Progress on the MHD closure with kinetic ions and drift kinetic electrons

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Model Motivation

- Current Gyrokinetic-Maxwell equations are not fully electromagnetic.
  - The $\boldsymbol{A}_{\parallel} - \phi$ model does not have $\delta\boldsymbol{B}_{\parallel}$.
- Gyrokinetic ordering may not be valid in some problems, such as Tokamak edge ETG or magnetic reconnection.
  - $\mathbf{E} \times \mathbf{B}$ flow comparable to the ion thermal speed
  - scale length of the equilibrium density or temperature profiles not much larger than ρ_i
  - weak guide-field reconnection
- By treating electrons as massless fluid, this simple hybrid model does not require a guide field, and it is capable of capturing MHD physics in a natural way.
- We are using the GEM code as a test bed for the model and algorithm.

Lorentz ion and fluid electron model

- Lorentz force ions:

$$\frac{d\boldsymbol{v}_i}{dt} = \frac{q}{m_i}(\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{B}), \qquad \frac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i$$

- Isothermal fluid electrons as a simple test:

$$\delta p_e = \gamma \delta n_e T_e = \gamma \delta n_i T_e.$$

Eventually we will add gyrokinetic electrons.

- Ampere's law:

$$\bigtriangledown \times \delta\boldsymbol{B} = \mu_0 e \left(n\boldsymbol{u_i} - n\boldsymbol{u_e}\right)$$

- Faraday's law

$$\bigtriangledown \times \boldsymbol{E} = -\frac{\partial\, \delta\boldsymbol{B}}{\partial t}.$$

Ohm's law

- Starting from the electron momentum equation:

$$\boldsymbol{E} = -\boldsymbol{u_i} \times \boldsymbol{B_0} + \frac{1}{\mu_0 en}(\bigtriangledown \times \delta \boldsymbol{B}) \times \boldsymbol{B_0} + \frac{\eta}{\mu_0} \bigtriangledown \times \delta \boldsymbol{B} - \frac{\nabla p_e}{en} - \frac{m_e}{en}\frac{\partial(n\boldsymbol{u_e})}{\partial t}.$$

- With Ampere's law and ion momentum equation

$$\bigtriangledown \times \delta \boldsymbol{B} = \mu_0 e \left(n\boldsymbol{u_i} - n\boldsymbol{u_e}\right)$$

$$\frac{\partial(n\boldsymbol{u_i})}{\partial t} = \frac{en}{m_i}(\boldsymbol{E} + \boldsymbol{u_i} \times \boldsymbol{B_0}) - \frac{1}{m_i} \bigtriangledown p_i.$$

- And neglect terms with m_e/M_i, we obtain Ohm's law

$$\boldsymbol{E} + \frac{c^2}{w_{pe}^2} \bigtriangledown \times (\bigtriangledown \times \boldsymbol{E}) = -\frac{\boldsymbol{J_i}}{en} \times \boldsymbol{B_0} + \frac{1}{\mu_0 en}(\bigtriangledown \times \delta \boldsymbol{B}) \times \boldsymbol{B_0} + \frac{\eta}{\mu_0} \bigtriangledown \times \delta \boldsymbol{B} - \frac{\gamma T_e \nabla n_i}{en}.$$

Implicit δf algorithm

- δf method for ions:

$$\frac{d}{dt} f_{i1} = -\frac{q}{m_i}(\boldsymbol{E} + \boldsymbol{v} \times \delta \boldsymbol{B}_1) \cdot \frac{\partial}{\partial \boldsymbol{v}} f_{i0}.$$

$$\frac{d}{dt} \omega_i = -\frac{q}{T_i} \boldsymbol{E} \cdot \boldsymbol{v}.$$

where the second equation comes from Maxwellian distribution.

- For ρ_i scale instabilities $k_\perp \rho_i \sim 1, \beta \sim 0.01$, the compressional wave frequency $\frac{\omega}{\Omega_i} \geq 10$, therefore $\Omega_i \Delta t \ll 0.01$ is needed. But in certain cases (e.g. NSTX), $\Omega_i \Delta t \sim 0.1$, which makes implicit method indispensable.

- A first-order scheme has been developed. Here we provide a second-order scheme with improved field solver.

"Particle-in-Cell simulation with Vlasov ions and drift kinetic electrons" by Yang Chen and Scott E Parker, accepted by Phys. of Plasma.

