Moments of the Boltzmann collision operator*

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Kinetic equation and Boltzmann operator

- Kinetic equation for $f_a(t, \mathbf{r}, \mathbf{v})$

$$\frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \frac{\partial f_a}{\partial \mathbf{r}} + \frac{q_a}{m_a}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_a}{\partial \mathbf{v}} = \sum_b C(f_a, f_b)$$

- Boltzmann collision operator

$$C(f_a, f_b) = \int d\mathbf{v}' \int \sigma d\Omega |\mathbf{v} - \mathbf{v}'| [f_a(\mathbf{v}_*) f_b(\mathbf{v}'_*) - f_a(\mathbf{v}) f_b(\mathbf{v}')]$$

$$(\mathbf{v}, \mathbf{v}') \xrightarrow{\text{collision}} (\mathbf{v}_*, \mathbf{v}'_*) = (\mathbf{v} + \Delta\mathbf{v}, \mathbf{v}' + \Delta\mathbf{v}')$$

- Landau operator: Taylor-expand and keep up to the second orders in $\Delta\mathbf{v}$ and $\Delta\mathbf{v}'$

$$f_a(\mathbf{v} + \Delta\mathbf{v}) f_b(\mathbf{v}' + \Delta\mathbf{v}')$$

  - ▷ Equivalent to the Fokker-Planck operator

$$C(f_a, f_b) = -\frac{\partial}{\partial \mathbf{v}} \cdot \left(f_a \frac{\langle \Delta\mathbf{v} \rangle_b}{\Delta t} \right) + \frac{1}{2} \frac{\partial}{\partial \mathbf{v}} \frac{\partial}{\partial \mathbf{v}} : \left(f \frac{\langle \Delta\mathbf{v} \Delta\mathbf{v} \rangle_b}{\Delta t} \right)$$

Landau operator

- Landau (1937) collision operator

$$C(f_a, f_b) = \frac{\gamma_{ab}}{2m_a} \frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{v}' \mathsf{U} \cdot \left[\frac{\partial f_a(\mathbf{v})}{\partial \mathbf{v}} f_b(\mathbf{v}') - f_a(\mathbf{v}) \frac{m_a}{m_b} \frac{\partial f_b(\mathbf{v}')}{\partial \mathbf{v}'} \right]$$

$$\mathsf{U} = \frac{|\mathbf{v} - \mathbf{v}'|^2 \mathsf{I} - (\mathbf{v} - \mathbf{v}')(\mathbf{v} - \mathbf{v}')}{|\mathbf{v} - \mathbf{v}'|^3}$$

$$\gamma_{ab} = \frac{q_a^2 q_b^2 \ln \Lambda_{ab}}{4\pi \epsilon_0^2 m_a} = \frac{3\sqrt{\pi} m_a v_{Ta}^3}{4 n_b \tau_{ab}}$$

- Rosenbluth, MacDonald, and Judd (1957)

$$C(f_a, f_b) = \frac{\gamma_{ab}}{2m_a} \frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{\partial}{\partial \mathbf{v}} \cdot \left(f_a \frac{\partial}{\partial \mathbf{v}} \frac{\partial G_b^+}{\partial \mathbf{v}} \right) - 2\left(1 + \frac{m_a}{m_b}\right) f_a \frac{\partial G_b^-}{\partial \mathbf{v}} \right]$$

where

$$G_b^{\pm}(\mathbf{v}) = \int d\mathbf{v}' |\mathbf{v} - \mathbf{v}'|^{\pm 1} f_b(\mathbf{v}')$$

Why Boltzmann?

- Scattering cross section for Coulomb collisions

$$\sigma = -\frac{d\Sigma}{d\Omega} = \frac{\rho_{90^\circ}^2}{4\sin^4(\theta/2)}, \; \rho_{90^\circ} = \frac{|\alpha|}{m_{ab}u^2}, \; \alpha = \frac{q_a q_b}{4\pi\epsilon_0}$$

- Velocity change due to a collision

$\mathbf{u} = \mathbf{v} - \mathbf{v}'$ (relative particle velocity)

$$\Delta\mathbf{u} = \Delta\mathbf{v} - \Delta\mathbf{v}' = \left(1 + \frac{m_a}{m_b}\right)\Delta\mathbf{v} = -\left(1 + \frac{m_b}{m_a}\right)\Delta\mathbf{v}'$$

$$(\Delta\mathbf{u})_\Sigma = \int \sigma \Delta\mathbf{u} d\Omega = -2\pi\rho_{90^\circ}^2 \mathbf{u} \ln(1 + \Lambda^2)$$

$$(\Delta\mathbf{u}\Delta\mathbf{u})_\Sigma = \int \sigma \Delta\mathbf{u}\Delta\mathbf{u} d\Omega \approx 4\pi\rho_{90^\circ}^2 (u^2\mathsf{I} - \mathbf{u}\mathbf{u}) \ln\Lambda$$

- For distribution functions with large $\mathbf{u}$, the Landau operator may be inaccurate
  - ▷ Large relative flow velocity
  - ▷ Runaway electrons

Why moments?

