Progress and Plans for the Nonlinear Relativistic Collision Operator

Tyler Markham, Eric Held, Jeong-Young Ji, Andrew Spenser

Utah State University

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Motivation

* Goal: Develop a self-consistent, numerically tractable formulation for RE modeling based on a continuum solution of the relativistic drift kinetic equation (rDKE) with a tight coupling to NIMROD's plasma fluid model.

Relativistic DKE for NIMROD

NORSE Code Kinetic Equation (Stahl 2017)[1]:

$$\frac{\partial f}{\partial t} - \frac{eE_{\parallel}}{m_e c} \cdot \nabla_s f + \nabla_s \cdot (F_s f) = C(f) + S$$

- $C(f)$: nonlinear relativistic collision operator.
- F_s: synchrotron radiation reaction force.
- S: sources and sinks (heat or particles)
- Verify with results from NORSE code.

* Is assumed to be zero for now.

** Main focus: Nonlinear Operator (reproduce Stahl's work)

[1] A. Stahl a,*, M. Landreman b, O. Embréus a, T. Fülöpa, "NORSE: A solver for the relativistic non-linear Fokker–Planck equation for electrons in a homogeneous plasma", Computer Physics Communications 212 (2017) 269–279

Relativistic Distributions

Initial Conditions

- RE distribution assumed to be small fraction of background electron number density.
  - $z \equiv mc^2/kT$, $n = \beta n_e$, $\tilde{f} \equiv f/m_e^3$, $s_{V\parallel} \equiv \gamma_{V\parallel} V_\parallel / c$, $\gamma = \sqrt{1+s^2}$
- Un-Shifted Maxwell-Jüttner Distribution (no initial flow velocity):

$$\tilde{f}^M = \frac{nz}{4\pi c^3 K_2(z)} e^{-z\gamma}$$

- Shifted Maxwell-Jüttner Distribution (parallel flow):

$$\tilde{f}^M = \frac{n_R z}{4\pi c^3 K_2(z)} e^{-z\left(\gamma\gamma_{V\parallel} - s_\parallel s_{V\parallel}\right)}$$

- Two Shifted Maxwell-Jüttner Distributions (+/- parallel flow):

$$\tilde{f}^{eq} = \frac{n_R z}{4\pi c^3 K_2(z)} \left(e^{-z\left(\gamma\gamma_{V\parallel} - s_\parallel s_{V\parallel}\right)} + e^{-z\left(\gamma\gamma_{V\parallel} + s_\parallel s_{V\parallel}\right)}\right)$$

Electric Field Acceleration and Synchrotron Radiation Reaction Force

- 2D case: $f = f(s, \xi, t)$ where $\xi = p_{\parallel}/p = s_{\parallel}/s$
- Guiding-center plasma kinetic equation with radiation reaction force and nonlinear relativistic collision operator:[2]

$$\frac{\partial f}{\partial \tilde{t}} - \tilde{\mathrm{E}}_{\parallel} \cdot \boldsymbol{\nabla}_s f + \boldsymbol{\nabla}_s \cdot (\mathrm{F}_s f) = C(f)$$

- Normalized variables: $\tilde{t} \equiv t/\tau_c$ and $\tilde{\mathrm{E}}_{\parallel} \equiv \mathrm{E}_{\parallel}/E_c$, where $E_c \equiv m_e c/e\tau_c$ is the critical electric field for REs.[3]

* Spatial dependence *is* implemented but has not been tested, and will not be discussed here.

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[2] A. Stahl a,*, M. Landreman b, O. Embréus a, T. Fülöpa, "NORSE: A solver for the relativistic non-linear Fokker–Planck equation for electrons in a homogeneous plasma", Computer Physics Communications 212 (2017) 269–279

[3] J.W. Connor and R.J. Hastie 1975 Nucl. Fusion 15 415

Beliaev-Budker: "Relativistic Landau" Collision Operator

$$C(f_a, f_b) = \frac{q_a^2 q_b^2 \ln \Lambda_{ab}}{8\pi\epsilon_0^2} \frac{\partial}{\partial p_1^i} \int U^{ij} \left(f_b \frac{\partial f_a}{\partial p_1^j} - f_a \frac{\partial f_b}{\partial p_2^j} \right) \mathrm{d}^3 p_2$$

$$U^{ij} = \mathcal{U} \left[\left(\gamma_{12}^2 - 1\right) \eta^{ij} - \frac{\gamma_1^2}{c^2} v_1^i v_1^j - \frac{\gamma_2^2}{c^2} v_2^i v_2^j + \frac{\gamma_{12}^2}{c^2} \left(\frac{v_1^i v_2^j + v_2^i v_1^j}{1 - \frac{v_1^n v_{2n}}{c^2}} \right) \right]$$

$$\mathcal{U} \equiv \frac{\gamma_{12}^2}{\gamma_1 \gamma_2 c \left(\gamma_{12}^2 - 1\right)^{\frac{3}{2}}}$$

$$\gamma_{12} \equiv \gamma_1 \gamma_2 \left(1 - \frac{v_1^n v_{2n}}{c^2} \right)$$

