{\mathrm{CGL}}^{2k(0)} - \mathsf{N}^{2k(0)} = 0$$

$$\mathsf{N}^{2k(0)} = \mathsf{N}_{\mathrm{CGL}}^{2k(0)} \equiv N_{\parallel}^{2k(0)}(-\tfrac{1}{2}\mathbf{e}_1\mathbf{e}_1 - \tfrac{1}{2}\mathbf{e}_2\mathbf{e}_2 + \mathbf{bb}) = \tfrac{3}{2}N_{\parallel}^{2k(0)}(\mathbf{bb} - \tfrac{1}{3}\mathsf{I})$$

- General-rank tensor moments $(l \geq 3)$: $\mathbf{b}\breve{\times}\mathsf{N}^{lk(0)} = 0$
  Coupled equations for $\mathsf{N}_{\parallel\cdots\times\cdots\perp\cdots}^{lk(0)} \Leftarrow N_{\parallel}^{lk(0)} \equiv (\mathbf{b}\cdots\mathbf{b})\cdot\mathsf{N}_{\parallel}^{lk(0)}$

$$\mathsf{N}_{\parallel\cdots\parallel\perp\cdots\perp}^{lk(0)} = 0 \text{ for odd number of } \perp$$

# Parallel $\delta^1$ order equations $\bar{D}_\parallel \bar{N}_\parallel^{(0)} = \hat{C}\bar{N}_\parallel^{(0)} + \bar{G}_\parallel^{(0)}$

$$(\mathbf{b}\cdots\mathbf{b})\cdot[\hat{D}\mathsf{N}^{(0)} + \Omega\mathbf{b}\breve{\times}\mathsf{N}^{(1)} = \hat{C}\mathsf{N}^{(0)} + \mathsf{G}^{(0)}]$$

$$\nabla\cdot\mathbf{b} = \nabla\cdot\frac{\mathbf{B}}{B} = -\partial_\ell \ln B$$

$$[\nabla\cdot(A_{ij\dots k}\underbrace{\mathbf{e}_i\mathbf{e}_j\cdots\mathbf{e}_k}_{l})]\cdot\underbrace{\mathbf{b}\cdots\mathbf{b}}_{l-1} = \partial_\parallel A_{\parallel\dots\parallel} - \frac{l+1}{2}A_{\parallel\dots\parallel}\partial_\ell \ln B$$

where $\bar{N}_\parallel^{lk} = \sqrt{\frac{\bar{\sigma}_l}{\sigma_l}}N_\parallel^{lk}$, $\bar{G}_\parallel^{lk} = \sqrt{\frac{\bar{\sigma}_l}{\sigma_l}}G_\parallel^{lk}$ and

$$\begin{aligned}(\bar{D}_\parallel\bar{N}_\parallel^{(0)})^{jp} \quad = \quad & v_T\left(\bar{\Psi}_{pk}^{j-}\partial_\ell\bar{N}_\parallel^{(0)j-1,k} + \bar{\Psi}_{pk}^{j+}\partial_\ell\bar{N}_\parallel^{(0)j+1,k}\right)\\ & + v_T\partial_\ell \ln B\left(\bar{\Psi}_{pk}^{j-}\frac{l-1}{2}\bar{N}_\parallel^{(0)j-1,k} - \bar{\Psi}_{pk}^{j+}\frac{l+2}{2}\bar{N}_\parallel^{(0)j+1,k}\right)\\ & + v_T\partial_\ell \ln T\left(\bar{\Phi}_{pk}^{j-}N^{(0)j-1,k} + \bar{\Phi}_{pk}^{j+}N^{(0)j+1,k}\right)\\ & + \frac{e_a}{mv_T}E_\parallel\left(\bar{\Theta}_{pk}^{j-}N^{(0)j-1,k} + \bar{\Theta}_{pk}^{j+}N^{(0)j+1,k}\right)\end{aligned}$$

$$\bar{D}_\parallel\bar{N}_\parallel^{(0)} \Rightarrow [\Psi]\partial_z\bar{N}_\parallel^{(0)} + \{\partial_z \ln B[\Psi_B] + [\Phi](\partial_z \ln T) + \hat{E}_\parallel[\Theta]\}\bar{N}_\parallel^{(0)}$$

