Updates on a transport study using general parallel moment equations in NIMROD

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Fluid equations and closures in NIMROD

- NIMROD solves a set of fluid equations

$$
\begin{aligned}
& d_t n + n\nabla\cdot\mathbf{V} = 0 \\
& \rho d_t\mathbf{V} - \mathbf{J}\times\mathbf{B} + \nabla p + \nabla\cdot\boldsymbol{\pi} = 0 \\
& \frac{3}{2} n_a d_t T_a + n_a T_a \nabla\cdot\mathbf{V}_a + \nabla\cdot\mathbf{h}_a + \nabla\mathbf{V}_a : \boldsymbol{\pi}_a = Q_a \\
& \mathbf{E} = -\mathbf{V}\times\mathbf{B} + \mathbf{R} + \frac{1}{ne}\mathbf{J}\times\mathbf{B} + \cdots - \frac{1}{ne}\nabla\cdot\boldsymbol{\pi}_{\mathrm{e}}
\end{aligned}
$$

- Closure relation: expressing $\mathbf{h}_a, \boldsymbol{\pi}_a, Q_a, \mathbf{R}$ in terms of $n_a, \mathbf{V}_a, T_a$

  e.g. $\mathbf{h} = \left(\kappa_{\|} - \kappa_{\perp}\right)\partial_{\|} T\,\hat{\mathbf{b}} + \kappa_{\perp}\nabla T$

- Accurate parallel closure relations can be obtained from the drift kinetic equation

Obtaining closure relations from the drift kinetic equation

- The first order drift kinetic equation

$$\frac{\partial \bar{f}_1}{\partial t} + \mathbf{v}_\parallel \cdot \nabla \bar{f}_1 + \mathbf{v}_\mathrm{D} \cdot \nabla \bar{f}_0 + q\mathbf{v}_\parallel \cdot \mathbf{E}^{(\mathbf{A})} \frac{\partial \bar{f}_0}{\partial U} = C\left(\bar{f}_1\right)$$

- Parallel moment expansion

$$\begin{aligned} \bar{f}_1 &= \hat{f}^\mathrm{m} \sum_{lk} \hat{P}^{lk} N_\parallel^{lk} \\ \hat{P}^{lk} &= \frac{1}{\sqrt{\sigma_{lk}}} s^l P_l(\xi) L_k^{l+1/2}(s^2) \end{aligned}$$

  - $P_l(\xi)$: Legendre polynomial of $\xi = \dfrac{v_\parallel}{v}$
  - $L_k^{l+1/2}(s^2)$: associated Laguerre polynomial of $s = \dfrac{v}{v_T}$
  - $N_\parallel^{lk}(t, \mathbf{x})$: parallel moments

- Maxwellian and non-Maxwellian parts of the distribution function
  - Maxwellian part $\left(f_1^\mathrm{M}\right)$: terms for $(l, k) = (0,0), (0,1), (1,0)$
  - non-Maxwellian part $\left(f_1^\mathrm{N}\right)$: higher order terms for $(l, k) = (0,2), \cdots$

Parallel moment equations for closure calculation

- Rearrange the equation for the non-Maxwellian part

$$\frac{\partial \overline{f_1^{\mathrm{N}}}}{\partial t} + \mathbf{v}_\parallel \cdot \nabla \overline{f_1^{\mathrm{N}}} = C\left(\overline{f_1}\right) - \left(\frac{\partial \overline{f_1^{\mathrm{M}}}}{\partial t} + \mathbf{v}_\parallel \cdot \nabla \overline{f_1^{\mathrm{M}}}\right) - \mathbf{v}_{\mathrm{D}} \cdot \nabla \overline{f_0} - q\mathbf{v}_\parallel \cdot \mathbf{E}^{(\mathbf{A})} \frac{\partial \overline{f_0}}{\partial U}$$

- Taking non-Maxwellian moments $\Rightarrow$ non-Maxwellian moment equations
  - non-Maxwellian moments are expressed in terms of $n, \mathbf{V}, T$

$$\frac{\partial N_\parallel^{jp}}{\partial t} + v_T \sum_{lk \neq \mathrm{M}} \left[\mathbf{\Psi}^{jp,lk} \frac{\partial N_\parallel^{lk}}{\partial \ell} + \mathbf{\Psi_B}^{jp,lk} \frac{\partial \ln B}{\partial \ell} N_\parallel^{lk} \right] = \frac{1}{\tau} \sum_{lk \neq \mathrm{M}} c^{jp,lk} N_\parallel^{lk} + g_\parallel^{jp}$$

