Implementing general moment equations for parallel closures in NIMROD*

Hankyu Lee, J.A. Spencer, E.D. Held, Jeong-Young Ji

Utah State University

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Solving the first order drift kinetic equation

- The first order drift kinetic equation

$$\mathbf{v}_{\|} \cdot \nabla \bar{f}_1 + \mathbf{v}_{\mathrm{D}} \cdot \nabla \bar{f}_0 + q \mathbf{v}_{\|} \cdot \mathbf{E}^{(\mathbf{A})} \frac{\partial \bar{f}_0}{\partial U} = C\left(\bar{f}_1\right)$$

- Parallel moment expansion to solve the equation

$$\bar{f}_1(\mathbf{x}, v, v_{\|}) = f_0 \sum_{lk} \hat{P}^{lk}(v, v_{\|}) M^{lk}(\mathbf{x})$$

$$\hat{P}^{lk}(v, v_{\|}) = \frac{1}{\sqrt{\sigma_{lk}}} s^l P_l(\xi) L_k^{l+1/2}(s^2)$$

  - $f_0 = \dfrac{n_0 \exp(-s^2)}{\left(\sqrt{\pi} v_T\right)^3}$, $v_T = \sqrt{\dfrac{2T}{m}}$, $s = \dfrac{v}{v_T}$, $\xi = \dfrac{v_{\|}}{v}$
  - $M^{lk}(\mathbf{x})$: Parallel moments

Non-Maxwellian parallel moment equation

- Maxwellian and non-Maxwellian parts of the first order distribution function

$$\bar{f}_1 = \bar{f}_1^{\mathrm{M}} + \bar{f}_1^{\mathrm{N}} = f_0 \left(\sum_{lk=\mathrm{M}} \hat{P}^{lk} M^{lk} + \sum_{lk\neq\mathrm{M}} \hat{P}^{lk} M^{lk} \right)$$

  - Maxwellian part $\left(\bar{f}_1^{\mathrm{M}}\right)$: terms for $(l,k) = (0,0)\,,(0,1)\,,(1,0)$
  - Non-Maxwellian part $\left(\bar{f}_1^{\mathrm{N}}\right)$: terms for $(l,k) = (0,2)\,,\cdots$

- Non-Maxwellian (j,p) parallel moment equation

$$\int d\mathbf{v} \hat{P}^{jp} \left[\mathbf{v}_{\|} \cdot \nabla \bar{f}_1 + \mathbf{v}_{\mathrm{D}} \cdot \nabla \bar{f}_0 + q \mathbf{v}_{\|} \cdot \mathbf{E}^{(\mathbf{A})} \frac{\partial \bar{f}_0}{\partial U} = C\left(\bar{f}_1\right) \right] \Rightarrow$$

$$\sum_{lk\neq\mathrm{M}} \left[\mathbf{\Psi}^{jp,lk} \partial_{\|} M^{lk} + \mathbf{\Psi_B}^{jp,lk} \left(\partial_{\|} \ln B\right) \hat{M}^{lk} \right] = \frac{1}{\lambda_{\mathrm{C}}} \sum_{lk\neq\mathrm{M}} c^{jp,lk} M^{lk} + g_{\|}^{jp}$$

  - $g_{\|}^{jp}$: Driving term from the Maxwellian part $\left(\mathbf{v}_{\mathrm{D}} \cdot \nabla f_0, \mathbf{v}_{\|} \cdot \nabla \bar{f}_1^{\mathrm{M}},\ C(\bar{f}_1^{\mathrm{M}})\right)$