- Integro-differential equation

$$\frac{\partial f_a}{\partial t} + \mathbf{v}\cdot\frac{\partial f_a}{\partial \mathbf{r}} + \frac{q_a}{m_a}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_a}{\partial \mathbf{v}} = \sum_b C(f_a, f_b) \qquad \text{(KE)}$$

$$C(f_a, f_b) = \int d\mathbf{v}' \int \sigma d\Omega |\mathbf{v}-\mathbf{v}'| [f_a(\mathbf{v}_*) f_b(\mathbf{v}'_*) - f_a(\mathbf{v}) f_b(\mathbf{v}')]$$

- Moment expansion

$$f_a = \sum_B w(t,\mathbf{r}) M_a^B(t,\mathbf{r}) P^B(\mathbf{v})$$

- $\int d\mathbf{v} P^A(\mathbf{v})$(KE) $\Rightarrow$ General moment equations

$$\sum_B D^{AB}\left(\frac{\partial}{\partial t}, \nabla, \mathbf{E}, \mathbf{B}\right) M_a^B = \sum_B \left(A^{AB} M_a^B + B^{AB} M_b^B\right) + \sum_{B,B'} C^{ABB'} M_a^B M_b^{B'}$$

General moment expansion [Ji and Held 2006 2008 2009]
Irreducible Hermite polynomials [Chapman 1916 Enskog 1917 Grad 1963]

$$\mathbf{c}_a = \frac{\mathbf{w}_a}{v_{Ta}} = \frac{\mathbf{v} - \mathbf{V}_a}{v_{Ta}}, \quad v_{Ta} = \sqrt{\frac{2T_a}{m_a}}$$

- Moment expansion: m_a^{lk}'s are symmetric traceless fluid moments

$$f_a(t, \mathbf{r}, \mathbf{v}) = f_a^{\mathrm{M}} \sum_{lk} \mathsf{m}_a^{lk}(t, \mathbf{r}) \cdot \hat{\mathsf{p}}_a^{lk}$$

$$\mathsf{n}_a^{lk}(t, \mathbf{r}) \equiv n_a \mathsf{m}_a^{lk} = \int d\mathbf{v} \hat{\mathsf{p}}_a^{lk} f_a$$

- $\hat{\mathsf{p}}^{lk}$'s are orthonormal tensorial polynomials and form a complete set

$$\int d\mathbf{v} \hat{\mathsf{p}}^{jp} \hat{\mathsf{p}}^{lk} \cdot \mathsf{m}^{lk} f^{\mathrm{M}} = \delta_{jl} \delta_{pk} \mathsf{n}^{jp}, \quad \hat{\mathsf{p}}^{lk} = \frac{1}{\sqrt{\sigma_{lk}}} \mathsf{p}^{lk}$$

$$\begin{aligned} \mathsf{p}_a^{lk} &= \mathsf{P}^l(\mathbf{c}_a) L_k^{(l+1/2)}(c_a^2) \\ &= \text{(harmonic tensor)(associated Laguerre polynomial)} \end{aligned}$$

$$f_a^{\mathrm{M}} = \frac{n_a}{\pi^{3/2} v_{Ta}^3} e^{-c_a^2} \text{ (Maxwellian distribution)}$$

Several low order moments (21 moments, for Braginskii)