  - Drives $g_\parallel^{jp}$
    - Thermodynamic drives: $\partial_\parallel T$, $(V_{\mathrm{e}} - V_{\mathrm{i}})_\parallel$,$W_\parallel$
    - Neoclassical drives: $\dfrac{dT}{d\psi}$,$\dfrac{dp}{d\psi}$

Benchmark test with a continuum DKE solver

- Continuum DKE solver in NIMROD[1]

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_\parallel \cdot \left[\nabla f_1 - \frac{1-\xi^2}{2\xi}\nabla \ln B \frac{\partial f_1}{\partial \xi}\right] - \sum_b C(f_1, f_b) =$$
$$- \mathbf{v}_\mathrm{D} \cdot \left[\nabla f_0 - \frac{s}{2}\nabla \ln T_0 \frac{\partial f_0}{\partial s}\right] + \frac{q}{2T_0 s}\mathbf{v}_\parallel \cdot \nabla \phi_1 \frac{\partial f_0}{\partial s}$$

- The steady state solutions from the parallel moment equation solver and the continuum DKE solver are compared

$$\frac{\partial N_\parallel^{jp}}{\partial t} + v_T \sum_{lk \neq \mathrm{M}} \left[\mathbf{\Psi}^{jp,lk} \frac{\partial N_\parallel^{lk}}{\partial \ell} + \mathbf{\Psi_B}^{jp,lk} \frac{\partial \ln B}{\partial \ell} N_\parallel^{lk}\right] - \frac{1}{\tau} \sum_{lk \neq \mathrm{M}} c^{jp,lk} N_\parallel^{lk} = g_\parallel^{jp}$$

  - Circular cross section tokamak (aspect ratio 10)
  - Core temperature: $T_\mathrm{e} = T_\mathrm{i} = 215\ \mathrm{eV}$
  - Constant number density: $10^{19}\ \mathrm{m}^{-3}$

[1] Held, E. D., *et al.* "Verification of continuum drift kinetic equation solvers in NIMROD." Physics of Plasmas 22.3 (2015)

Implementing general parallel moment equations

- $\texttt{numL} \times \texttt{numK}$: number of non-Maxwellian parallel moments $[N]$ to be solved
- Time discrete equation constructed by 2D finite-element basis functions

$$
\begin{aligned}
&(1 + \Delta t(v_T\,[\Psi]\,\partial_{\|} + v_T[\Psi_{\mathbf{B}}](\partial_{\|} \ln B) - \frac{1}{\tau}\,[\mathtt{C}]))\,[\Delta N] \\
&= \Delta t(\frac{1}{\tau}\,[\mathtt{C}]\,[N] - v_T\,[\Psi]\,\partial_{\|}\,[N] - v_T[\Psi_{\mathbf{B}}](\partial_{\|} \ln B)\,[N] + [g])
\end{aligned}
$$

- $\frac{1}{\tau}[\mathtt{C}]$: Collision matrix $\Rightarrow$ Not coupled with l index
- $v_T[\Psi]\partial_{\|}$: Streaming matrix $\Rightarrow$ Coupled with neighboring l indices