Solving the parallel moment equations

- Truncated non-Maxwellian parallel moment equations
  - With increasing the number of moments, a convergent solution is obtained
  - $\partial_\parallel^{j+} = \partial_\parallel - \dfrac{j+2}{2}\partial_\parallel \ln B$, $\partial_\parallel^{j-} = \partial_\parallel + \dfrac{j-1}{2}\partial_\parallel \ln B$

$$
\begin{bmatrix}
 & \mathbf{0} & \psi_{21}^{0+}\partial_\parallel^{0+} & \psi_{22}^{0+}\partial_\parallel^{0+} & & \mathbf{0} & \cdots \\
 & & & \psi_{32}^{0+}\partial_\parallel^{0+} & & & \\
\psi_{12}^{1-}\partial_\parallel^{1-} & & & \mathbf{0} & \psi_{10}^{1+}\partial_\parallel^{1+} & \psi_{11}^{1+}\partial_\parallel^{1+} & \mathbf{0} \\
\psi_{22}^{1-}\partial_\parallel^{1-} & \psi_{23}^{1-}\partial_\parallel^{1-} & & & & \psi_{21}^{1+}\partial_\parallel^{1+} & \\
 & & \psi_{01}^{2-}\partial_\parallel^{2-} & & & & \\
 & \mathbf{0} & \psi_{11}^{2-}\partial_\parallel^{2-} & \psi_{12}^{2-}\partial_\parallel^{2-} & & \mathbf{0} & \cdots \\
 & \vdots & & \mathbf{0} & & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix} M^{02} \\ M^{03} \\ M^{11} \\ M^{12} \\ M^{20} \\ \vdots \\ M^{\mathrm{LK}} \end{bmatrix}
$$

$$
= \frac{1}{\lambda_{\mathrm{C}}}
\begin{bmatrix}
c^{02,02} & c^{02,03} & & \mathbf{0} & & \cdots \\
c^{03,02} & c^{03,03} & & & & \\
 & \mathbf{0} & c^{11,11} & c^{11,12} & \mathbf{0} & \cdots \\
 & & c^{12,11} & c^{12,12} & & \\
 & \vdots & & \mathbf{0} & & \ddots
\end{bmatrix}
\begin{bmatrix} M^{02} \\ M^{03} \\ M^{11} \\ M^{12} \\ M^{20} \\ \vdots \\ M^{\mathrm{LK}} \end{bmatrix}
+
\begin{bmatrix} g_\parallel^{02} \\ 0 \\ g_\parallel^{11} \\ g_\parallel^{12} \\ g_\parallel^{20} \\ g_\parallel^{21} \\ \vdots \\ g_\parallel^{\mathrm{LK}} \end{bmatrix}
$$

Fourier method of solving parallel moment equations [1]

- Physical quantities are expanded in Fourier series
- $\partial_{\|} = \frac{\mathbf{B}\cdot\nabla\theta}{B}\frac{\partial}{\partial\theta} = \frac{B^\theta}{B}\frac{\partial}{\partial\theta}$

$$\sum_{lk\neq \mathrm{M}}\left[\boldsymbol{\Psi}^{jp,lk}\partial_{\|}M^{lk} + \boldsymbol{\Psi}_{\mathbf{B}}{}^{jp,lk}\left(\partial_{\|}\ln B\right)M^{lk}\right] = \frac{1}{\lambda_{\mathrm{C}}}\sum_{lk\neq \mathrm{M}} c^{jp,lk}M^{lk} + g_{\|}^{jp}$$

$$\sum_{lk\neq \mathrm{M}}\left[\boldsymbol{\Psi}^{jp,lk}\partial_{\theta} + \boldsymbol{\Psi}_{\mathbf{B}}{}^{jp,lk}\left(\partial_{\theta}\ln B\right) - \frac{B}{B^\theta\lambda_{\mathrm{C}}}c^{jp,lk}\right]M^{lk} = g_{\theta}^{jp}$$

- $M^{lk}\left(\theta\right) = M^{lk}_{(0)} + M^{lk}_{(1-)}\sin\theta + M^{lk}_{(1+)}\cos\theta + M^{lk}_{(2-)}\sin 2\theta + \cdots$
- $g_{\theta}^{jp} = \frac{\partial_\theta \ln B}{B/B_0}\left(g_p^{jp}\hat{p}_{0,\psi} + g_T^{jp}\hat{T}_{0,\psi}\right) + G_\theta^{jp}$

Obtaining Fourier coefficients

- Fourier modes

$$\varphi_{(0)} = 1, \ \varphi_{(2n-1)} = \varphi_{(n-)} = \sin n\theta, \ \varphi_{(2n)} = \varphi_{(n+)} = \cos n\theta$$