Second order implicit scheme

- Particle push

$$\frac{\boldsymbol{x}^{n+1}-\boldsymbol{x}^{n}}{\Delta t}=(1-\theta)\ \boldsymbol{v}^{n}+\theta\ \boldsymbol{v}^{n+1},$$

$$\frac{\boldsymbol{v}^{n+1}-\boldsymbol{v}^{n}}{\Delta t}=\frac{q}{m}\left((1-\theta)\ (\boldsymbol{E}^{n}+\boldsymbol{v}^{n}\times\boldsymbol{B}_0)+\theta\ (\boldsymbol{E}^{n+1}+\boldsymbol{v}^{n+1}\times\boldsymbol{B}_0)\right),$$

$$\frac{\omega^{n+1}-\omega^{n}}{\Delta t}=\frac{q}{T_{i0}}\left((1-\theta)\ (\boldsymbol{E}^{n}\cdot\boldsymbol{v}^{n})+\theta\ (\boldsymbol{E}^{n+1}\cdot\boldsymbol{v}^{n+1})\right).$$

- Faraday's law

$$\frac{\delta\boldsymbol{B}^{n+1}-\delta\boldsymbol{B}^{n}}{\Delta t}=-[(1-\theta)\ \nabla\times\boldsymbol{E}^{n}+\theta\ \nabla\times\boldsymbol{E}^{n+1}].$$

- Ohm's law:

$$\boldsymbol{E}^{n+1}+\alpha\ \nabla\times\nabla\times\boldsymbol{E}^{n+1}+\theta\,\frac{\Delta t}{\beta_e}\left(\nabla\times\nabla\times\boldsymbol{E}^{n+1}\right)\times\boldsymbol{B}_0$$

$$=-\gamma\,\nabla n_i-\boldsymbol{J}^{\star}\times\boldsymbol{B}_0+\frac{1}{\beta_e}\ (\nabla\times\delta\boldsymbol{B}^{n})\times\boldsymbol{B}_0+\frac{\eta}{\beta_e}\ \nabla\times\delta\boldsymbol{B}^{n}$$

$$-(1-\theta)\,\frac{\Delta t}{\beta_e}(\nabla\times\nabla\times\boldsymbol{E}^{n})\times\boldsymbol{B}_0-(1-\theta)\,\frac{\eta\Delta t}{\beta_e}\ \nabla\times\nabla\times\boldsymbol{E}^{n}.$$

$\alpha=\frac{m_e}{m_i}\frac{1}{\beta_e}+\theta\eta\Delta t$

Ion current

- First half push cycle

$$\begin{aligned}
\boldsymbol{v}^\star &= \boldsymbol{v}^n + (1-\theta)\Delta t \, \frac{q}{m} \left(\boldsymbol{E}^n + \boldsymbol{v}^n \times \boldsymbol{B}_0\right), \\
\boldsymbol{x}^\star &= \boldsymbol{x}^n + (1-\theta)\Delta t \, \boldsymbol{v}^n, \\
\omega^\star &= \omega^n + (1-\theta)\Delta t \, \frac{q}{T_{i0}} \left(\boldsymbol{E}^n \cdot \boldsymbol{v}^n\right).
\end{aligned}$$

- Dependence of $\boldsymbol{J}_i^{n+1}$ on $\boldsymbol{E}_1^{n+1}$

$$\begin{aligned}
\boldsymbol{J}_i^{n+1} &= \boldsymbol{J}_i^\star + \theta \, \Delta t \frac{V}{N} \sum_j \frac{1}{\Delta V} \frac{q}{T_i} \boldsymbol{v}_j \boldsymbol{E}^{n+1}(\boldsymbol{x}_j^{n+1}) \cdot \boldsymbol{v}_j S(\boldsymbol{x} - \boldsymbol{x}_j^{n+1}) \\
&\simeq \boldsymbol{J}_i^\star + \theta \, \Delta t \, \frac{q^2}{m} \, \boldsymbol{E}^{n+1} \equiv \boldsymbol{J}_i'.
\end{aligned}$$

  Iterate on the differences between $\boldsymbol{J}_i^{n+1}$ and $\boldsymbol{J}_i'$ while solving Ohm's law to obtain $\boldsymbol{E}^{n+1}$ (5 iterations is enough).

- Once we have $\boldsymbol{E}^{n+1}$, we could proceed to update $\delta \boldsymbol{B}$ and fulfill the second half push cycle.

Field solver

- Zero-order B field

$$\boldsymbol{B}_0 = \boldsymbol{e}_y \boldsymbol{B}_{0y} + \boldsymbol{e}_z \boldsymbol{B}_{0z}.$$

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- To solve Ohm's law
  - If $\boldsymbol{B}_{0y}$ and the guide field $\boldsymbol{B}_{0z}$ are constants, solve directly in frequency space.

