Closures in toroidal geometry for arbitrary collisionality with no flux surface average

Jeong-Young Ji and Eric Held

Utah State University

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Moment expansion of a distribution function

- Landau (Fokker-Planck) kinetic equation

$$\frac{\partial f_a}{\partial t} + \mathbf{v}\cdot\nabla f_a + \frac{q_a}{m_a}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\partial_{\mathbf{v}} f_a = \sum_b C(f_a, f_b)$$

- Moment expansion: expansion coefficients, m^{lk}'s, are symmetric traceless fluid moments

$$\begin{aligned} f_a(t,\mathbf{x},\mathbf{v}) &= f_a^{(0)} \sum_{lk} \frac{1}{\sqrt{\sigma_k^l}} \mathsf{m}_a^{lk}(t,\mathbf{x}) \cdot \mathsf{p}_a^{lk} \\ \mathsf{n}_a^{lk} \equiv n_a \mathsf{m}_a^{lk}(t,\mathbf{x}) &= \int d\mathbf{v} \frac{1}{\sqrt{\sigma_k^l}} \mathsf{p}_a^{lk} f_a(t,\mathbf{x},\mathbf{v}) \end{aligned}$$

- General moment equations (Ji and Held PoP 2006, 2008, 2009)

$$\hat{D}_a\left[d_a, d_a \ln T_a, \nabla, \nabla \ln T_a, \nabla\mathbf{V}_a, \mathbf{a}_a\right]\mathsf{n}_a + \Omega_a \mathbf{b}\breve{\times}\mathsf{n}_a = \sum_b (\hat{A}_{ab}\mathsf{n}_a + \hat{B}_{ab}\mathsf{n}_b)$$

where $d_a = \dfrac{\partial}{\partial t} + \mathbf{V}_a\cdot\nabla, \quad \mathbf{a}_a = \dfrac{q_a}{m_a}(\mathbf{E}+\mathbf{V}_a\times\mathbf{B}) - d_a\mathbf{V}_a$

Tensorial Hermite polynomial

$$\mathsf{p}_a^{lk} = \mathsf{P}^l(\mathbf{c}_a) L_k^{(l+1/2)}(c_a^2) \quad \Rightarrow \quad \mathbf{v}^{l+2k}$$

$$\mathbf{c}_a = \frac{\mathbf{v} - \mathbf{V}_a}{v_{Ta}}, \quad v_{Ta} = \sqrt{\frac{2T_a}{m_a}}$$

- Harmonic tensor: spherical harmonics

$$\begin{aligned}
\mathsf{P}^0(\mathbf{c}) &= 1 \\
\mathsf{P}^1(\mathbf{c}) &= \mathbf{c} \\
\mathsf{P}^2(\mathbf{c}) &= \mathbf{cc} - \frac{c^2}{3}\mathsf{I} \quad (\mathsf{I} = \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3 \equiv \mathsf{ii}) \\
\mathsf{P}^3(\mathbf{c}) &= \mathbf{ccc} - \frac{c^2}{5}(\mathbf{c}\mathsf{ii} + \mathsf{i}\mathbf{c}\mathsf{i} + \mathsf{ii}\mathbf{c})
\end{aligned}$$

- Heat flow $(\mathbf{h})$ and viscosity tensor $(\boldsymbol{\pi})$

$$\mathsf{m}^{11} = -\sqrt{\frac{4}{5}}\frac{1}{n v_T T}\mathbf{h} \qquad \mathsf{m}^{20} = \frac{\sqrt{2}}{2p}\boldsymbol{\pi}$$

General closures for arbitrary collisionality with $\delta = \rho/L \ll 1$

Fluid moments to be solved: $\mathsf{F} = \{n_a, \mathbf{V}_a, T_a\}$

$\Rightarrow$Equations require closures $\mathsf{C} = \{\mathbf{h}_a, \boldsymbol{\pi}_a, \mathbf{R}_a, Q_a\}$

Moment equations for closures: $\hat{D}_a \mathsf{n}_a + \underbrace{\Omega_a}_{\delta^{-1}} \mathbf{b}\breve{\times}\mathsf{n}_a = \hat{C}_a \mathsf{n}_a + \mathsf{G}_a \qquad (\mathsf{C} \subset \{\mathsf{n}_a\})$

G_a: thermodynamic drives, expressed in terms of F_a $(\nabla T_a, \nabla \mathbf{V}_a, \mathbf{V}_{\mathrm{ei}})$