- RHS term: $\underbrace{\frac{\partial_\theta \ln B}{B/B_0}}_{A(\theta)} \left(g_p^{jp} \hat{p}_{0,\psi} + g_T^{jp} \hat{T}_{0,\psi} \right)$

$$A_{(m)} = \frac{1}{\sigma_{(m)}} \int d\theta \varphi_{(m)} A(\theta)$$

- Matrix elements: $\mathcal{O} = \partial_\theta, \ \partial_\theta \ln B, \ B/B^\theta \lambda_{\mathrm{C}}$

$$\left(\mathcal{O} M^{lk}\right)_{(i)} = \frac{1}{\sigma_{(i)}} \int d\theta \varphi_{(i)} \mathcal{O} \sum_j M^{lk}_{(j)} \varphi_{(j)} = \sum_j \mathcal{O}_{(i,j)} M^{lk}_{(j)}$$

$$\mathcal{O}_{(i,j)} = \frac{1}{\sigma_{(i)}} \int d\theta \varphi_{(i)} \mathcal{O} \varphi_{(j)}$$

Finite element method of solving parallel moment equations in NIMROD

$$\sum_{lk\neq \mathrm{M}}\left[\mathbf{\Psi}^{jp,lk}\partial_\theta+\mathbf{\Psi_B}^{jp,lk}\left(\partial_\theta \ln B\right)-\frac{B}{B^\theta\lambda_\mathrm{C}}c^{jp,lk}\right]M^{lk}=g_\theta^{jp}$$

- Basis function $\alpha\left(R,Z\right)$

$$M^{lk}(R,Z)=\sum_j M_j^{lk}\alpha_j\left(R,Z\right)$$

- Matrix elements

$$\left(\mathcal{O}M^{lk}\right)_{(i)}=\int dRdZ\alpha_i\mathcal{O}\sum_j M_j^{lk}\alpha_j=\sum_j\mathcal{O}_{(i,j)}M_j^{lk}$$

$$\mathcal{O}_{(i,j)}=\int dRdZ\alpha_i\mathcal{O}\alpha_j$$

In a circular magnetic field,

$$\partial_\theta=-Z\frac{\partial}{\partial R}+\left(R-R_0\right)\frac{\partial}{\partial Z},\ \partial_\theta\ln B=-\frac{Z}{B}\frac{\partial B}{\partial R}+\frac{\left(R-R_0\right)}{B}\frac{\partial B}{\partial Z}$$

Benchmark problem

- Comparing ion parallel closures in NIMROD (Finite Element Method) to MATLAB (Fourier Method)
- Axisymmetric, high aspect ratio tokamak (major radius: $R_0 = 5\ \mathrm{m}$)
  - Circular magnetic field [4]
- Comparing the results at minor radius $r = 0.45\ \mathrm{m}$
- Knudsen number : $\dfrac{B^\theta \lambda_C}{B} = 100$
- Comparing the results from the $\hat{p}_{0,\psi}$ and $\hat{T}_{0,\psi}$ drives

$$
\begin{aligned}
\hat{p}_{0,\psi} &= \frac{I}{q v_T B_0 n_0} \frac{dp_0}{d\psi} \\
\hat{T}_{0,\psi} &= \frac{I}{q v_T B_0} \frac{dT_0}{d\psi}
\end{aligned}
$$