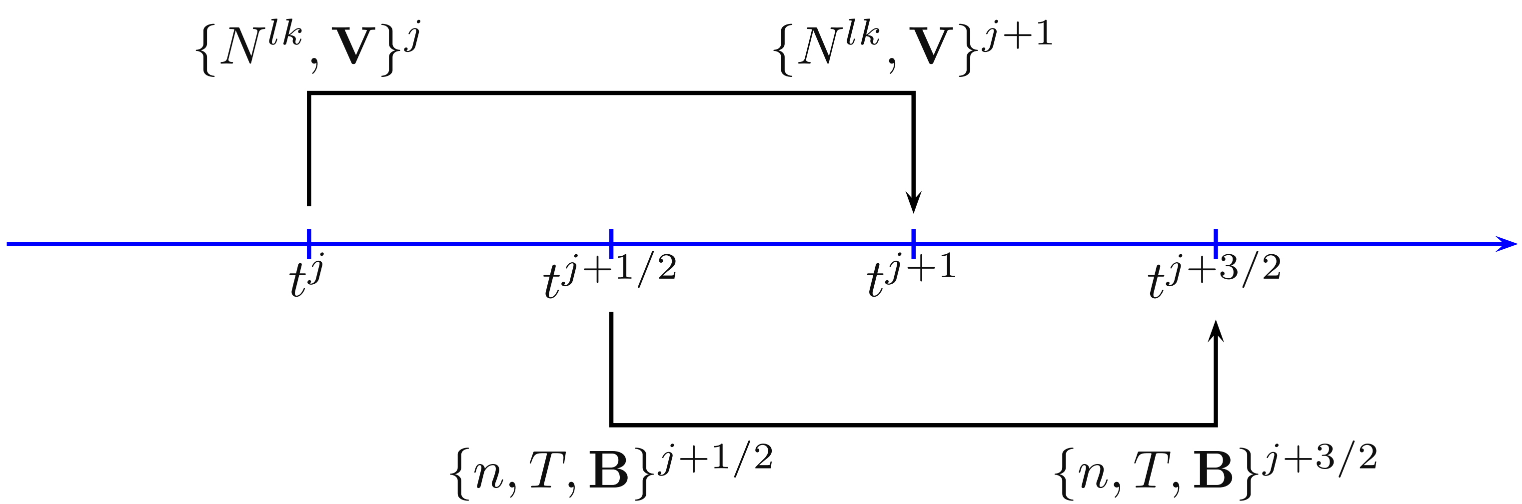
$$
\begin{bmatrix}
C^{022} & C^{023} & 0 & 0 & & & \\
C^{032} & C^{033} & 0 & 0 & & & \\
0 & 0 & C^{111} & C^{112} & 0 & 0 & \\
0 & 0 & C^{121} & C^{122} & 0 & 0 & \\
 & & 0 & 0 & C^{200} & C^{201} & \\
 & & 0 & 0 & C^{210} & C^{211} & \\
 & & & & & & \ddots
\end{bmatrix}
\begin{bmatrix}
N^{02} \\ N^{03} \\ N^{11} \\ N^{12} \\ N^{20} \\ N^{21} \\ \vdots
\end{bmatrix}
\qquad
\begin{bmatrix}
0 & 0 & \psi_{21}^{0+} & \psi_{22}^{0+} & 0 & 0 & \cdots \\
0 & 0 & 0 & \psi_{32}^{0+} & 0 & 0 & \cdots \\
\psi_{12}^{1-} & 0 & 0 & 0 & \psi_{10}^{1+} & \psi_{11}^{1+} & \cdots \\
\psi_{22}^{1-} & \psi_{23}^{1-} & 0 & 0 & 0 & \psi_{21}^{1+} & \cdots \\
0 & 0 & \psi_{01}^{2-} & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
N^{02} \\ N^{03} \\ N^{11} \\ N^{12} \\ N^{20} \\ N^{21} \\ \vdots
\end{bmatrix}
$$

Applying closure relations to MHD equations

- Parallel heat flux, viscosity and resistivity are calculated from the parallel moments
  - Using Braginskii closures for perpendicular, cross closures

$$
\begin{aligned}
h_{\|} &= -\frac{\sqrt{5}}{2} v_T T N^{11} \\
\pi_{\|} &= \frac{2}{\sqrt{3}} T N^{20} \\
ne\eta J_{\|} &\Rightarrow \frac{m_{\mathrm{e}} v_{T,\mathrm{e}}}{\tau_{\mathrm{ei}}} \left[-n_{\mathrm{e}} \frac{V_{\mathrm{ei},\|}}{v_{T,\mathrm{e}}} + \frac{1}{\sqrt{2}} \sum_{k=1} a_{\mathrm{ei}}^{10k} N^{1k} \right]
\end{aligned}
$$

- Parallel moments are calculated first $\Rightarrow$ then implicit leapfrog for MHD equations
  - Numerical instability from the viscosity term



Time derivative effect on Braginskii closures

- Braginskii closure is obtained from 8 moment ($\texttt{numL} = 4, \texttt{numK} = 2$) equations below

$$\cancel{\frac{\partial N_{\parallel}^{jp}}{\partial t}} = \frac{1}{\tau} \sum_{lk \neq \mathrm{M}} c^{jp,lk} N_{\parallel}^{lk} + g_{\parallel}^{jp}$$

- Comparing with the (4,2) moment solution which keeps the transient behavior
- Viscosity from the moment equation is not applied on the MHD
- poly_deg=4, (mx,my)=(64,48)