## Parallel $\delta^1$ order equations $\Rightarrow N_\parallel^{(0)}$

$$[\psi]\partial_z \bar{N}_\parallel^{(0)} = [c]\bar{N}_\parallel^{(0)} + g_\parallel^{(0)} - \{\partial_z \ln B[\Psi_B] + [\Phi](\partial_z \ln T) + \hat{E}_\parallel[\Theta]\}\bar{N}_\parallel^{(0)}$$

- Equivalent to $v_\parallel \partial_\parallel \bar{f}_{a0} = C(\bar{f}_{a0})$ with $\bar{f}_a = \sum_{lk} f_a^{(0)} \hat{P}_l(\xi) s_a^l \hat{L}_k^l(s_a^2) M_{a\parallel}^{lk}$
- Solution of $[\psi]\partial_z \bar{N}_\parallel^{(0)} = [c]\bar{N}_\parallel^{(0)} + g_\parallel^{(0)}$ $\Rightarrow$ integral closures

$$\begin{aligned}
\bar{N}_\parallel^{1p(0)}(y) &= \int [K^{1p,1k}(y-z) g_\parallel^{1k(0)}(z) + K^{1p,20}(y-z) g_\parallel^{20(0)}(z)] dz \\
\bar{N}_\parallel^{20(0)}(y) &= \int [K^{20,1k}(y-z) g_\parallel^{1k(0)}(z) + K^{20,11}(y-z) g_\parallel^{11(0)}(z)] dz
\end{aligned}$$

where $dz = d\ell / v_T \tau = d\ell / L_{\mathrm{C}}$ [Ji Held Sovinec 2009 PoP]

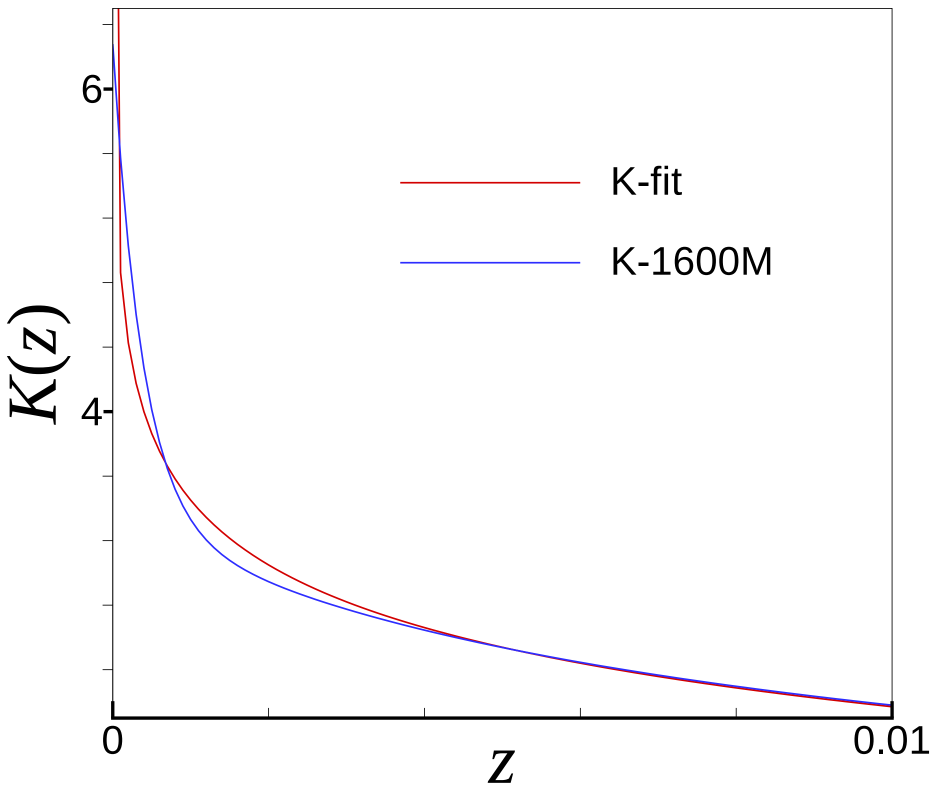
$$g_{a\parallel}^{1k} = \delta_{k1} \frac{\sqrt{5}}{2} \frac{n_a}{T_a} \frac{\partial T_a}{\partial z} + \delta_{a\mathrm{e}} Z \sqrt{2} a_{\mathrm{ei}}^{10k} \frac{V_{\mathrm{ei}\parallel}}{v_{T\mathrm{e}}}, \quad g_{a\parallel}^{20} = -\tau_{aa} \frac{\sqrt{3}}{2} n_a \mathbf{bb} : \mathsf{W}_a$$