* $(\gamma_{12} - 1) m_a c^2 = kT_a$ is assumed to pull out of integral
* γ_{12} is a bi-product of the relative velocity constructed from an arbitrary Lorentz boost.
  - Time-dilation projection factors

- Braams and Karney Formulation:[4]

$$\Upsilon_0 \equiv -\frac{1}{4\pi}\int \left(\gamma_{12}^2 - 1\right)^{-\frac{1}{2}} f_b\left(\boldsymbol{x}_2, \boldsymbol{s}_2, t\right)\frac{\mathrm{d}^3 s_2}{\gamma_2}$$

$$\Upsilon_1 \equiv -\frac{1}{8\pi}\int \left(\gamma_{12}^2 - 1\right)^{\frac{1}{2}} f_b\left(\boldsymbol{x}_2, \boldsymbol{s}_2, t\right)\frac{\mathrm{d}^3 s_2}{\gamma_2}$$

$$\Upsilon_2 \equiv -\frac{1}{32\pi}\int \left[\gamma_{12}\cosh^{-1}\left(\gamma_{12}\right) - \left(\gamma_{12}^2 - 1\right)\right] f_b\left(\boldsymbol{x}_2, \boldsymbol{s}_2, t\right)\frac{\mathrm{d}^3 s_2}{\gamma_2}$$

$$\Pi_0 \equiv -\frac{1}{4\pi}\int \gamma_{12}\left(\gamma_{12}^2 - 1\right)^{-\frac{1}{2}} f_b\left(\boldsymbol{x}_2, \boldsymbol{s}_2, t\right)\frac{\mathrm{d}^3 s_2}{\gamma_2}$$

$$\Pi_1 \equiv -\frac{1}{4\pi}\int \cosh^{-1}\left(\gamma_{12}\right) f_b\left(\boldsymbol{x}_2, \boldsymbol{s}_2, t\right)\frac{\mathrm{d}^3 s_2}{\gamma_2}$$

[4] Bastiaan J. Braams and Charles F. F. Karney Phys. Rev. Lett. 59, 1817

Differential Formulation of the Beliaev-Budker Operator

Relativistic Analogue to Rosenbluth Potentials

$$L_0 \Upsilon_0 = f_b \qquad L_2 \Upsilon_1 = \Upsilon_0 \qquad L_2 \Upsilon_2 = \Upsilon_1$$

$$L_1 \Pi_0 = f_b \qquad L_1 \Pi_1 = \Pi_0$$

$$L_a \Psi \equiv (\boldsymbol{I} + \boldsymbol{s}_1 \boldsymbol{s}_1) : \boldsymbol{\nabla}_{s_1} \boldsymbol{\nabla}_{s_1} \Psi + 3 \boldsymbol{s}_1 \cdot \boldsymbol{\nabla}_{s_1} \Psi + \left(1 - a^2\right) \Psi$$

- If the pitch-angle basis for Υ_a, Π_a, and f_b are the Legendre polynomials $P_l(\xi)$, then this can be re-expressed as:

$$L_{0,l} \Upsilon_{0,l} = f_{b,l} \qquad L_{2,l} \Upsilon_{1,l} = \Upsilon_{0,l} \qquad L_{2,l} \Upsilon_{2,l} = \Upsilon_{1,l}$$

$$L_{1,l} \Pi_{0,l} = f_{b,l} \qquad L_{1,l} \Pi_{1,l} = \Pi_{0,l}$$

$$L_{a,l} \Psi \equiv \gamma_1^2 \frac{\partial^2 \Psi}{\partial s_1^2} + \left(\frac{2}{s_1} + 3 s_1 \right) \frac{\partial \Psi}{\partial s_1} + \left(1 - a^2 - \frac{l(l+1)}{s_1^2} \right) \Psi$$

- $C(f) = C_{ee}(f) + C_{ei}(f)$
- $\nu \equiv \frac{ne^4 \ln \Lambda_{ee}}{4\pi\epsilon_0^2 m_e^2 c^3}$
- $\alpha \equiv \frac{4\pi\nu}{n}$
- Assume a stationary, Maxwellian ion population.
- Utilizing $m_e \ll m_i$ and quasi-neutrality, $C_{ei}(f)$ is the given by:

$$C_{ei}(f) \approx Z_{eff} \frac{\nu\gamma}{s^3} \left[\frac{1}{2} \frac{\partial}{\partial \xi} \left(\left(1 - \xi^2\right) \frac{\partial f}{\partial \xi} \right] \right.$$

$$\frac{C_{ee}(f)}{\alpha} = C^{(s^2)} \frac{\partial^2 f}{\partial s^2} + C^{(s)} \frac{\partial f}{\partial s} + C^{(\xi^2)} \frac{\partial^2 f}{\partial \xi^2} + C^{(\xi)} \frac{\partial f}{\partial \xi} + C^{(s\xi)} \frac{\partial^2 f}{\partial s \partial \xi} + C^{(f)} f$$