$\mathsf{p}^{lk} = \mathsf{P}^l(\mathbf{c})L_k^l(c^2)$	$L_k^l = L_k^{(l+\frac{1}{2})}$	n^{lk}	fluid moment equation	indep.
$\mathsf{P}^0 = 1$	$L_0^0 = 1$	n	density (n)	1
	$L_1^0 = \frac{3}{2} - c^2$	0	temperature (T)	1
$\mathsf{P}^1 = \mathbf{c}$	$L_0^1 = 1$	0	flow velocity $(\mathbf{V})$	3
	$L_1^1 = \frac{5}{2} - c^2$	n^{11}	heat flow $(\mathbf{h})$	3
	$L_2^1 = \frac{35}{8} - \frac{7}{2}c^2 + \frac{1}{2}c^4$	n^{12}	heat w. heat flow $(\mathbf{r})$	3
$\mathsf{P}^2 = \mathbf{cc} - \frac{c^2}{3}\mathsf{I}$	$L_0^2 = 1$	n^{20}	viscosity $(\boldsymbol{\pi})$	5
	$L_1^2 = \frac{7}{2} - c^2$	n^{21}	heat viscosity $(\boldsymbol{\theta})$	5

$$
\begin{aligned}
f = & f^{\mathrm{M}\{n,\mathbf{V},T\}}\Big(1 + \cdots && \leftarrow \text{scalar moments} \\
& + \sigma_{\mathbf{h}}\mathsf{p}^{11}\cdot\mathbf{h} + \sigma_{\mathbf{r}}\mathsf{p}^{12}\cdot\mathbf{r} + \cdots && \leftarrow \text{vector moments} \\
& + \sigma_{\boldsymbol{\pi}}\mathsf{p}^{20}\cdot\boldsymbol{\pi} + \sigma_{\boldsymbol{\theta}}\mathsf{p}^{21}\cdot\boldsymbol{\theta} + \cdots && \leftarrow \text{rank-2 tensor moments} \\
& + \cdots\Big) && \leftarrow \text{higher rank tensor momemnts}
\end{aligned}
$$

Moments of the Boltzmann collision operator

- $C(f_a, f_b) = \int d\mathbf{v}' \int \sigma d\Omega |\mathbf{v} - \mathbf{v}'| [f_a(\mathbf{v}_*) f_b(\mathbf{v}'_*) - f_a(\mathbf{v}) f_b(\mathbf{v}')]$

$$\int d\mathbf{v} \mathsf{P}^{jp}(\mathbf{c}_a) C\left(f_a^{\mathrm{M}}(\mathbf{c}_a) \mathsf{m}_a^{lk} \cdot \mathsf{P}^{lk}(\mathbf{c}_a), f_b^{\mathrm{M}}(\mathbf{c}_b)\right) = \mathsf{A}_{ab}^{jp,lk}$$

$$\int d\mathbf{v} \mathsf{P}^{jp}(\mathbf{c}_a) C\left(f_a^{\mathrm{M}}(\mathbf{c}_a), f_b^{\mathrm{M}}(\mathbf{c}_b) \mathsf{m}_b^{lk} \cdot \mathsf{P}^{lk}(\mathbf{c}_b)\right) = \mathsf{B}_{ab}^{jp,lk}$$

- Use

$$\int d\mathbf{v} \mathsf{Q}(\mathbf{v}) \int d\mathbf{v}' \int \sigma d\Omega |\mathbf{v} - \mathbf{v}'| [f_a(\mathbf{v}_*) f_b(\mathbf{v}'_*) - f_a(\mathbf{v}) f_b(\mathbf{v}')]$$

$$= \int d\mathbf{v} \int d\mathbf{v}' \int \sigma d\Omega |\mathbf{v} - \mathbf{v}'| \left[\mathsf{Q}(\mathbf{v}_*) - \mathsf{Q}(\mathbf{v})\right] f_a(\mathbf{v}) f_b(\mathbf{v}')$$

- Calculate

$$\mathsf{A}_{ab}^{jp,lk} \to \int d\mathbf{v} \int d\mathbf{v}' \int \sigma d\Omega |\mathbf{v} - \mathbf{v}'| \left[\mathbf{c}_{a*}^j c_{a*}^{2q} - \mathbf{c}_a^j c_a^{2q}\right] f_a^{\mathrm{M}} \boxed{\mathbf{c}_a^l \cdot \mathsf{m}_a^{lk} c_a^{2m}} f_b^{\mathrm{M}}$$

$$\mathsf{B}_{ab}^{jp,lk} \to \int d\mathbf{v} \int d\mathbf{v}' \int \sigma d\Omega |\mathbf{v} - \mathbf{v}'| \left[\mathbf{c}_{a*}^j c_{a*}^{2q} - \mathbf{c}_a^j c_a^{2q}\right] f_a^{\mathrm{M}} f_b^{\mathrm{M}} \boxed{\mathbf{c}_b'^l \cdot \mathsf{m}_b^{lk} c_b'^{2m}}$$