- Iteration for more accurate solutions

$$\bar{N}_\parallel^{jp(0)}(y) = \int [K^{jp,mq}(y-z) \{\partial_z \ln B[\Psi_B] + [\Phi](\partial_z \ln T) + \hat{E}_\parallel[\Theta]\}^{mq,lk} \bar{N}_\parallel^{lk(0)}(z)] dz$$

# Simple fitting functions for kernels

Heat flow $q_{e\|} = -\frac{1}{2} v_{Te} T_e \int K(z - z') \frac{n_e}{T_e} \frac{dT_e(z')}{dz'} dz'$




$$K_{1600M}(z) = \sum_{k_A<0, A=1}^{1600} a_A \exp(-k_A|z|)$$

Collisionless limit

$$q_{\|} \propto \int \frac{T(\ell')}{\ell'} d\ell' \propto -\int \ln \ell' \frac{dT(\ell')}{d\ell'} d\ell'$$

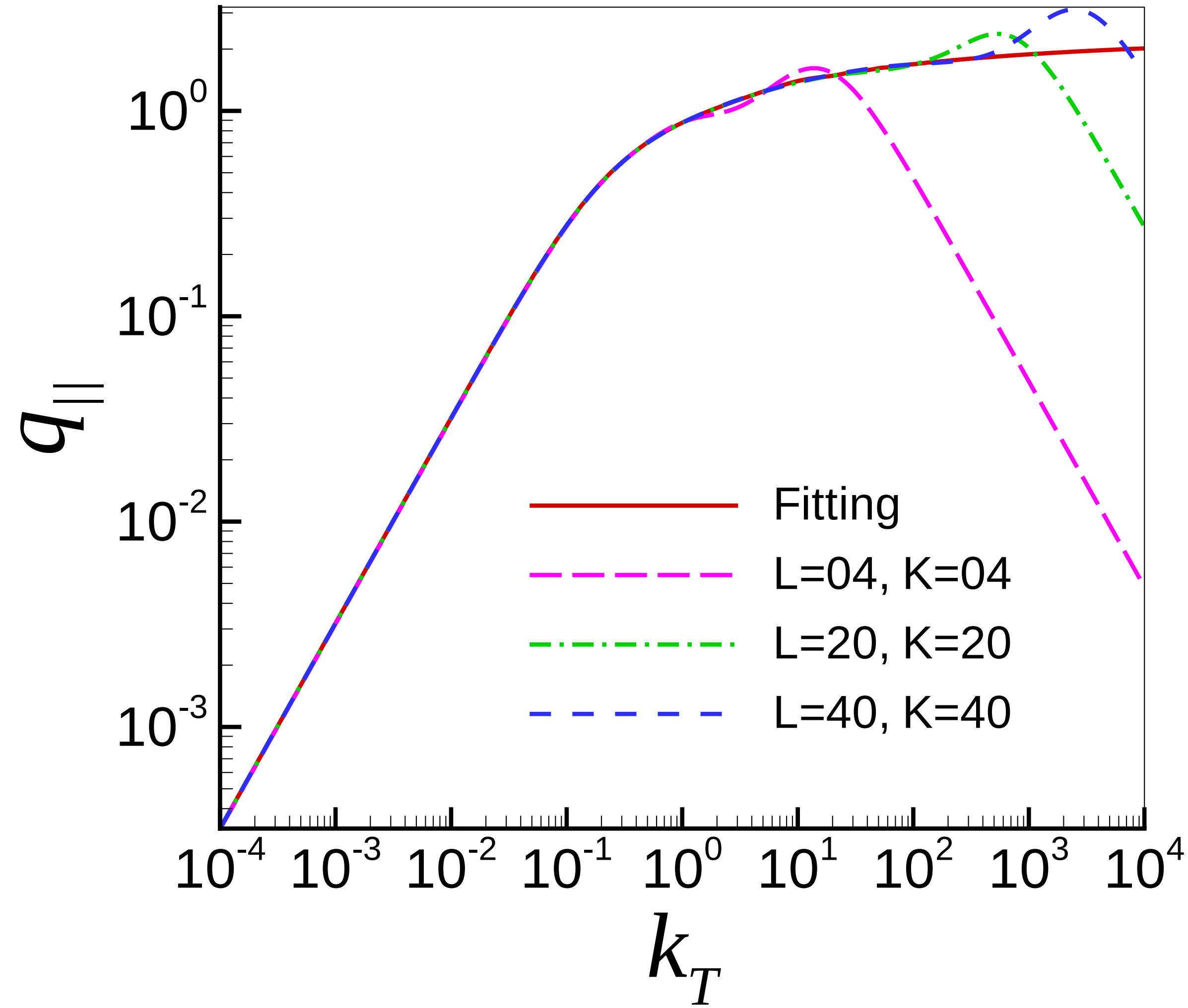
$$K(z \to 0) = -a_0 \ln z$$

$$K_{\text{fit}}(z) = -a(z) \ln[1 - \exp(-z^{0.5466})]$$

$$a(z) = \begin{cases} 0.4260 + 0.7568 \exp(-z^{0.1174})], & \text{for } z \leq 0.3040 \\ a(0.3040) = 0.7432, & \text{otherwise} \end{cases}$$

# Electron heat flow for $T(z) = T_0 + T_1 \sin k_T z$

Longer collision length requires more moments ($k_T = 2\pi L_\mathrm{C}/L_T$)




$K_\mathrm{fit}$ is accurate within 1.3% error for any collisionality