  - If $\boldsymbol{B}_{0y}$ and the guide field $\boldsymbol{B}_{0z}$ are space-dependent, as in the Harris sheet equilibrium, we could solve $\boldsymbol{E}^{n+1}(x_i, k_y, k_z)$ by writting Ohm's law in matrix form.

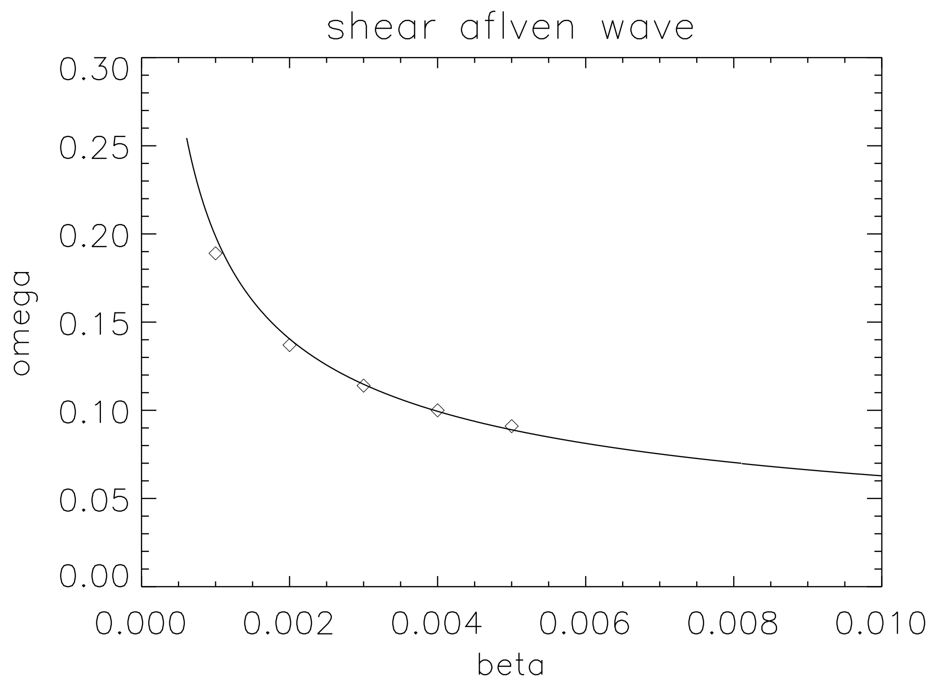
Matrix solver

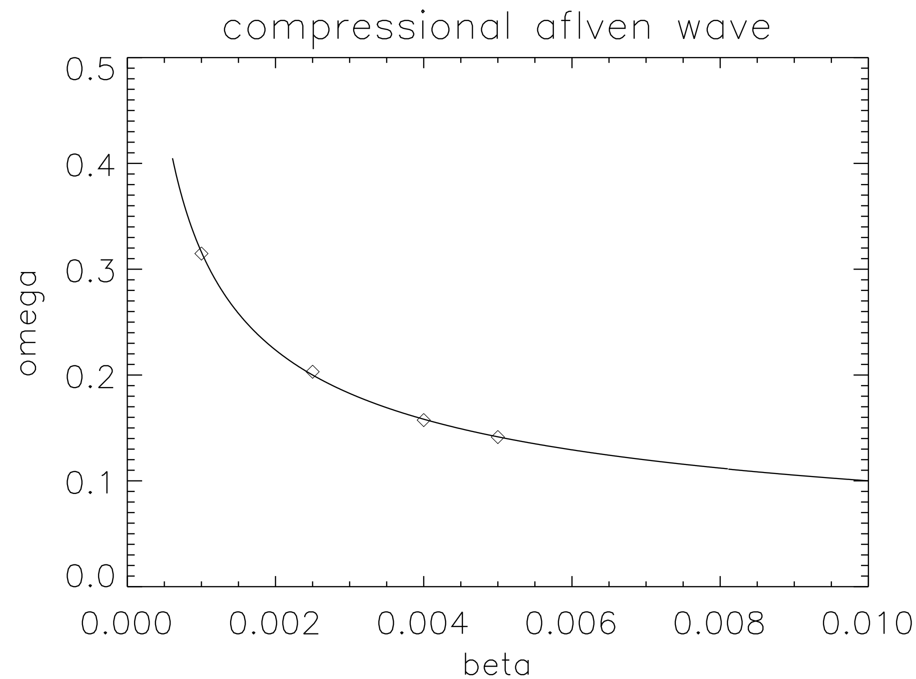
- Matrix equations

$$\begin{pmatrix} M_{xx} & M_{xy} & M_{xz} \\ M_{yx} & M_{yy} & M_{yz} \\ M_{zx} & M_{zy} & M_{zz} \end{pmatrix} \begin{pmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{pmatrix} = \begin{pmatrix} N_x \\ N_y \\ N_z \end{pmatrix}$$