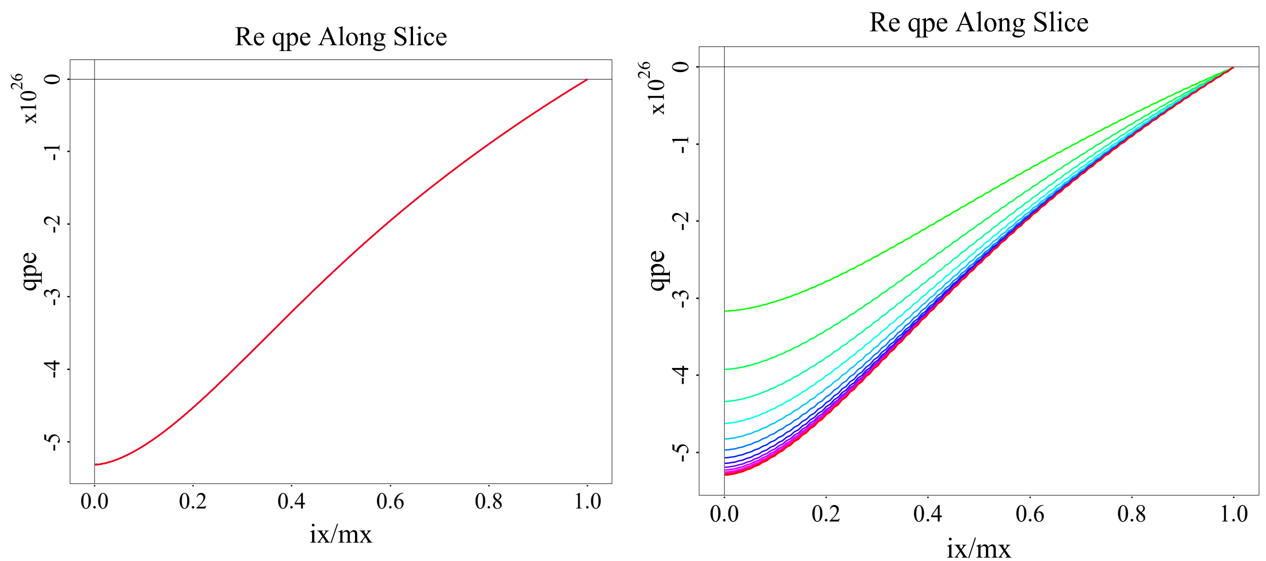


Figure 1: Radial time slice profile of electron parallel heat flux $h_{\mathrm{e},\parallel}$ from Braginskii closure (left) and time dependent moment solution (right)