Temperature, electric potential and flow velocity on the flux surfaces (ψ)

$$\begin{aligned} T &= T_0(\psi) + T_1(\mathbf{x}) \\ \phi &= \phi_0(\psi) + \phi_1(\mathbf{x}) \quad \Rightarrow \quad \mathbf{a}_0 = \frac{q}{m}(-\nabla\phi_0 + \mathbf{V}_1 \times \mathbf{B}) \\ \mathbf{V} &= \mathbf{V}_1(\mathbf{x}) \end{aligned}$$

$$\begin{aligned} \hat{D}_0^{jp,lk}\mathsf{n}^{lk} &= v_{T0}[\hat{\Psi}^{j-}_{pk}\nabla + \hat{\Phi}^{j-}_{pk}\nabla \ln T_0]\mathsf{n}^{j-1,k} + v_{T0}^{-1}\hat{\Theta}^{j-}_{pk}\mathbf{a}_0\mathsf{n}^{j-1,k} \\ &\quad + v_{T0}[\hat{\Psi}^{j+}_{pk}\nabla + \hat{\Phi}^{j+}_{pk}\nabla \ln T_0]\cdot\mathsf{n}^{j+1,k} + v_{T0}^{-1}\hat{\Theta}^{j+}_{pk}\mathbf{a}_0\cdot\mathsf{n}^{j+1,k} \end{aligned}$$

Expand $\mathsf{n}_a = \mathsf{n}_a^{(0)} + \mathsf{n}_a^{(1)} + \mathsf{n}_a^{(2)} + \cdots, \quad \mathsf{G}_a = \mathsf{G}_a^{(0)} + \mathsf{G}_a^{(1)}$

- δ^{-1}: $\Omega\mathbf{b}\breve{\times}\mathsf{n}^{(0)} = 0$
- δ^{0}: $\hat{D}_0\mathsf{n}^{(0)} + \Omega\mathbf{b}\breve{\times}\mathsf{n}^{(1)} = \hat{C}\mathsf{n}^{(0)} + \mathsf{G}^{(0)}$
- δ^{1}: $\hat{D}_0\mathsf{n}^{(1)} + \hat{D}_1\mathsf{n}^{(0)} + \Omega\mathbf{b}\breve{\times}\mathsf{n}^{(2)} = \hat{C}\mathsf{n}^{(1)} + \mathsf{G}^{(1)}$

Solutions of $\mathsf{b}\breve{\times}\mathsf{n}^{lk(0)} = 0$

$$\mathsf{b}_{\|}\mathbf{V} = \mathbf{bb}\cdot\mathbf{V} \equiv \mathbf{V}_{\|},\ \mathsf{b}_{\times}\mathbf{V} \equiv \mathbf{b}\times\mathbf{V} = \mathbf{V}_{\times},\ \mathsf{b}_{\perp}\mathbf{V} \equiv (\mathsf{I}-\mathbf{bb})\cdot\mathbf{V} = \mathbf{V}_{\perp}$$

$$\mathsf{b}_{\times\perp}\mathsf{W} \equiv \mathsf{W}_{\times\perp},\ \mathsf{K}^{-1} \equiv \tfrac{1}{2}\mathsf{b}_{\times\perp} + 2\mathsf{b}_{\|\times}$$

- Vector moments: $\mathbf{b}\times\mathsf{n}^{1k(0)} = 0 \quad\Rightarrow\quad \mathsf{n}_{\perp}^{1k(0)} = 0$
- Rank 2 tensor moments: $\mathbf{b}\breve{\times}\mathsf{n}^{2k(0)} = \mathbf{b}\times\mathsf{n}^{2k(0)} - \mathsf{n}^{2k(0)}\times\mathbf{b} = 0$

$$\mathsf{K}^{-1}(\mathbf{b}\times\mathsf{n}^{2k(0)} - \mathsf{n}^{2k(0)}\times\mathbf{b}) = \mathsf{n}_{\mathrm{CGL}}^{2k(0)} - \mathsf{n}^{2k(0)} = 0$$

$$\mathsf{n}^{2k(0)} = \mathsf{n}_{\mathrm{CGL}}^{2k(0)} \equiv n_{zz}^{2k(0)}(-\tfrac{1}{2}\mathbf{e}_1\mathbf{e}_1 - \tfrac{1}{2}\mathbf{e}_2\mathbf{e}_2 + \mathbf{bb}) = \tfrac{3}{2}n_{zz}^{2k(0)}(\mathbf{bb} - \tfrac{1}{3}\mathsf{I})$$

- General-rank tensor moments $(l \geq 3)$: solve coupled equations for $\mathsf{n}_{\|\cdots\times\cdots\perp\cdots}^{lk(0)}$

