Linear Fluid Runaway Electron Beam Calculations

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Linear simulations have revealed the effects of RE current on resistive MHD instabilities.

- Prior analytic and computational results have shown that the linear behavior tearing and resistive kink modes are affected by the presence of runaway electron current.
- In a cylindrical $(2,1)$ tearing mode case, NIMROD calculations agree with published results for Lundquist number $S \gtrsim 10^4$.
- For $S \leq 10^3$, there is a distinct, faster growing mode that is localized near the origin, away from the rational surface.
- A reduced model is presented to analyze the mechanism of instability, and it is posited that it is a manifestation of the 'resistive hose' instability

The linearized equations of the model are resistive MHD with zero pressure + RE continuity.

Lowercase letter variables are perturbed quantities, capital letter variables are equilibrium quantities:

$$\partial_t n_r + \boldsymbol{\nabla} \cdot (n_r \boldsymbol{U}_r + N_r \boldsymbol{u}_r) = 0, \;\; \boldsymbol{U}_r = -c_r \frac{\boldsymbol{B}}{B}, \; \boldsymbol{u}_r = -c_r \boldsymbol{b}_\perp$$

$$\rho \partial_t \boldsymbol{v} = \boldsymbol{j} \times \boldsymbol{B} + \boldsymbol{J} \times \boldsymbol{b}, \quad \partial_t \boldsymbol{b} = -\boldsymbol{\nabla} \times \boldsymbol{e}$$

$$\boldsymbol{\nabla} \times \boldsymbol{b} = \mu_0 \boldsymbol{j},$$

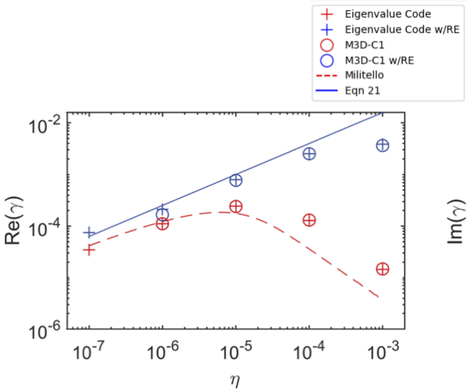
$$\boldsymbol{e} = -\boldsymbol{v} \times \boldsymbol{B} + \eta \left(\boldsymbol{j} - \boldsymbol{j}_r\right)$$

$$\boldsymbol{j}_r = -e \left(n_r \boldsymbol{U}_r + N_r \boldsymbol{u}_{\boldsymbol{r}}\right)$$

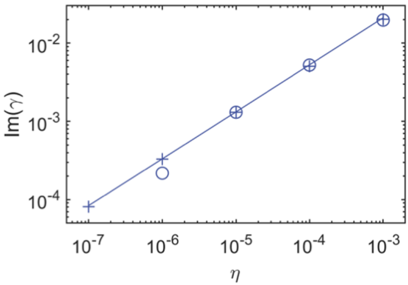
$$\boldsymbol{\nabla} \cdot \boldsymbol{b} = 0$$

For this work we fix the plasma density, ρ, to be constant, and assume the equilibrium flow $\boldsymbol{V} = 0$. $c_r > 0$ is the parallel speed of runaway electrons along magnetic field lines.

Results from Liu, et al. compare the analytic growth rate scaling with linear M3D-C1 calculations for the (2,1) tearing mode.[1]