- $\boldsymbol{N}$ stands for the right hand side of Ohm's law and $\tilde{E}_x, \tilde{E}_y, \tilde{E}_z$ are column vectors of $\boldsymbol{E}^{n+1}(x_i, k_y, k_z)$.
- $M_{xx} \cdots$ are $l_x \times l_x$ matrices (almost tridiagonal matrices).
- For every k_y and k_z mode, we have to invert a matrix of size $3l_x \times 3l_x$. l_x is the number of grids on x.

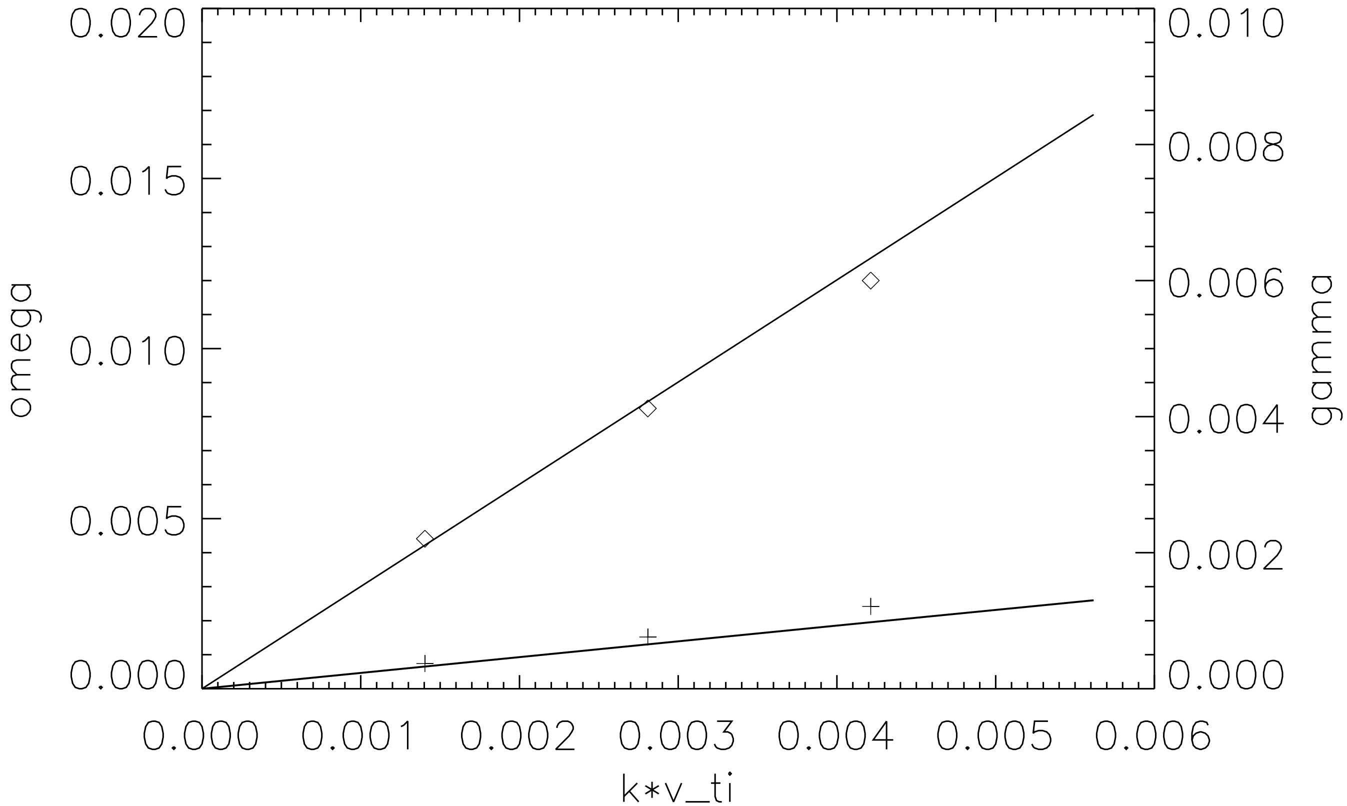
3-D Shearless Slab Alfven waves





$2 \times 32 \times 32$ grids, 131072 particles.
For shear Alfven wave, $k_\perp = 0$, $k_\parallel \rho_i = 0.00628$, initialize with $\delta \boldsymbol{B}_\perp$.
For compressional Alfven wave, $k_\parallel = 0$, $k_\perp \rho_i = 0.01$, initialize with $\delta \boldsymbol{B}_\parallel$. These simulations are done in a tilted B_0 field.