Ion parallel closures with increasing Fourier modes

- Results are normalized by $\hat{p}_{0,\psi}$ and $\hat{T}_{0,\psi}$

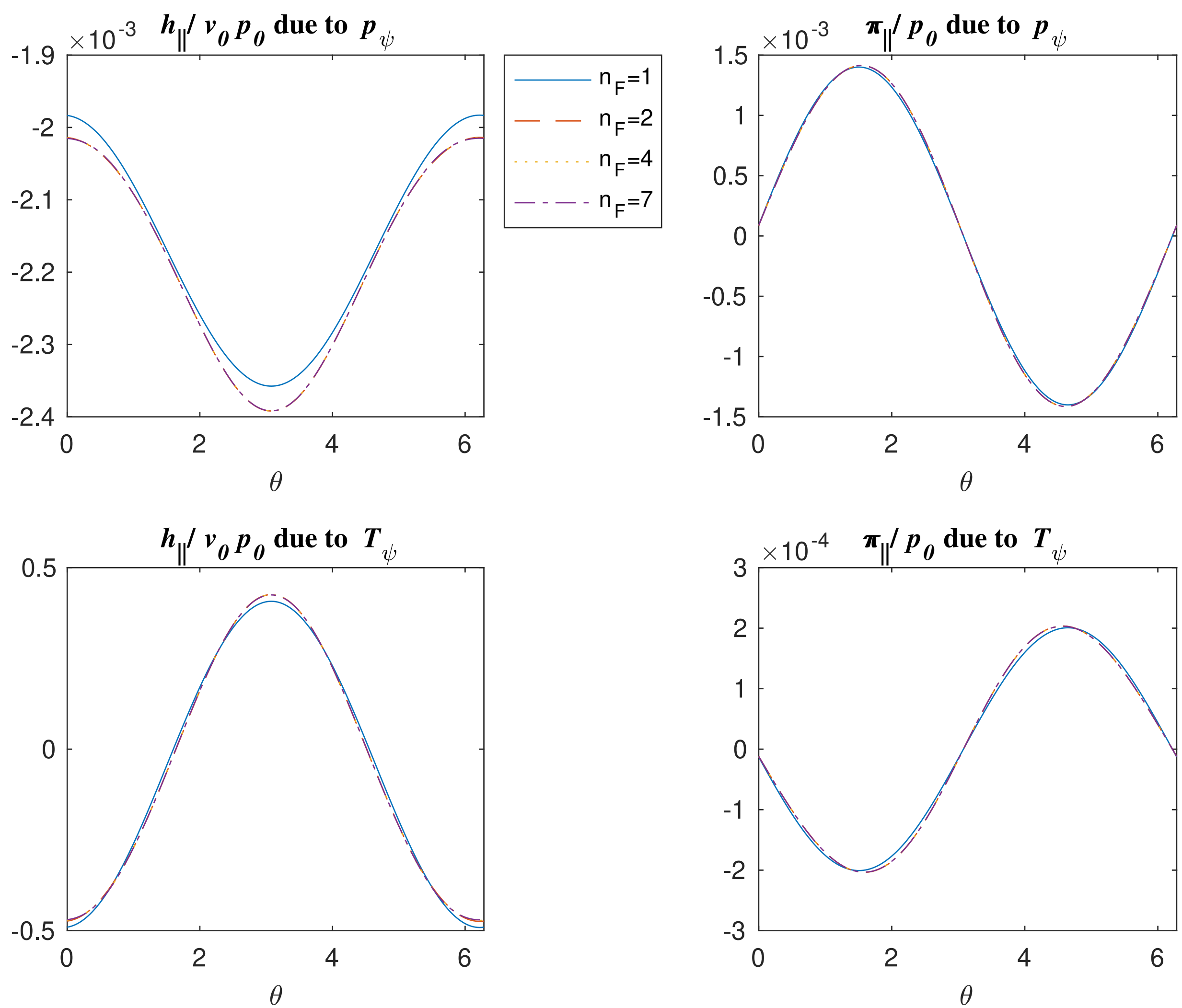

Figure 1: Parallel heat flux and viscosity due to $\frac{dp_0}{d\psi}$ and $\frac{dT_0}{d\psi}$

Ion parallel closures with increasing polynomial degrees of basis functions

- (mx,my)=(64,48)