# Perpendicular $\delta^1$ order equations $\Rightarrow \mathsf{N}_\perp^{(1)}$

$$\Omega \mathbf{b} \breve{\times} \mathsf{N}^{l(1)} = C\mathsf{N}^{l(0)} + \mathsf{G}^{l(0)} - (\hat{D}\mathsf{N})^{l(0)}$$

- Vector moments $\mathsf{N}_\perp^{1(1)} = -\Omega^{-1}\mathbf{b} \times [C\mathsf{N}^{1(0)} + \mathsf{G}^{1(0)} - (\hat{D}\mathsf{N})^{1(0)}]$

$$\begin{aligned} \mathsf{N}_\perp^{1p(1)} &= -\Omega^{-1}\mathbf{b} \times \Big[\mathsf{G}^{1p(0)} - \hat{\Psi}_{pk}^{1-}\nabla\mathsf{N}^{0k(0)} - \hat{\Phi}_{pk}^{1-}\nabla T\mathsf{N}^{0k(0)} - \hat{\Theta}_{pk}^{1-}\mathbf{E}\mathsf{N}^{0k(0)} \\ &\quad -\hat{\Psi}_{pk}^{1+}\nabla\cdot\mathsf{N}^{2k(0)} - \hat{\Phi}_{pk}^{1+}\nabla T\cdot\mathsf{N}^{2k(0)} - \hat{\Theta}_{pk}^{1+}\mathbf{E}\cdot\mathsf{N}^{2k(0)}\Big] \end{aligned}$$

$$\begin{aligned} \mathbf{b} \times (\nabla \cdot \mathsf{N}^{2(0)}) &= -\tfrac{1}{2}\mathbf{b} \times \nabla_\perp N_{zz}^{2(0)} \\ \mathbf{b} \times (\nabla T \cdot \mathsf{N}^{2(0)}) &= -\tfrac{1}{2}N_{zz}^{2(0)}\mathbf{b} \times \nabla T \\ \mathbf{b} \times (\mathbf{E} \cdot \mathsf{N}^{2(0)}) &= -\tfrac{1}{2}N_{zz}^{2(0)}\mathbf{b} \times \mathbf{E} \end{aligned}$$