Ion acoustic wave



$2 \times 32 \times 32$ grids, 131072 particles. $k_\perp = 0$.

Whistler wave

- By neglecting ion current and electron inertia, the Ohm's law yields

$$\boldsymbol{E} = \frac{1}{\beta_e} (\nabla \times \delta \boldsymbol{B}) \times \boldsymbol{B}_0.$$

- Numerical form

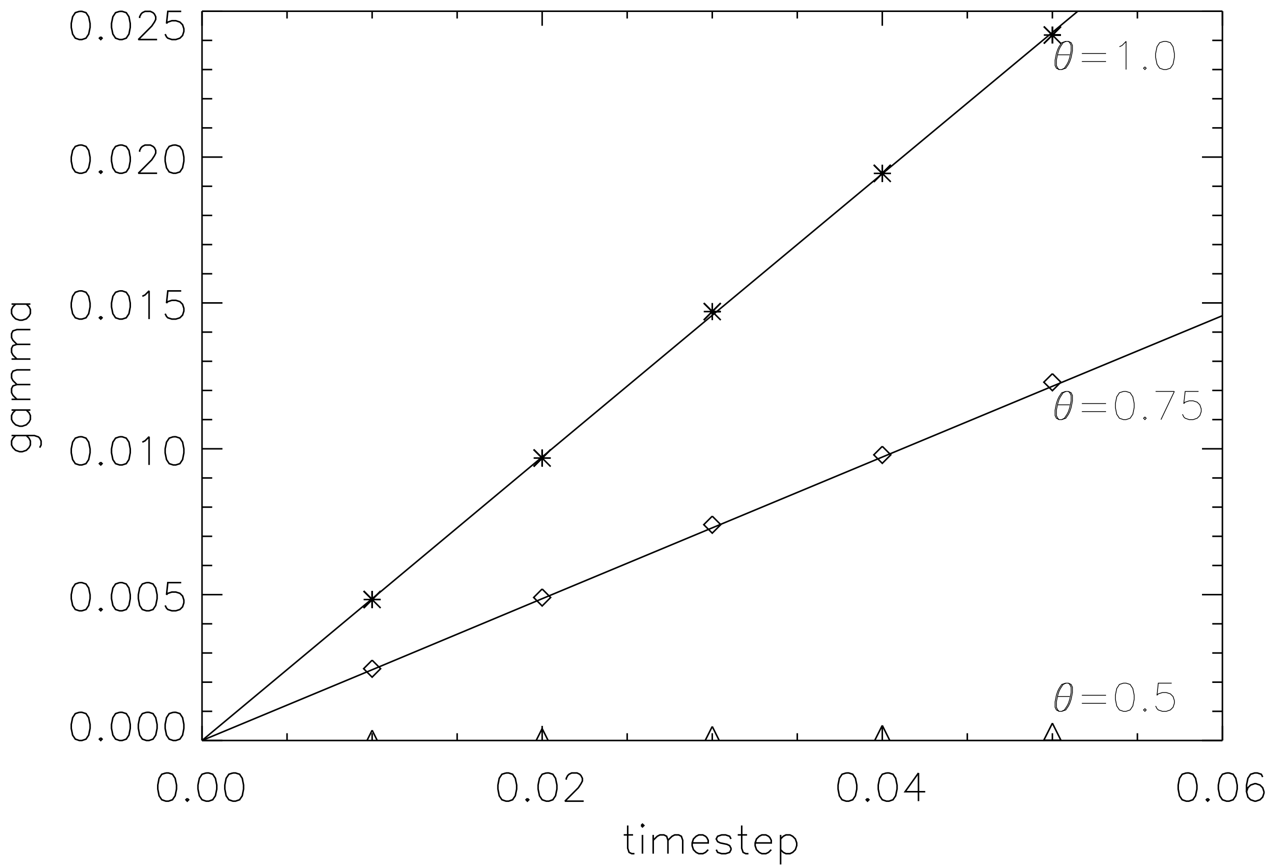
$$\boldsymbol{E}^{n+1} + \theta \, \frac{\Delta t}{\beta_e} \left(\nabla \times \nabla \times \boldsymbol{E}^{n+1}\right) \times \boldsymbol{B}_0$$
$$= \frac{1}{\beta_e} \; (\nabla \times \delta \boldsymbol{B}^n) \times \boldsymbol{B}_0 - (1-\theta) \; \left(\frac{\Delta t}{\beta_e} \nabla \times \nabla \times \boldsymbol{E}^n\right) \times \boldsymbol{B}_0$$