Calculation steps

- Change velocity variables

$$\begin{array}{c}\mathbf{w}_\star = X_{ba}\mathbf{w}_a + X_{ab}\mathbf{w}'_b \\ \mathbf{w}_\dagger = \mathbf{w}_a - \mathbf{w}'_b\end{array} \leftrightarrow \begin{array}{c}\mathbf{w}_a = v_\star(\mathbf{c}_\star + \chi_a \mathbf{c}_\dagger) \\ \mathbf{w}'_b = v_\star(\mathbf{c}_\star + \chi'_b \mathbf{c}_\dagger)\end{array}, \qquad \begin{array}{c} v_\star^{-2} = v_{Ta}^{-2} + v_{Tb}^{-2} \\ v_\dagger^2 = v_{Ta}^2 + v_{Tb}^2\end{array}$$

where $X_{ab} = (1 + v_{Ta}^2/v_{Tb}^2)^{-1}$, $X_{ba} = 1 - X_{ab}$, $\chi_a = v_{Ta}/v_{Tb}$, $\chi'_b = -v_{Tb}/v_{Ta}$

$$\int d\mathbf{v} \int d\mathbf{v}' f_a^{\mathrm{M}} f_b^{\mathrm{M}} |\mathbf{v} - \mathbf{v}'| = n_a n_b v_\dagger^{\pm 1} \int d\mathbf{c}_\star \int d\mathbf{c}_\dagger \frac{e^{-c_\star^2}}{\pi^{3/2}} \frac{e^{-c_\dagger^2}}{\pi^{3/2}} |\mathbf{c}_\dagger - \mathbf{x}|$$

where $\mathbf{u} = v_\dagger(\mathbf{c}_\dagger - \mathbf{x})$, $\mathbf{x} = (\mathbf{V}_b - \mathbf{V}_a)/v_\dagger$

- Perform σ integration $\int \sigma d\Omega \left[\mathbf{c}_{a*}^j c_{a*}^{2q} - \mathbf{c}_a^j c_a^{2q}\right] \to \int \sigma d\Omega (\Delta \mathbf{u})^n |\Delta \mathbf{u}|^{2m}$
- Perform $\mathbf{c}_\star$ integration $\displaystyle\int d\mathbf{c}_\star \mathbf{c}_\star^n c^{2u} \frac{e^{-c_\star^2}}{\pi^{3/2}} = \frac{2[u + (n+1)/2]!}{\pi^{1/2}(n+1)} \{\mathsf{I}^{n/2}\}$
- Simplify $(\mathbf{c}_\dagger \cdot \mathbf{x})^\beta \left\{ \left\{ \mathsf{I}^{P_0} \mathbf{c}_\dagger^{Q_0} \mathbf{x}^\alpha \right\} \cdot^{y_0} \left\{ \mathsf{I}^{P_1} \mathbf{c}_\dagger^{Q_1} \mathsf{m}^{R_1} \right\} \right\}$
- Perform $\mathbf{c}_\dagger$ integration (an overline denotes traceless symmetrization)

$$\mathsf{A}_{ab}^{jp,lk} = \sum \delta_{j,\alpha+l-2\beta+2\gamma} \mathcal{A}_{\alpha,\beta,\gamma}^{j,p,l,k} \left(\frac{m_b}{m_a}, \frac{T_b}{T_a}, x\right) \overline{\mathbf{x}^\alpha \cdot^\beta \mathsf{m}_a^{lk} \mathsf{I}^\gamma}$$

$$\mathsf{B}_{ab}^{jp,lk} = \sum \delta_{j,\alpha+l-2\beta+2\gamma} \mathcal{B}_{\alpha,\beta,\gamma}^{j,p,l,k} \left(\frac{m_b}{m_a}, \frac{T_b}{T_a}, x\right) \overline{\mathbf{x}^\alpha \cdot^\beta \mathsf{m}_b^{lk} \mathsf{I}^\gamma}$$

Examples: moments due to relative flow

- Landau

$$\mathsf{A}_{ab}^{10,00} = \frac{3\sqrt{\pi}n_a X}{4\tau_{ab}\mu_b}\frac{E - xE'}{x^2}\hat{\mathbf{x}}$$

$$\mathsf{A}_{ab}^{20,00} = \frac{3\sqrt{\pi}\sqrt{X}}{8\tau_{ab}}\left[\left(\frac{3+6X\mu_b^{-1}}{x^3} - \frac{2}{x}\right)E - \left(\frac{3+6X\mu_b^{-1}}{x^2} + 4X\mu_b^{-1}\right)E'\right]\overline{\hat{\mathbf{x}}\hat{\mathbf{x}}}$$

where $E = \mathrm{erf}(x)$, $\mu_b = m_b/(m_a + m_b)$, $X = (1 + v_{Tb}^2/v_{Ta}^2)^{-1}$, $\mathbf{x} = (\mathbf{V}_b - \mathbf{V}_a)/v_\dagger$