- Viscous stress $\mathbf{b}\breve{\times}\mathsf{N}^{20(1)} = C^{20k}\mathsf{N}^{2k(0)} + \mathsf{G}^{20(0)} - (\hat{D}\mathsf{N}^{(0)})^{20}$

$$\mathsf{N}^{20(1)} = \mathsf{N}_{\mathrm{CGL}}^{20(1)} - \Omega^{-1}\mathsf{K}^{-1}\mathsf{G}^{20(0)}$$

# Parallel $\delta^2$ order $(\hat{D}\mathsf{N})^{l(1)} + \mathbf{b}\breve{\times}\mathsf{N}^{l(2)} = C\mathsf{N}^{l(1)} + \mathsf{G}^{l(1)} \Rightarrow N_\parallel^{(1)}$

- Parallel $\delta^1$ moments $N_\parallel^{(1)}$

$$[\psi]\partial_z \bar{N}_\parallel^{(1)} = [c]\bar{N}_\parallel^{(1)} + g_{\parallel\text{eff}}^{(1)} - \{\partial_z \ln B[\Psi_B] + [\Phi](\partial_z \ln T) + \hat{E}_\parallel[\Theta]\}\bar{N}_\parallel^{(1)}$$

$$\bar{G}_{\parallel\text{eff}}^{(1)} = \bar{G}_\parallel^{(1)} - (\hat{D}\mathsf{N}_\perp^{(1)})_\parallel \;\;\Rightarrow\;\; g_{\parallel\text{eff}}^{(1)} = \tau_{aa}\bar{G}_{\parallel\text{eff}}^{(1)}$$

$$\nabla\cdot\mathsf{N}_\perp^1 = \nabla\cdot\mathsf{N}_\parallel^1 + \nabla\cdot\mathsf{N}_\perp^1 \Rightarrow \partial_z N_\parallel^1 - \bar{G}_{\parallel\psi}^{1(1)}$$

- On flux surfaces $T(\mathbf{x}) = T^{(0)}(\psi) + T^{(1)}(\mathbf{x})$ and $\mathbf{V} = \mathbf{V}^{(1)} \Rightarrow N_\parallel^{(0)} = 0$

$$\mathsf{N}_\perp^{11(1)} = -\frac{\sqrt{5}}{2} n_a v_{Ta} \Omega_a^{-1} \mathbf{b}\times\nabla\ln T_a^{(0)}$$

Axi-symmetric geometry $\mathbf{B} = I(\psi)\nabla\varphi + \nabla\varphi\times\nabla\psi$

$$\bar{G}_{\parallel\psi}^{A(1)} = -\gamma_A\nabla\cdot\frac{n\mathbf{B}\times\nabla T^{(0)}}{e_a B^2} = \gamma_A \mathbf{B}\cdot\nabla\left(\frac{nI}{e_a B^2}\frac{dT^{(0)}}{d\psi}\right)$$

where $\gamma_{02} = -\sqrt{\frac{10}{3}},\ \gamma_{20} = \frac{1}{\sqrt{3}},\ \gamma_{21} = -\sqrt{\frac{7}{6}}$

$$\int K^{11,A}(y-z)\nabla\cdot\frac{n\mathbf{B}\times\nabla T^{(0)}}{e_a B^2} = \int K(y-z)\mathbf{B}\cdot\nabla B^{-2}\frac{nI}{e_a}\frac{dT^{(0)}}{d\psi}$$

# Perpendicular $\delta^2$ order $(\hat{D}\mathsf{N})^{l(1)} + \mathbf{b}\breve{\times}\mathsf{N}^{l(2)} = C\mathsf{N}^{l(1)} + \mathsf{G}^{l(1)} \Rightarrow \mathsf{N}_\perp^{(2)}$

- Perpendicular vector moments

$$\mathsf{N}_\perp^{1(2)} = -\Omega^{-1}\mathbf{b} \times [C\mathsf{N}^{1(1)} + \mathsf{G}^{1(1)} - (\hat{D}\mathsf{N})^{1(1)}]$$

- On axi-symmetric flux surfaces
  - Braginskii

$$-\Omega^{-1}\mathbf{b}\times\mathsf{N}^{11(1)} = \frac{\sqrt{5}}{2} n v_T \Omega^{-2}\mathbf{b}\times(\mathbf{b}\times\nabla \ln T^{(0)}) = -\frac{\sqrt{5}}{2} n v_T \Omega^{-2}\nabla_\perp \ln T^{(0)}$$