- The numerical dispersion relation from a Von Neumann stability analysis

$$\tan(\omega_r \Delta t) = \frac{\frac{k^2}{\beta}\Delta t}{1 - (\frac{k^2}{\beta}\Delta t)^2 \; \theta(1-\theta)}$$

$$\omega_i \Delta t = -\frac{1}{2} \ln \left(\frac{\left(1 - (\frac{k^2}{\beta}\Delta t)^2 \; \theta(1-\theta)\right)^2 + (\frac{k^2}{\beta}\Delta t)^2}{\left(1 + (\frac{k^2}{\beta}\Delta t)^2 \; (1-\theta)^2\right)^2} \right)$$

Numerical dispersion relation



$16 \times 16 \times 32$ grids, 131072 particles, $k_\perp = 0$, $k_\| = 0.0628$, $\beta = 0.004$.

Harris sheet equilibrium 1

- Zero-order $\boldsymbol{B}$

$$\boldsymbol{B}(\boldsymbol{x}) = B_{y0}\tanh(\frac{x}{L})\ \hat{\boldsymbol{y}} + B_G\ \hat{\boldsymbol{z}}$$

- The equilibrium distribution function is

$$f_{0s} = n_h\ \mathrm{sech}^{\,2}(\frac{x}{L})\left(\frac{2\pi T_s}{m_s}\right)^{-\frac{3}{2}} \exp\left[-\frac{m(v_x^2+v_y^2+(v_z-v_{ds})^2)}{2T_s}\right]$$

$$+n_b\ \left(\frac{2\pi T_s}{m_s}\right)^{-\frac{3}{2}} \exp\left(-\frac{mv^2}{2T_s}\right)$$

- Load particles as Maxwellian

$$g_s = n_0\left(\frac{2\pi T_s}{m_s}\right)^{-\frac{3}{2}}\ \exp\left(-\frac{m_s\boldsymbol{v}^2}{2T_s}\right)$$

- Weight equation

$$\frac{d\omega_i}{dt} = \frac{q_s}{T_s}\ \left(\boldsymbol{E}\cdot\boldsymbol{v}(\frac{f_h}{g_s}+\frac{n_b}{n_0}) - \boldsymbol{v}_d\cdot(\boldsymbol{E}+\boldsymbol{v}\times\delta\boldsymbol{B})\ \frac{f_h}{g_s}\right)$$

$$\frac{f_h}{g_s} = \frac{n_h}{n_0}\ \mathrm{sech}\,(\frac{x}{L})^2\ \exp\left(\frac{m_s}{2T_s}\ (2\boldsymbol{v}_d\cdot\boldsymbol{v} - \boldsymbol{v}_d^2)\right).$$

Harris sheet equilibrium 2

- Now that the zero order density and $\boldsymbol{B}_0$ are nonuniform, the Ohm's law becomes

$$\tilde{n}\,\boldsymbol{E} = -\boldsymbol{J}_i \times \boldsymbol{B}_0 + \frac{1}{\beta_e}(\nabla \times \delta\boldsymbol{B}) \times \boldsymbol{B}_0 + \frac{1}{\beta_e}(\nabla \times \boldsymbol{B}_0) \times \delta\boldsymbol{B} + \frac{\eta}{\beta_e}\tilde{n}\,\nabla \times \delta\boldsymbol{B} - \gamma \nabla n_i.$$