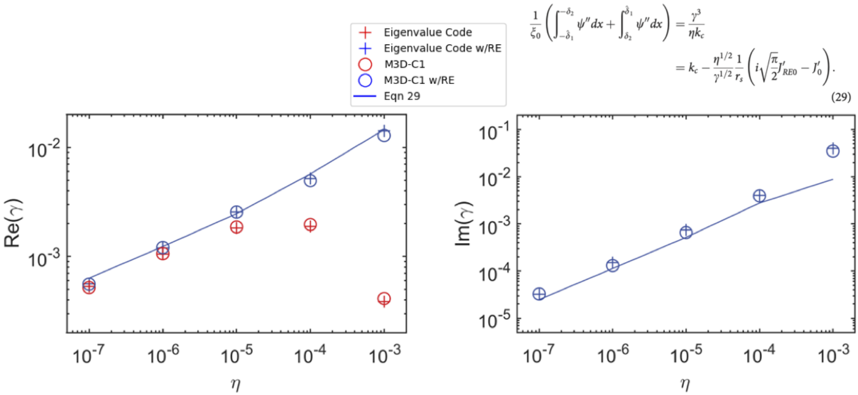


$$\frac{\gamma^{5/4}}{\eta^{3/4}k_c^{1/2}}\frac{2\pi\Gamma(3/4)}{\Gamma(1/4)} = \Delta' - i\pi\frac{mJ'_{RE0}}{|k_c|r_s}, \qquad (21)$$



[1] C. Liu et al., Physics of Plasmas **27**, 10.1063/5.0018559 (2020).

Liu et al. also analyze the effect of runaways on the linear growth rate of the (1,1) resistive kink.[2]

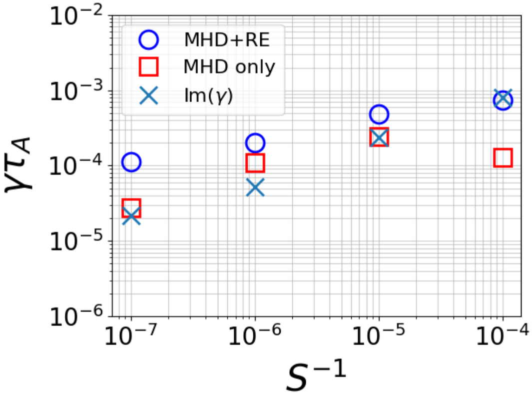


$$\frac{1}{\xi_0}\left(\int_{-\delta_1}^{-\delta_2}\psi'' dx + \int_{\delta_2}^{\delta_1}\psi'' dx\right) = \frac{\gamma^3}{\eta k_c} = k_c - \frac{\eta^{1/2}}{\gamma^{1/2}}\frac{1}{r_s}\left(i\sqrt{\frac{\pi}{2}}J'_{RE0} - J'_0\right). \quad (29)$$

[2] C. Liu et al., Physics of Plasmas **27**, 10.1063/5.0018559 (2020).

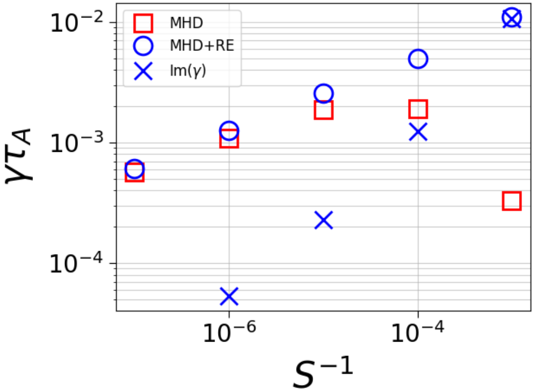
NIMROD results from the cylindrical geometry tearing mode case reproduce Liu et. al analysis.

Equilibrium setup:
$q = 1.15\left(1 + \frac{(r/a)^2}{0.6561}\right)$
$\boldsymbol{J} \times \boldsymbol{B} = 0$, $\boldsymbol{B}(0) = 1$.
$c_r = 20.0$, $V_A = 1.0$
Uniform background plasma density and resistivity
For the case with runaway electrons, all the equilibrium current is carried by runaways.



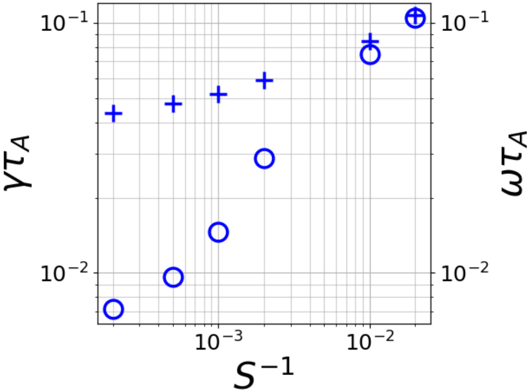
NIMROD also observes the modified scaling of the resistive kink in the presence of runaway current.