- Boltzmann

$$\mathsf{A}_{ab}^{20,00} = \frac{3\sqrt{\pi}n_a\sqrt{X}}{8\tau_{ab}}\left[\left(\frac{3-9\lambda^{-1}/2+6X\mu_b^{-1}}{x^3} - \frac{2-3\lambda^{-1}}{x}\right)E - \left(\frac{3-9\lambda^{-1}/2-6X\mu_b^{-1}}{x^2} + 4X\mu_b^{-1}\right)E'\right]\overline{\hat{\mathbf{x}}\hat{\mathbf{x}}},$$

Boltzmann $\rightarrow$ Landau as $\lambda = \ln\Lambda \rightarrow \infty$

Collision matrix element: heat flux - heat flux

$$\mathsf{A}_{ab}^{11,11} = \frac{3\sqrt{\pi}X^{3/2}}{16\tau_{ab}}\Bigg[\mathsf{n}_a^{11}\left(A_{0E}^{11,11}\frac{E}{x^3}+A_{0e}^{11,11}\frac{E'}{x^2}\right)$$
$$+\hat{\mathbf{x}}\hat{\mathbf{x}}\cdot\mathsf{n}_a^{11}\left(A_{1E}^{11,11}\frac{E}{x^3}+A_{1e}^{11,11}\frac{E'}{x^2}\right)\Bigg],$$