- Numerical form:

$$\tilde{n}\,\boldsymbol{E}^{n+1} + \tilde{\alpha}\ \nabla \times \nabla \times \boldsymbol{E}^{n+1} + \theta\,\frac{\Delta t}{\beta_e}\{\left(\nabla \times \nabla \times \boldsymbol{E}^{n+1}\right) \times \boldsymbol{B}_0 + (\nabla \times \boldsymbol{B}_0) \times (\nabla \times \boldsymbol{E}^{n+1})\}$$

$$= -\gamma\,\nabla\,n_i - \boldsymbol{J}^{\star} \times \boldsymbol{B}_0 + \frac{1}{\beta_e}\ (\nabla \times \delta\boldsymbol{B}^n) \times \boldsymbol{B}_0 + \frac{1}{\beta_e}\ (\nabla \times \boldsymbol{B}_0) \times \delta\boldsymbol{B}^n + \frac{\eta}{\beta_e}\,\tilde{n}\ \nabla \times \delta\boldsymbol{B}^n$$

$$-(1-\theta)\,\frac{\Delta t}{\beta_e}\{(\nabla \times \boldsymbol{B}_0) \times (\nabla \times \boldsymbol{E}^n) + (\nabla \times \nabla \times \boldsymbol{E}^n) \times \boldsymbol{B}_0 + \eta\,\tilde{n}\ \nabla \times \nabla \times \boldsymbol{E}^n\}$$

  - $\tilde{\alpha} = \frac{1}{\beta_e}\,(\frac{m_e}{m_i} + \theta\eta\,\Delta t\,\tilde{n})$ and $\tilde{n} = \mathrm{sech}^2(\frac{x}{L}) + \frac{n_b}{n_h}$ for Harris sheet equilibrium.

- Currently we are looking into linear instabilities (e.g. linear tearing mode).

The Lorentz ion/Drift kinetic electron model

Lorentz ions:

$$\frac{d\mathbf{v}_i}{dt} = \frac{q}{m_i}(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}), \qquad \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i$$

Drift kinetic electrons: $\varepsilon = \frac{1}{2} m_e v^2$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}_G \equiv v_\parallel \left(\mathbf{b} + \frac{\delta \mathbf{B}_\perp}{B_0} \right) + \mathbf{v}_D + \mathbf{v}_E$$

$$\frac{d\varepsilon}{dt} = -e\mathbf{v}_G \cdot \mathbf{E} + \mu \frac{\partial B}{\partial t}, \qquad \frac{d\mu}{dt} = 0$$

Ampere's equation

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J}_i - e n_e (\mathbf{V}_{\mathrm{e}\perp} + u_{\parallel \mathrm{e}} \mathbf{b}))$$

$$\mathbf{V}_{\mathrm{e}\perp} = \frac{1}{B} \mathbf{E} \times \mathbf{b} - \frac{1}{enB} \mathbf{b} \times \nabla P_{\perp \mathrm{e}}$$

$$\mathbf{J}_i = \int f_i \mathbf{v}\, d\mathbf{v}, \qquad u_{\parallel \mathrm{e}} = \int f_e\, v_\parallel\, d\mathbf{v}, \qquad P_{\perp \mathrm{e}} = \int f_e \frac{1}{2} m_e v^2\, d\mathbf{v}$$

Faraday's equation,

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

"Particle-in-Cell simulation with Vlasov ions and drift kinetic electrons" by Yang Chen and Scott E Parker, accepted by Phys. of Plasma.

- Quasi-neutrality
  - No displacement current.

- No transverse electron inertia (no electron polarization current). Electron FLR and polarization current can be added for reconnection problems.

- The magnetic field perturbation is 3-D, whereas in the $A_\| - \phi$ model $\delta \mathbf{B} = \nabla \times (A_\| \mathbf{b})$ is 2-D

- Unable to combine $A_\| - \phi$ field model with Vlasov ions. With GK ions ϕ is obtained from GK Poisson equation. With Vlasov ions the equation

$$n_i = n_e$$

  does not determine ϕ!

Including gyrokinetic electrons

- Gyrokinetic equations are usually derived in terms of $\mathbf{A}$ and ϕ, to make explicit the ordering

$$\frac{\partial \mathbf{A}}{\partial t} \sim \epsilon_\delta \nabla_\perp \phi$$

- The Frieman-Chen gyrokinetic equation, assuming isotropy ($\partial F_0/\partial \mu = 0$),

$$\hat{L}_g \delta H_0 \equiv \left(\frac{\partial}{\partial t} + v_\parallel \mathbf{b} \cdot \nabla + \mathbf{v}_D \cdot \nabla \right) \delta H_0 = -\frac{q}{m} (S_L + \langle R_{\mathrm{NL}} \rangle),$$

where δH_0 is related to the perturbed distribution δF through $\delta F = \frac{q}{m} \phi \frac{\partial F_0}{\partial \epsilon} + \delta H_0$

$$S_L = \frac{\partial}{\partial t} \langle \phi - \mathbf{v} \cdot \mathbf{A} \rangle \frac{\partial F_0}{\partial \epsilon} - \nabla \langle \phi - \mathbf{v} \cdot \mathbf{A} \rangle \times \frac{\mathbf{b}}{\Omega} \cdot \nabla F_0,$$

$$\langle R_{\mathrm{NL}} \rangle = -\nabla \langle \phi - \mathbf{v} \cdot \mathbf{A} \rangle \times \frac{\mathbf{b}}{\Omega} \cdot \nabla \delta H_0.$$