Kink equilibrium:
$q = 0.9(1 + 1/2(r/a)^2)$
$\boldsymbol{J} \times \boldsymbol{B} = 0,\ \boldsymbol{B}(0) = 1.$
$c_r = 20.0,\ V_A = 1.0$
Uniform background plasma density and resistivity
For the case with runaway electrons, all the equilibrium current is carried by runaways.

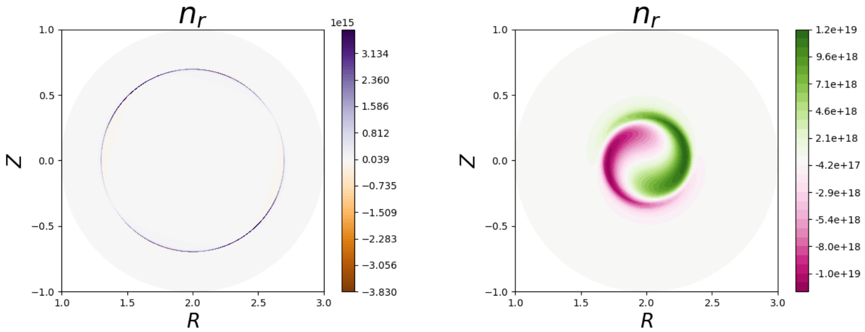


With the runaway current, a different eigenmode grows faster than the tearing mode when resistivity is sufficiently large.

Growth rate and frequency($\mathrm{Im}\{\gamma\}$) of the fastest growing mode with RE current in the range $S \lesssim 10^4$. In this regime, the distinct scaling of the frequency with the resistivity suggests a different instability branch is present



Radial structure of eigenmode differentiates it from the tearing mode



The linear eigenfunction of the tearing mode (left) is localized near the $q = 2$ rational surface with poloidal mode number $m = 2$. The new mode has $m = 1$, and is not localized near the rational surface.

A reduced model captures the key features observed in the simulation

Neglect velocity fluctuations, and change variables to $\Lambda_r = \mu_0 e c_r N_r / B$, $\lambda_r = \mu_0 e c_r n_r / B$, then since $\partial_t \boldsymbol{B} = 0$:

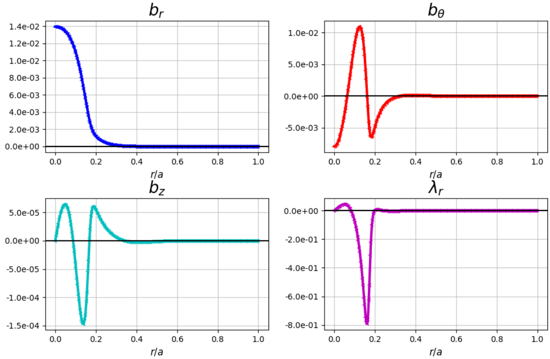
$$\partial_t \lambda_r - \frac{c_r}{B} \left(\boldsymbol{B} \cdot \nabla \lambda_r + \boldsymbol{\nabla} \cdot (\Lambda_r \boldsymbol{b}_\perp) \right) = 0, \quad (1)$$

$$\partial_t \boldsymbol{b} = \frac{\eta}{\mu_0} \boldsymbol{\nabla} \times (\lambda_r \boldsymbol{B} + \Lambda_r \boldsymbol{b}_\perp) - \frac{\eta}{\mu_0} \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{b}. \quad (2)$$

This system was solved in NIMROD without $\boldsymbol{v}$ evolution as a time-dependent, initial value calculation, and also with a 1D spectral eigenvalue code CYL-SPEC.

Eigenvalue calculations observe the same mode structure and growth rates as in the NIMROD calculation.

Eigenvalues of the system are sought for the a single mode of the form $\exp(im\theta + ikz + \gamma t)$. A spectral method is applied to the resulting ODE in the radial coordinate.