where $\chi = v_{Tb}/v_{Ta}$

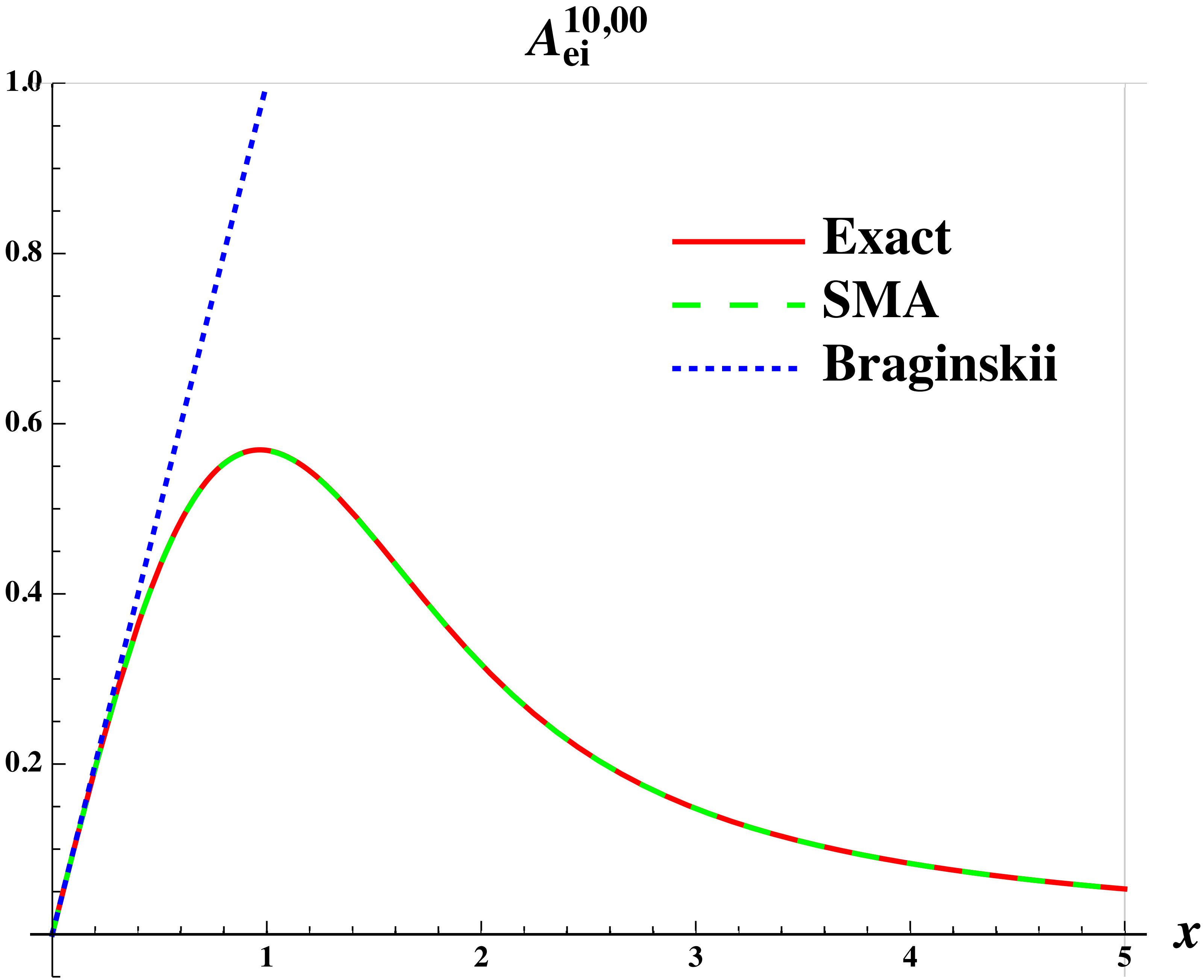
$$\begin{aligned}
A_{0E}^{11,11} &= -8+7\lambda^{-1}-9\mu_b^{-1}+2\lambda^{-1}\mu_b\\
A_{0e}^{11,11} &= 8-7\lambda^{-1}+9\mu_b^{-1}-2\lambda^{-1}\mu_b\\
&+[6X-2-2\lambda^{-1}-4X\lambda^{-1}+\left(-4+22X-15X^2\right)\mu_b^{-1}]x^2\\
&+2X(-2+3X)\mu_b^{-1}x^4\\
A_{1E}^{11,11} &= 3\left(8-7\lambda^{-1}+9\mu_b^{-1}-2\lambda^{-1}\mu_b\right),\\
A_{1e}^{11,11} &= \left(-24+21\lambda^{-1}-27\mu_b^{-1}+6\lambda^{-1}\mu_b\right)\\
&+\left(-16+14\lambda^{-1}-18\mu_b^{-1}+4\lambda^{-1}\mu_b\right)x^2\\
&+2X\left(-6+4\lambda^{-1}-16\mu_b^{-1}+21X\mu_b^{-1}\right)x^4-12\mu_b^{-1}X^2x^6
\end{aligned}$$