Time derivative effect on Braginskii closures

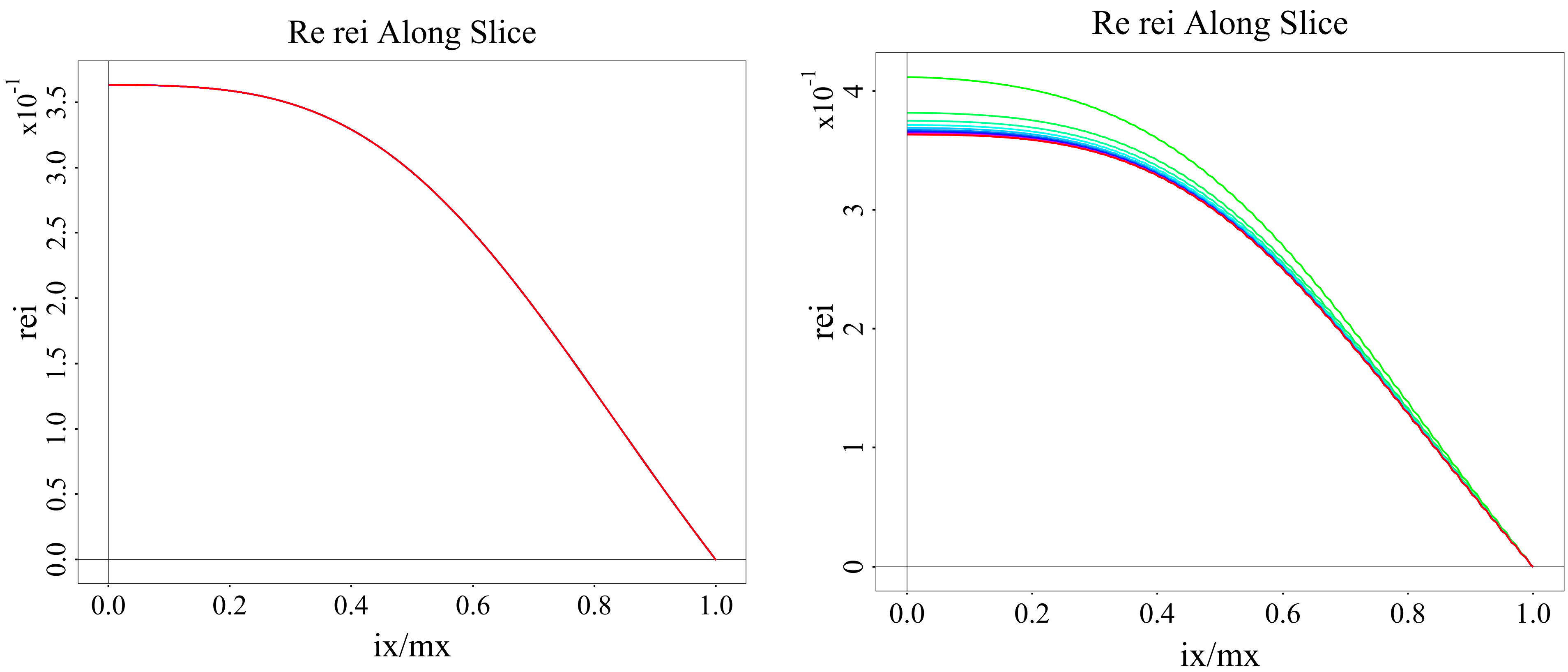


Figure 2: Braginskii electron closures $R_{\mathrm{e},\parallel}$(left) and time dependent moment closure $R_{\mathrm{e},\parallel}$(right)

Time derivative effect on Braginskii closures

- All electron closures reach the same steady state
- Ion closures need more time to reach the steady state