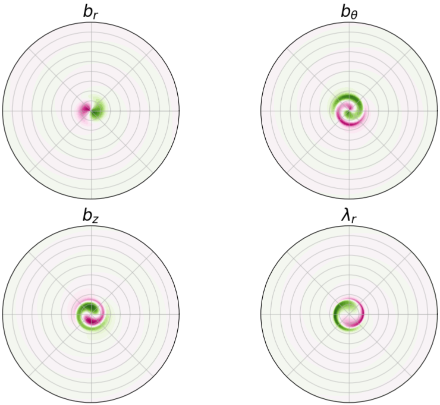
Radial profiles of the fastest growing eigenmode mode using the equilibrium profiles from the tearing mode case for $m = 1, k = -0.1$. $c_r = 20.0$, and $\eta/\mu_0 = 10^3$ which corresponds to the $S = 10^3$ case in the tearing mode case. Eigenvalue:

$$\gamma = 1.54 \times 10^{-2} + 3.338 \times 10^{-1} i$$

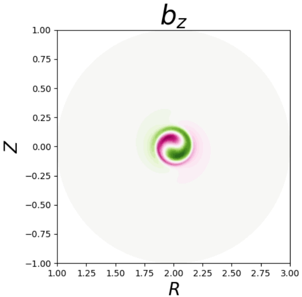
The eigenvalue from the NIMROD calculation with the same parameters is

$$\gamma = 1.46 \times 10^{-2} + 3.254 \times 10^{-1} i$$

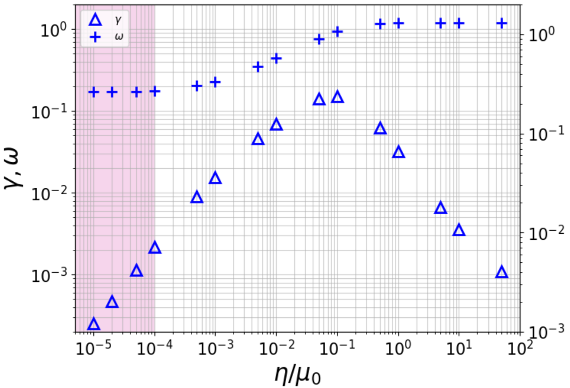
Eigenvalue calculations from the CYL-SPEC code observe the same mode structure and growth rates as in the NIMROD calculation.



Contour plots reconstructed from the radial profiles of the fastest growing eigenmodes from the eigenvalue code (left). The mode structure identical to the NIMROD simulation below:



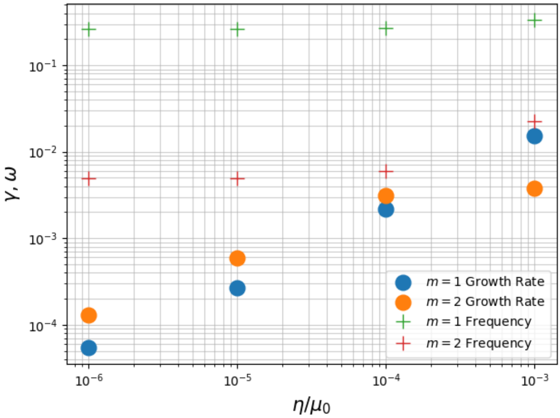
At large resistivity, eigenvalues scale with η the same way observed in the MHD simulation in the high-η regime.



- $\omega > \gamma$ for all η values for this mode
- In the tearing mode, $\omega \lesssim \gamma$ at low η.
- The shaded region indicates where the MHD tearing mode growth rates would be larger for this equilibrium.

Comparing η scaling between (2,1) and (1,1) growth rates in CYL-SPEC clarifies the situation.

This version of CYL-SPEC is MHD + RE's (velocity is included).



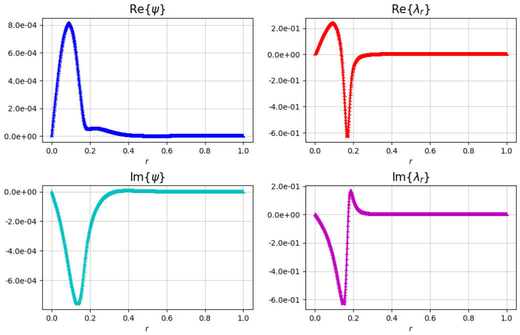
A further reduced model reveals that the instability is driven by the gradient in the equilibrium runaway density.