Collision matrix element: viscosity - viscosity

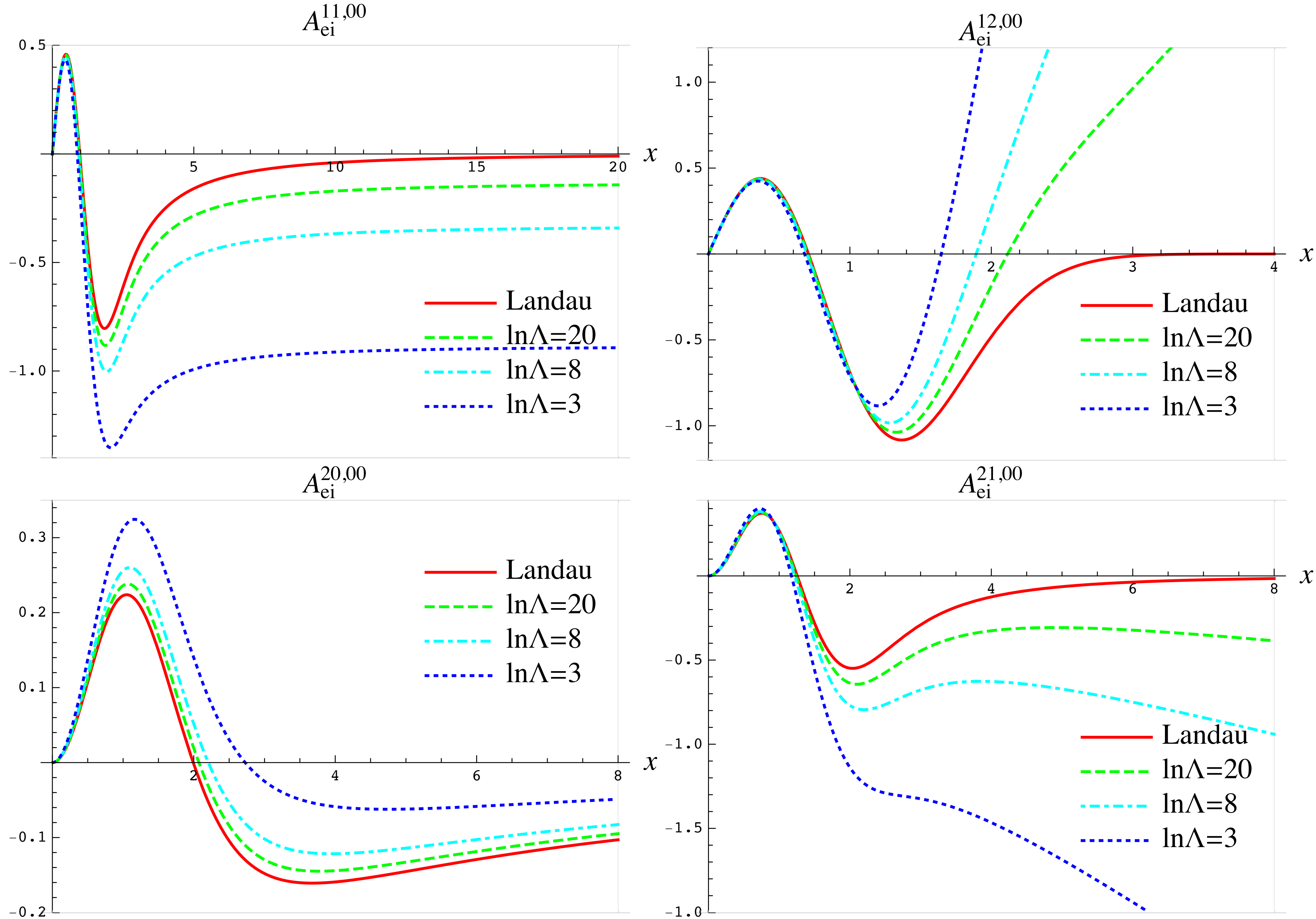
$$
\begin{aligned}
\mathsf{A}_{ab}^{20,20} \quad = \quad & \frac{3\sqrt{\pi}X^{3/2}}{32\tau_{ab}}\Bigg[\mathsf{n}_a^{20}\left(A_{0E}^{20,20}\frac{E}{x^5}+A_{0e}^{20,20}\frac{E'}{x^4}\right)\\
&+\overline{\hat{\mathbf{x}}\hat{\mathbf{x}}}\cdot\mathsf{n}_a^{20}\left(A_{1E}^{20,20}\frac{E}{x^5}+A_{1e}^{20,20}\frac{E'}{x^4}\right)\\
&+\overline{\hat{\mathbf{x}}\hat{\mathbf{x}}}\hat{\mathbf{x}}\hat{\mathbf{x}}:\mathsf{n}_a^{20}\left(A_{2E}^{20,20}\frac{E}{x^5}+A_{2e}^{20,20}\frac{E'}{x^4}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
A_{0E}^{20,20} &= 6-9\lambda^{-1}+12X\mu_b^{-1}+(-4+6\lambda^{-1}-8\mu_b^{-1})x^2,\\
A_{0e}^{20,20} &= -6+9\lambda^{-1}-12X\mu_b^{-1}+8X\chi^2\mu_b^{-1}x^2,\\
A_{1E}^{20,20} &= -30(2-3\lambda^{-1})-120X\mu_b^{-1}+\left(12(2-3\lambda^{-1})+24\mu_b^{-1}\right)x^2\\
A_{1e}^{20,20} &= 30(2-3\lambda^{-1})+120X\mu_b^{-1}+\left(8(2-3\lambda^{-1})+8(-3+10X)\mu_b^{-1}\right)x^2\\
&\quad +8(-2+4X)\mu_b^{-1}x^4,\\
A_{2E}^{20,20} &= 105(2-3\lambda^{-1})/2+210X\mu_b^{-1}-15(2-3\lambda^{-1})x^2,\\
A_{2e}^{20,20} &= -105(2-3\lambda^{-1})/2-210X\mu_b^{-1}+\left(-20(2-3\lambda^{-1})-140X\mu_b^{-1}\right)x^2\\
&\quad +\left(-4(2-3\lambda^{-1})-56X\mu_b^{-1}\right)x^4-16X\mu_b^{-1}x^6
\end{aligned}
$$