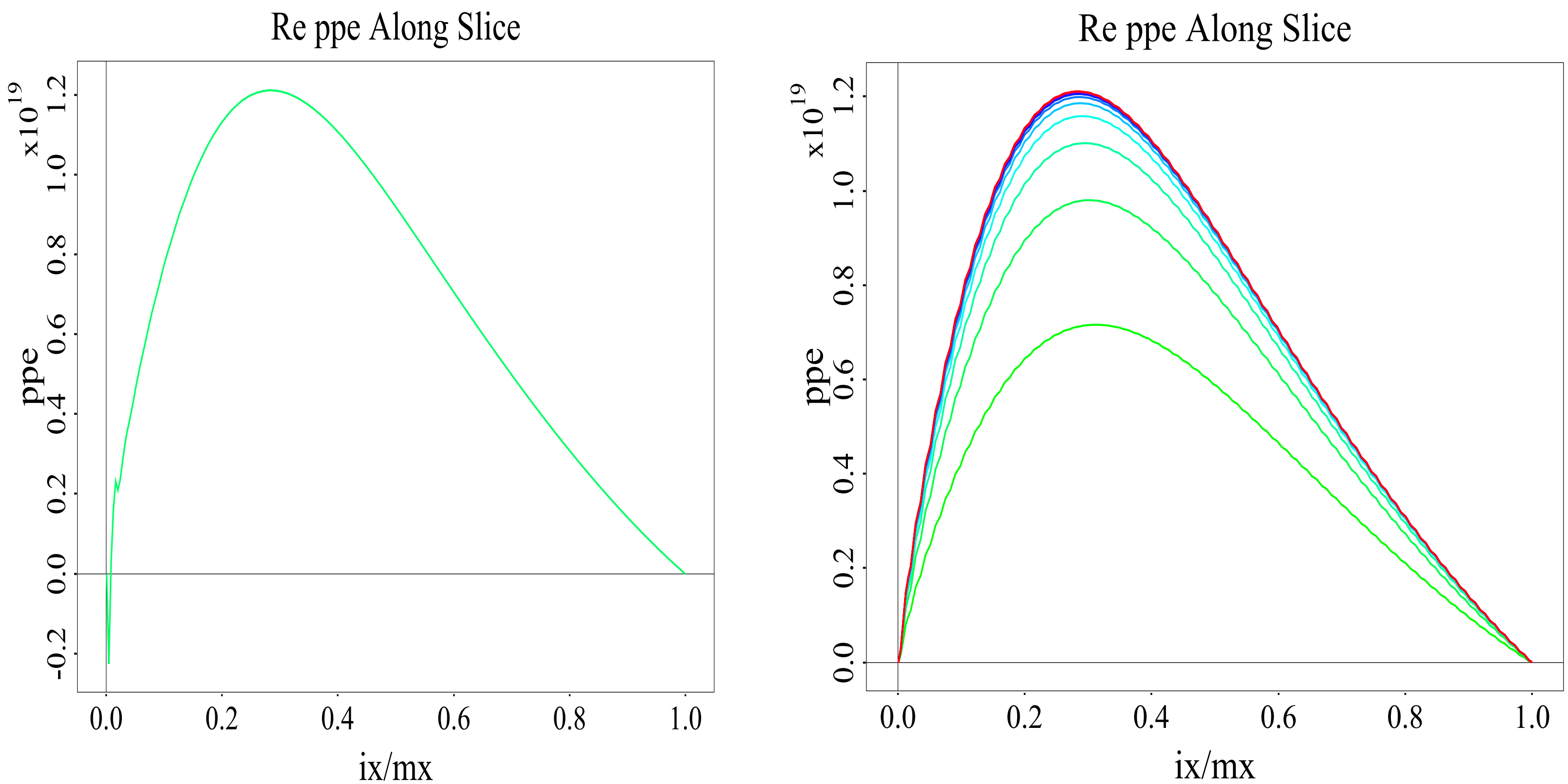


Figure 3: Braginskii electron closures $\pi_{e,\parallel}$(left) and time dependent moment closure $\pi_{e,\parallel}$(right)

Effects of the streaming operator on steady state solutions

- Comparing closures from 8 moment ($\texttt{numL} = 4, \texttt{numK} = 2$) equations with and without the streaming operator term

$$\frac{\partial N_{\parallel}^{jp}}{\partial t} + v_T \sum_{lk \neq \mathrm{M}} \left[\mathbf{\Psi_B}^{jp,lk} \frac{\partial \ln B}{\partial \ell} N_{\parallel}^{lk} + \mathbf{\Psi}^{jp,lk} \frac{\partial N_{\parallel}^{lk}}{\partial \ell} \right] = \frac{1}{\tau} \sum_{lk \neq \mathrm{M}} c^{jp,lk} N_{\parallel}^{lk} + g_{\parallel}^{jp}$$

- Viscosity from the moment equation is not applied on the MHD

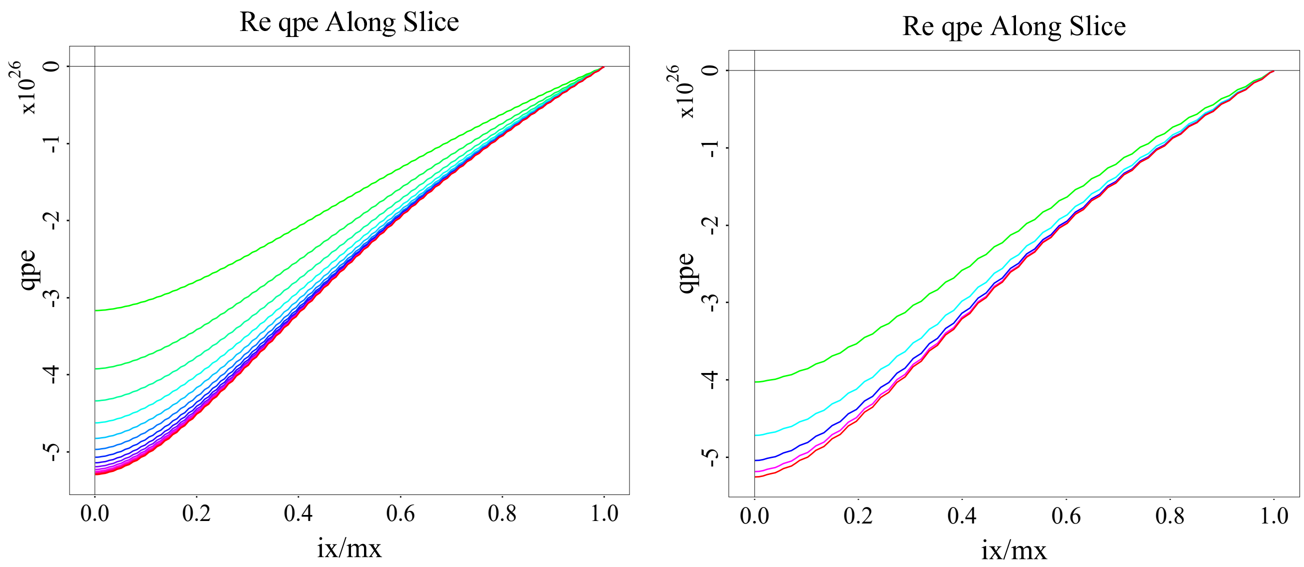


Figure 4: Electron parallel heat flux $h_{e,\parallel}$ excluding (left) / including (right) the streaming operator