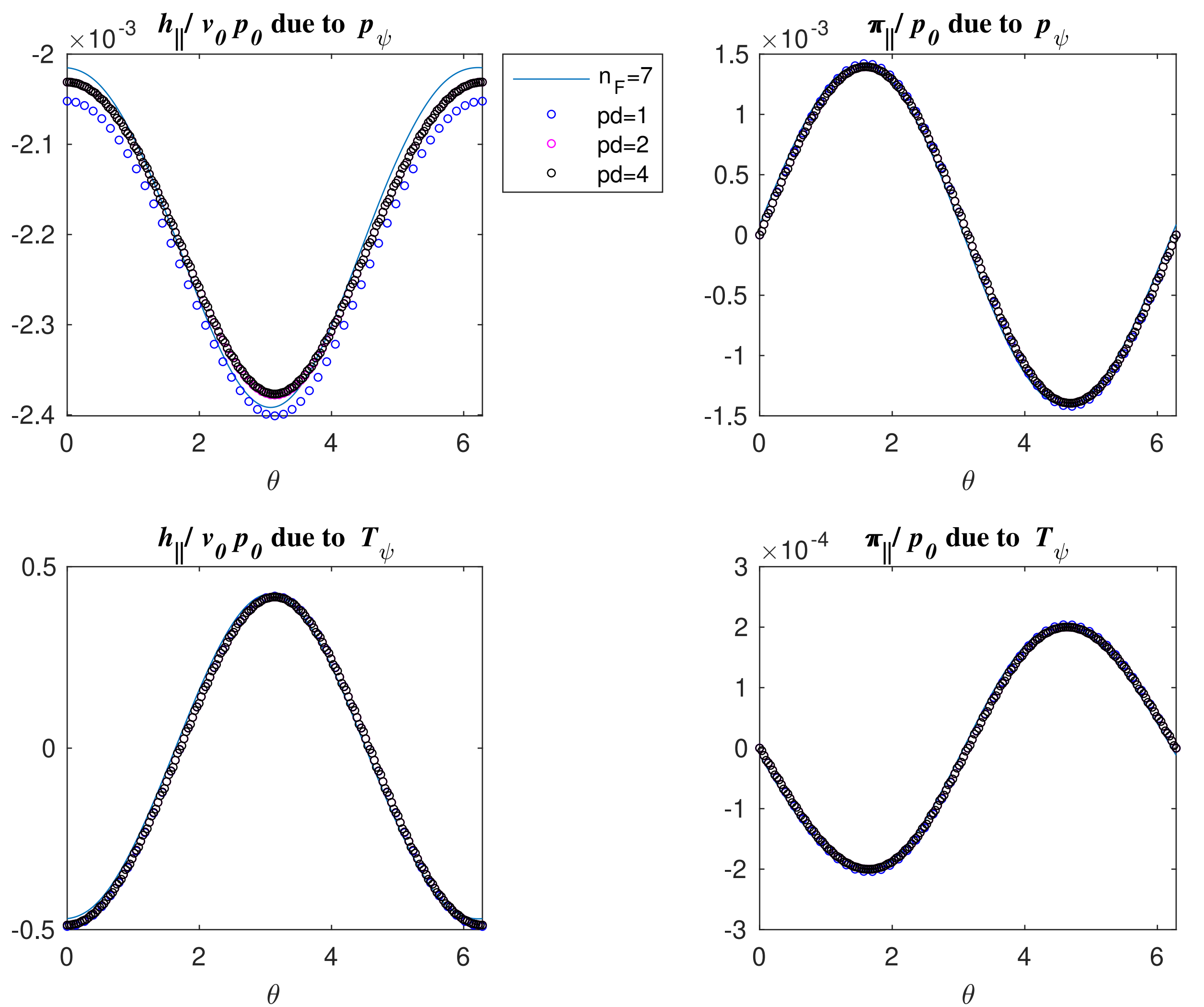

Figure 2: Parallel heat flux and viscosity due to $\frac{dp_0}{d\psi}$ and $\frac{dT_0}{d\psi}$

Ion parallel closures with increasing number of moments from 8 to 32

- (mx,my,poly_degree)=(64,48,2)