Since the axial component $|b_z| \ll |b_r|, |b_\theta|$, simplify the system via $\boldsymbol{b} = \nabla\psi \times \hat{\boldsymbol{z}}$. Additionally, we neglect terms of order $r/R \sim B_\theta/B$, and assume $B_z \sim B_0 = \text{const.}$ (large-aspect ratio expansion).

$$(\omega + c_r F(r))\lambda_r = -c_r \frac{m\Lambda_r'}{r}\frac{\psi}{B} \quad (3)$$

$$\omega\psi - i\frac{\eta}{\mu_0}\left(\frac{1}{r}(r\psi')' - \frac{m^2}{r^2}\psi\right) = i\frac{\eta}{\mu_0}B_z\lambda_r \quad (4)$$

The eigenvalues of this reduced system are also sought with a 1D spectral method, resulting in the same growth rate for the fastest growing mode.



This description of the RE-plasma interaction is similar to theory of resistive beam instabilities[4]

A self-pinched electron beam propagating through a neutralizing background plasma experiences a long-wavelength instability due to the decoupling of the beam motion from the magnetic field lines. Weinberg[3] gives the field equation describing the instability as

$$\frac{1}{r}(rE'_{1z})' - \frac{m^2}{r^2}E_{1z} + \frac{4\pi i\sigma\omega}{c^2}E_{1z} = -\frac{4\pi iev\omega}{c^2}n_1. \tag{5}$$

E_{1z} is the axial electric field, n_1 is the perturbed beam density, σ is the plasma conductivity, v is the axial beam velocity, and c is the speed of light. This equation bears resemblance to (4).:

$$\omega\psi - i\frac{\eta}{\mu_0}\left(\frac{1}{r}(r\psi')' - \frac{m^2}{r^2}\psi\right) = i\frac{\eta}{\mu_0}B_z\lambda_r$$

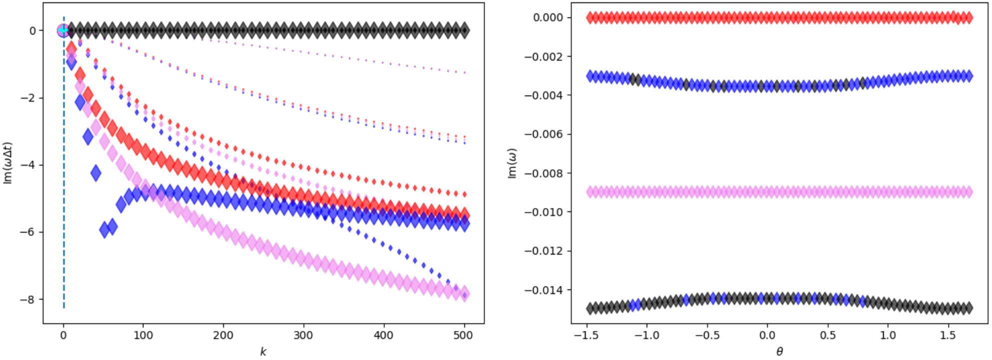
[3] S. Weinberg, Journal of Mathematical Physics **8**, 614–641 (1967).
[4] M. N. Rosenbluth, Physics of Fluids **3**, 932–936 (1960).

Conclusions

- The linearized fluid runaway electron model agrees with published results from Liu in the large S regime.
- At low S, a second branch of linear instability appears that is believed to be related to the resistive hose instability.
- The forms of equations (5) and (4) indicate the fundamental mechanism for instability is the perturbed beam density driving oscillations in the electromagnetic field.
- Work remains to extend the analytic dispersion relations from[5] to the large-guide field/massless beam particle regime.

[5] S. Weinberg, Journal of Mathematical Physics **8**, 614–641 (1967), M. N. Rosenbluth, Physics of Fluids **3**, 932–936 (1960), S. Weinberg, Journal of Mathematical Physics **5**, 1371–1386 (1964).

Von Neumann stability analysis of time stepping scheme suggests numerical stability.



Long wavelength instability is a feature of the inconsistency in the slab equilibrium.