Effects of the streaming operator on steady state solutions

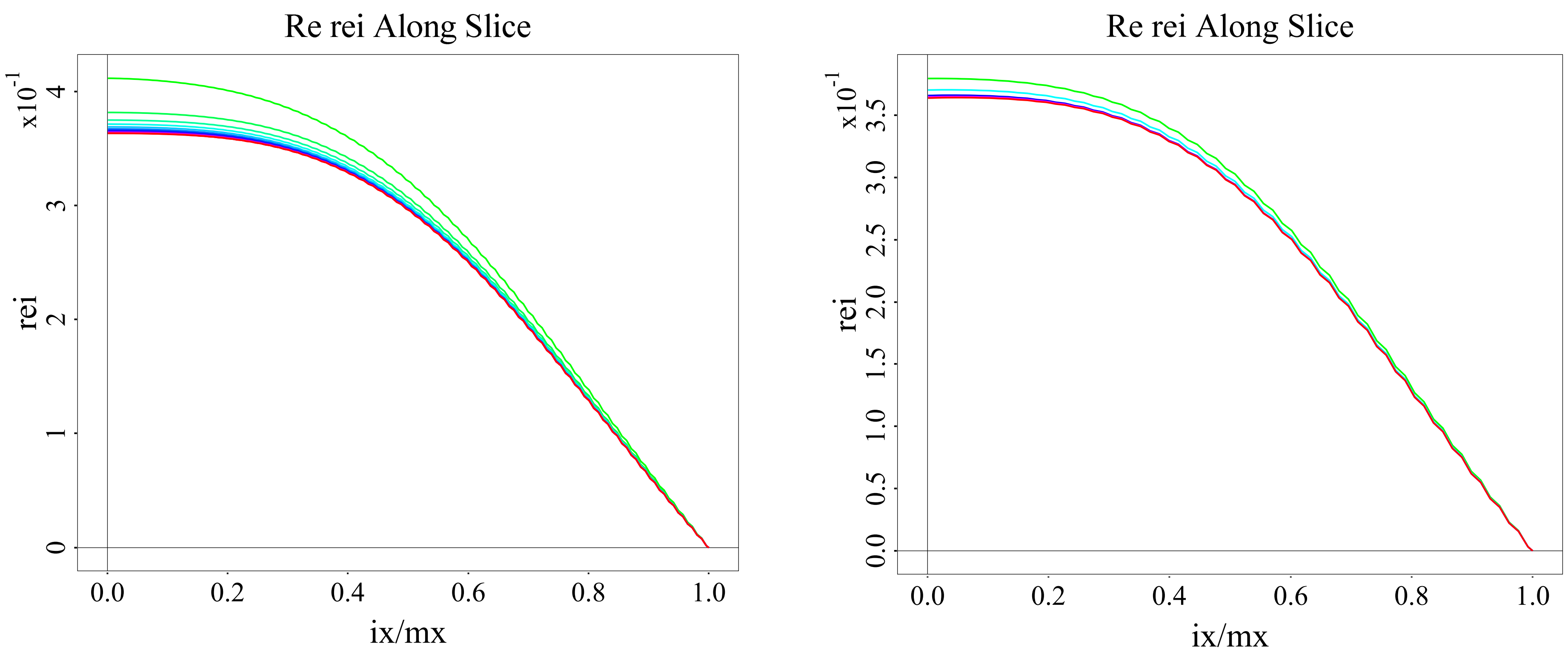


Figure 5: Electron parallel friction force density $R_{\mathrm{e},\parallel}$ excluding (left) / including (right) the streaming operator

Effects of the streaming operator on steady state solutions

- Electron parallel viscosity is reduced by a factor of ten with the streaming operator