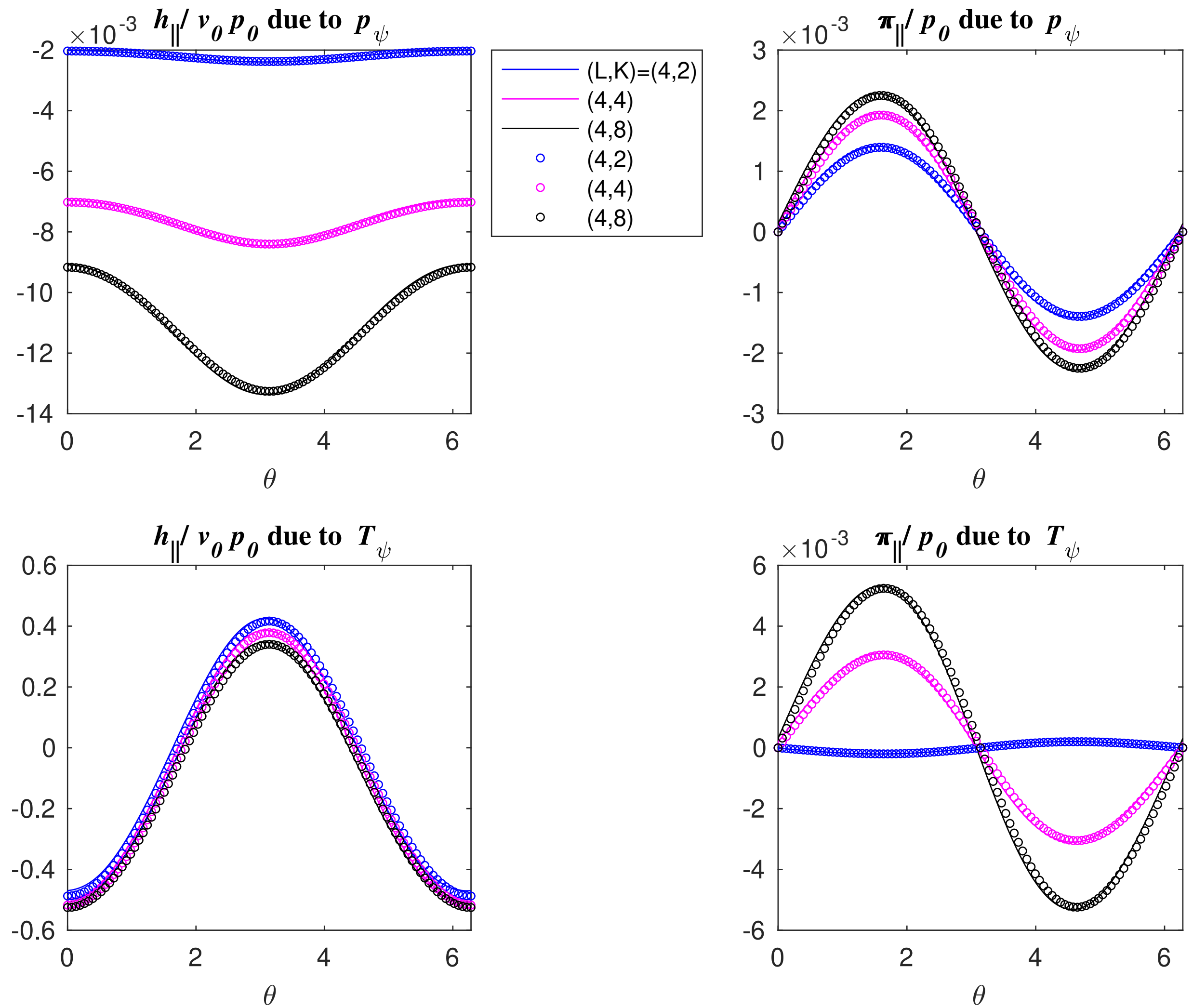

Figure 3: Parallel heat flux and viscosity due to $\frac{dp_0}{d\psi}$ and $\frac{dT_0}{d\psi}$