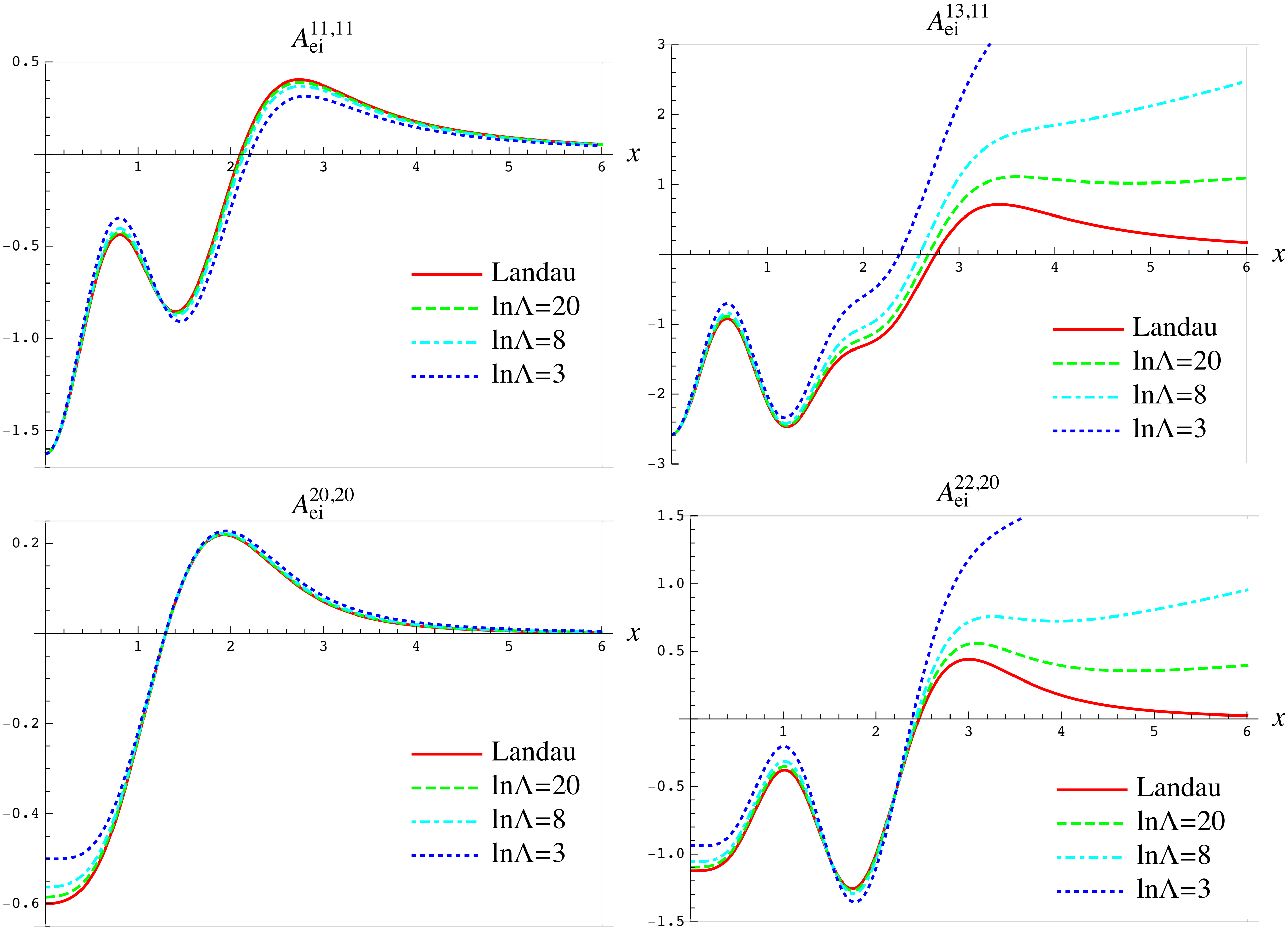
Momentum $(1, 0)$ moment due to relative flow: Dreicer's friction



$(1,1)$, $(1,2)$, $(2,0)$, and $(2,1)$ moments due to $\mathbf{V}_{\rm ei}$ $(x = |\mathbf{V}_{\rm ei}|/v_{Te})$



Collision matrix elements



Summary and future work

- Formulas are implemented in `Mathematica` and verified against existing results
- Obtain closures in high collisionality
  - ▷ Parallel closures
  - ▷ Additional drive terms are from $\mathbf{x}$, e.g. $\overline{\mathbf{x}\mathbf{x}}$
  - ▷ Different rank tensors with $\mathbf{x}$, e.g. $\mathsf{A}^{20,11} \rightarrow \mathbf{x}\mathsf{n}^{11}$, $\mathsf{A}^{20,30} \rightarrow \mathbf{x} \cdot \mathsf{n}^{30}$
  - ▷ Check convergence with increasing moments