- Define $\delta f = \frac{q}{m} \langle \phi \rangle \frac{\partial F_0}{\partial \epsilon} + \delta H_0$. The gyrokinetic equation for δf is, written in terms of $\mathbf{E}_1$ and $\mathbf{B}_1$

$$\frac{D}{Dt} \delta f = -\left(\frac{1}{B_0} \langle \mathbf{E}_1 \rangle \times \mathbf{b} + v_\parallel \frac{\langle \mathbf{B}_{1\perp} \rangle}{B_0} \right) \cdot \nabla F_0 + \frac{1}{m} \dot{\epsilon} \frac{\partial F_0}{\partial \epsilon}$$

$$\frac{D}{Dt} = \hat{L}_g + \left(\frac{1}{B_0} \langle \mathbf{E}_1 \rangle \times \mathbf{b} + v_\parallel \frac{\langle \mathbf{B}_{1\perp} \rangle}{B_0} \right) \cdot \nabla, \qquad \dot{\epsilon} = q \left(v_\parallel \mathbf{b} + \mathbf{v}_D + v_\parallel \frac{\langle \mathbf{B}_{1\perp} \rangle}{B_0} \right) \cdot \langle \mathbf{E}_1 \rangle + q \langle \mathbf{v}_\perp \cdot \mathbf{E}_{1\perp} \rangle$$

- The perturbed electron diamagnetic flow comes from δf,

$$n_0 \mathbf{V}_D(\mathbf{x}) = \int (v_\parallel \mathbf{b} + \mathbf{v}_\perp(\mathbf{R}', \epsilon, \mu, \alpha))\delta f(\mathbf{R}', \epsilon, \mu)\delta(\mathbf{x} - \mathbf{R}' - \boldsymbol{\rho})\, J d\mathbf{R}'\, d\epsilon\, d\mu\, d\gamma$$

$n_0\mathbf{V}_D$ is computed by depositing the particle current along the gyro-ring. In the drift-kinetic limit $\mathbf{V}_D$ reduces to the electron diamagnetic flow.

- The electron $\mathbf{E} \times \mathbf{B}$ flow comes from the first term in δF,

$$n_0 \mathbf{V}_E(\mathbf{x}) = \frac{\mathbf{q}}{\mathbf{m}} \int \mathbf{v}\left(\phi(\mathbf{x}) - \langle\phi\rangle(\mathbf{x} - \boldsymbol{\rho}, \epsilon, \mu)\right)\frac{\partial \mathbf{F_0}}{\partial \epsilon}\, \mathbf{J}\, \mathbf{d}\epsilon\, \mathbf{d}\mu\, \mathbf{d}\gamma$$

in eikonal form,

$$n_0 \mathbf{V}_E = n_0 \frac{h}{B_0} \delta \mathbf{E}_k \times \mathbf{b}$$

with $b = k_\perp^2 v_T^2 / \Omega^2$ and

$$h(b) = -\frac{1}{b^2} \int_0^\infty e^{-x^2/2b} J_0(b) J_0'(b) x^2\, dx$$

In the limit of small $k\rho \ll 1$ the factor $h(b)$ become unity, so that $n_0\mathbf{V}_E$ become the total guiding center $\mathbf{E}\times\mathbf{b}$ flow.

Summary

1. We implemented an implicit algorithm with Lorentz force ions and isothermal fluid electrons which is
   - Quasi-neutral and fully electromagnetic.
   - Suitable for MHD scale plasmas.

2. Second order implicit scheme allows bigger time step, $\Omega_i \Delta t \gtrsim 0.1$.
   - Compared the first order and second order scheme with Whistler waves

3. With the matrix solver,the code is now capable of dealing with nonuniform zero-order $\boldsymbol{B}$ field.

4. Demonstrated 3-D slab simulation for compressional and shear Alfven waves, Whistler wave, and the ion acoustic wave.

5. Work is underway to apply this model in Harris sheet equilibrium.