  - Neoclassical

$$-\Omega^{-1}\mathbf{b}\times\nabla\mathsf{N}^{02} = -\Omega^{-1}\mathbf{b}\times\nabla\left[\left(\gamma_A \int K^{02,A}(y-z) B\partial_\ell B^{-2} dz\right)\frac{nI}{e_a}\frac{dT^{(0)}}{d\psi}\right]$$

Kernel integral becomes $\theta$ dependent coefficient

- Viscous stress $\mathbf{b}\breve{\times}\mathsf{N}^{20(2)} = C^{20k}\mathsf{N}^{2k(1)} + \mathsf{G}^{20(1)} - (\hat{D}\mathsf{N})^{20(1)}$

$$\mathsf{N}^{20(2)} = \mathsf{N}_{\mathrm{CGL}}^{20(2)} - \Omega^{-1}\mathsf{K}^{-1}[C^{20k}\mathsf{N}^{2k(1)} + \mathsf{G}^{20(1)} - (\hat{D}\mathsf{N}^{(1)})^{20}]$$

# Collisional heating and friction, and $\cdots$

$$
\begin{aligned}
Q_\mathrm{e} &= -Q_\mathrm{i} = 3\mu\frac{n_\mathrm{e}}{\tau_\mathrm{ei}}(T_\mathrm{i}-T_\mathrm{e}) - \mathbf{V}_\mathrm{ei}\cdot\mathbf{R}_\mathrm{e} - \sqrt{\frac{3}{2}}\frac{T_\mathrm{e}}{\tau_\mathrm{ei}}\sum_{k=2}\hat{a}_\mathrm{ei}^{01,0k}\mathsf{n}_\mathrm{e}^{0k}, \\
\mathbf{R}_\mathrm{e} &= -\mathbf{R}_\mathrm{i} = \frac{m_\mathrm{e}n_\mathrm{e}}{\tau_\mathrm{ei}}\mathbf{V}_\mathrm{ei} + \frac{m_\mathrm{e}v_{T\mathrm{e}}}{\tau_\mathrm{ei}}\sum_{k=1}\frac{1}{\sqrt{2}}\hat{a}_\mathrm{ei}^{10k}\mathsf{N}_\mathrm{e}^{1k}
\end{aligned}
$$

$$
\begin{aligned}
R_\parallel &= \frac{m_\mathrm{e}n_\mathrm{e}}{\tau_\mathrm{ei}}V_{\mathrm{ei}\parallel} + \frac{m_\mathrm{e}v_{T\mathrm{e}}}{\tau_\mathrm{ei}}\int[K^{R,V}V_{\mathrm{ei}\parallel} + K^{R,11}\nabla_\parallel T + K^{R,20}\nabla_\parallel V_\parallel] \\
\bar{N}_{a\parallel}^{11} &= \int[\delta_{a\mathrm{e}}K^{11,V}V_{\mathrm{ei}\parallel} + K_a^{11,11}\nabla_\parallel T + K_a^{11,20}\nabla_\parallel V_\parallel] \\
\bar{N}_{a\parallel}^{20} &= \int[\delta_{a\mathrm{e}}K^{20,V}V_{\mathrm{ei}\parallel} + K_a^{20,11}\nabla_\parallel T + K_a^{20,20}\nabla_\parallel V_\parallel]
\end{aligned}
$$

- Transport theory $\Leftarrow$ Closures and

$$
m_a n_a d_a \mathbf{V}_a - n_a e_a(\mathbf{E} + \mathbf{V}_a\times\mathbf{B}) + \nabla p_a + \nabla\cdot\boldsymbol{\pi}_a = \mathbf{R}_a
$$

# Summary and future work

- Unified closures for general collisionality/magnetic geometry
  - Braginskii's theory in the high collision limit
  - More accurate neoclassical transport theory
  - Transport theory without flux surface average
- Find fitting functions for kernels
- Compare with neoclassical transport theory
  - Flux surfaces
  - Axi-symmetric geometry
- Apply to interesting fusion devices