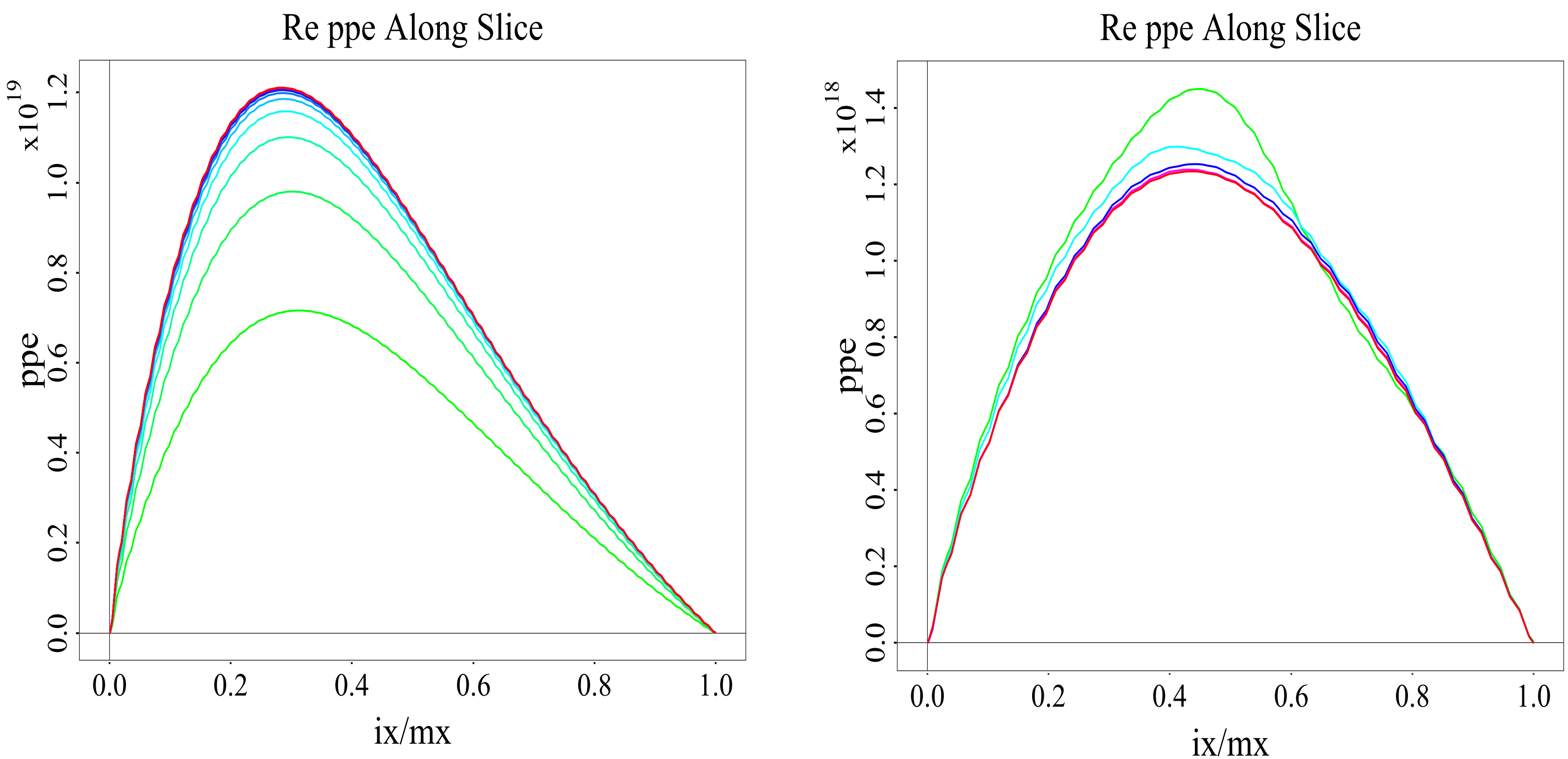


Figure 6: Electron parallel viscosity $\pi_{e,\parallel}$ excluding (left) / including (right) the streaming operator

Future work

- Improving the numerical stability of MHD advance when applying the parallel viscosity
- Checking the convergence of solutions with an increasing number of moment equations
- Comparing the ion parallel flow results with the continuum DKE solver
- Calculating the bootstrap current from the steady state solution
  - Using the electron momentum equation to distinguish the bootstrap current from the Ohmic current
- Applying the closure calculation in other tokamak setups