- Let $\Upsilon_- \equiv 4\Upsilon_2 - \Upsilon_1$, $\Upsilon_+ \equiv 4\Upsilon_2 + \Upsilon_1$, and $\Pi \equiv 2\Pi_1 - \Pi_0$

$$
\begin{aligned}
C^{(s^2)} \equiv & \gamma\left(8\Upsilon_2 - \Upsilon_0\right) - 2\frac{\gamma^3}{s}\frac{\partial \Upsilon_-}{\partial s} - \frac{\gamma(1-\xi^2)}{s^2}\frac{\partial^2 \Upsilon_-}{\partial \xi^2} + 2\frac{\gamma\xi}{s^2}\frac{\partial \Upsilon_-}{\partial \xi} \\
C^{(s)} \equiv & \frac{1}{\gamma s}(2+3s^2)\left(8\Upsilon_2 - \Upsilon_0\right) - 16\gamma\frac{\partial \Upsilon_2}{\partial s} + 6\gamma\frac{\partial \Upsilon_1}{\partial s} - \gamma\frac{\partial \Upsilon_0}{\partial s} \\
& - 2\frac{\gamma^3}{s}\left(\frac{\partial^2 \Upsilon_-}{\partial s^2} + \frac{1}{s}\frac{\partial \Upsilon_-}{\partial s}\right) \\
& + \frac{1}{\gamma s}\left(2 + \frac{1}{s^2}\right)\left(2\xi\frac{\partial \Upsilon_-}{\partial \xi} - (1-\xi^2)\frac{\partial^2 \Upsilon_-}{\partial \xi^2}\right) - \gamma\frac{\partial \Pi}{\partial s}
\end{aligned}
$$

$$C^{(\xi^2)} \equiv \frac{1-\xi^2}{\gamma s^2}\left(\frac{\gamma^2}{s}\frac{\partial \Upsilon_-}{\partial s} + \frac{1}{s^2}\left[(1-\xi^2)\frac{\partial^2\Upsilon_-}{\partial \xi^2} - \xi\frac{\partial \Upsilon_-}{\partial \xi}\right] - \Upsilon_+\right)$$

$$\begin{aligned} C^{(\xi)} \equiv & -\frac{\xi(1-\xi^2)}{\gamma s^4}\frac{\partial^2\Upsilon_-}{\partial \xi^2} - 2\frac{\gamma(1-\xi^2)}{s^3}\frac{\partial^2\Upsilon_-}{\partial s\partial \xi} - 2\frac{\gamma\xi}{s^3}\frac{\partial \Upsilon_-}{\partial s} \\ & + \left(\frac{2}{\gamma s^4} + 3\frac{(1-\xi^2)}{\gamma s^2}\right)\frac{\partial \Upsilon_-}{\partial \xi} \\ & - \frac{(1-\xi^2)}{\gamma s^2}\left(4\frac{\partial \Upsilon_-}{\partial \xi} - 3\frac{\partial \Upsilon_1}{\partial \xi} + \frac{\partial \Upsilon_0}{\partial \xi} + \frac{\partial \Pi}{\partial \xi}\right) + 2\frac{\xi}{\gamma s^2}\Upsilon_+ \end{aligned}$$

$$C^{(s\xi)} \equiv 2\frac{\gamma(1-\xi^2)}{s^3}\left(s\frac{\partial^2\Upsilon_-}{\partial s\partial\xi} - \frac{\partial\Upsilon_-}{\partial\xi}\right)$$

$$C^{(f)} \equiv -\gamma\frac{\partial^2\Pi}{\partial s^2} - \frac{1}{\gamma s}(2+3s^2)\frac{\partial\Pi}{\partial s} - \frac{(1-\xi^2)}{\gamma s^2}\frac{\partial^2\Pi}{\partial\xi^2} + 2\frac{\xi}{\gamma s^2}\frac{\partial\Pi}{\partial\xi}$$

- 10 new ξ coupling matrices need to be pre-computed
- Call potential solver routine for each time-step to feed into the RHS

Plans and Progress

Progress:

- Potential solver routine has been created
- Currently verifying the integrity of the solution vector
  - Implementation of $L_{a,l}$
  - Residual calculation

Plans:

- Build and vet the new ξ coupling matrices
- Create new routine in collision_op.f90 to calculate collision coupling matrix.
- Add diagnostic to check 2D phase-space contour plots of the potentials as function of time

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