Ion parallel closures with increasing number of moments from 4×8 to 64×128

- Working on parallelism for L and K indices in NIMROD

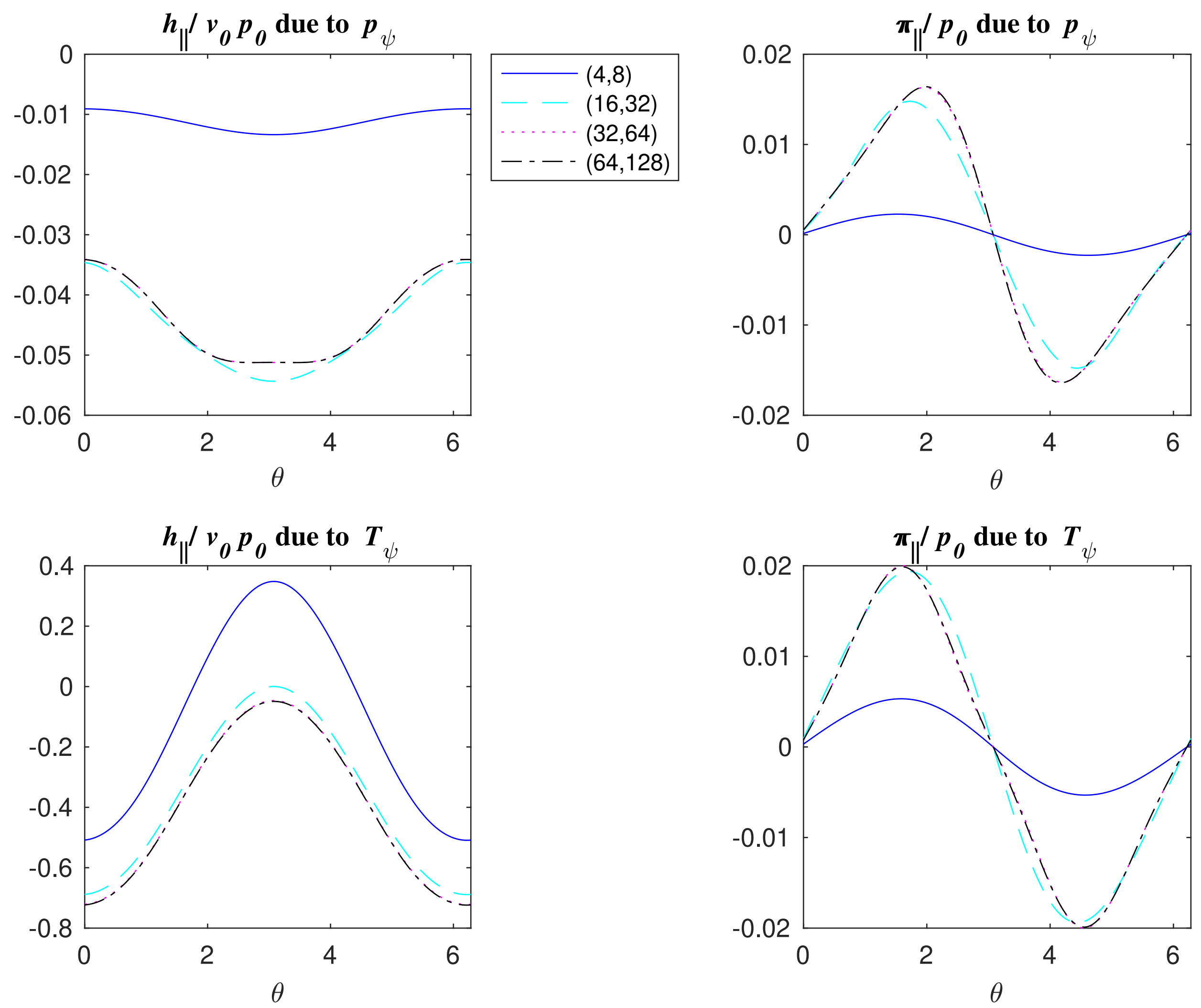

Figure 4: Parallel heat flux and viscosity due to $\frac{dp_0}{d\psi}$ and $\frac{dT_0}{d\psi}$

Future plans

- Testing parallelism for L and K indices
- Obtaining electron closures
- Solving neoclassical transport
  - Including Maxwellian moments
  - Calculating ion parallel flows [4]
- Applying closures for time dependent problems

References

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[2] J.-Y. Ji, H. Q. Lee, and E. D. Held, “Ion parallel closures”, Phys. Plasmas 24, 022127 (2017).

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[4] E. D. Held, S. E. Kruger, J.-Y. Ji, E. A. Belli, and B. C. Lyons, “Verification of continuum drift kinetic equation solvers in nimrod," Physics of Plasmas, vol. 22, 032511 (2015).