  $\mathsf{n}_{\|\cdots\times\cdots\perp\cdots}^{lk(0)} = 0$ for odd number of $\perp$ or $\times$

$$\mathsf{n}^{lk(0)} = \frac{(2l-1)!!}{l!}n_{\|}^{lk(0)}\mathsf{P}^{l}(\mathbf{b})$$

$n_{\|}^{lk(0)} \equiv n_{zz\cdots z}^{lk(0)}$ can be found from the δ^0-order parallel moment equations

δ^0-order equations $\hat{D}_0\mathsf{n}^{(0)} + \Omega\mathbf{b}\breve{\times}\mathsf{n}^{(1)} = \hat{C}\mathsf{n}^{(0)} + \mathsf{G}^{(0)}$

- Parallel moments

$$D_\parallel n_\parallel^{(0)} = C n_\parallel^{(0)} + \underbrace{G_\parallel^{(0)}}_{=0}$$

$$n_\parallel^{(0)} = 0 \qquad \Rightarrow \qquad \mathsf{n}^{(0)} = 0$$

- Perpendicular moments

$$\mathsf{G}_a^{10(0)} \propto \nabla p_{a0} = \nabla\psi\frac{dp_{a0}}{d\psi}, \quad \mathsf{G}_a^{11(0)} \propto \nabla T_{a0} = \nabla\psi\frac{dT_{a0}}{d\psi}, \quad \mathsf{G}_a^{lk(0)} = 0$$

$$\mathbf{V}_{a\perp}^{(1)} \propto \mathbf{b}\times\nabla p_{a0}, \quad \mathbf{h}_{a\perp}^{(1)} \propto \mathbf{b}\times\nabla T_{a0}, \quad \mathsf{n}_{a\perp}^{lk(1)} = 0 \; (l,k \neq 1,1)$$

$$\mathsf{n}^{lk(1)} = \frac{(2l-1)!}{l!} n_\parallel^{lk(1)} \mathsf{P}^l(\mathbf{b}) \quad (l,k \neq 1,1)$$

$n_\parallel^{lk(1)} \equiv n_{zz\ldots z}^{lk(1)}$ can be found from the δ^1-order parallel moment equations

Parallel δ^1 order equations $[\hat{D}_0\mathsf{n}^{(1)} + \Omega\mathbf{b}\breve{\times}\mathsf{n}^{(2)} = \hat{C}\mathsf{n}^{(1)} + \mathsf{G}^{(1)}]_{\|}$

$$\hat{\Psi}^{l-}_{pk}\left(\overline{\nabla\mathsf{n}^{l-1,k}}\right)_{\|} \quad\Rightarrow\quad \bar{\Psi}^{l-}_{pk}\left(\partial_\ell \bar{n}^{l-1,k}_{\|} + \frac{l-1}{2}\partial_\ell \ln B\bar{n}^{l-1,k}_{\|}\right)$$

$$\hat{\Psi}^{l+}_{pk}\left(\nabla\cdot\mathsf{n}^{l+1}\right)_{\|} \quad\Rightarrow\quad \bar{\Psi}^{l+}_{pk}\left(\partial_\ell \bar{n}^{l+1}_{\|} - \frac{l+2}{2}\partial_\ell \ln B\bar{n}^{l+1}_{\|}\right)$$

$$[\Psi]\partial_z\bar{n}^{(1)}_{\|} = [c]\bar{n}^{(1)}_{\|} + g^{(1)}_{\|} - \{\partial_z \ln B[\Upsilon] + \partial_z \ln T_0[\Phi] + a_{0\|}[\Theta]\}\bar{n}^{(1)}_{\|}$$

- $\delta^1 : (\mathbf{x}, w, \mu) : \dfrac{\partial f_1}{\partial t} + v_{\|}\mathbf{b}\cdot\nabla f_1 + \mathbf{v}_{\mathrm{D}}\cdot\nabla f_0 + q(v_{\|}E_{1\|} + \mathbf{v}_{\mathrm{D}}\cdot\mathbf{E}_0)\dfrac{\partial f_0}{\partial w} = C(f_1)$

$$f_0 = \frac{n_0}{\pi^3 v_0^3}\exp(-\frac{v^2}{v_0^2}) \Rightarrow \frac{\partial f_1}{\partial t} + v_{\|}\frac{\partial f_1}{\partial \ell} - C(f_1) = G_{\mathrm{t}}$$

where $G_{\mathrm{t}} = (G^{00}_{\mathrm{t}} + G^{01}_{\mathrm{t}}P^{01} + G^{02}_{\mathrm{t}}P^{02} + G^{10}_{\mathrm{t}}P^{10} + G^{20}_{\mathrm{t}}P^{20} + G^{21}_{\mathrm{t}}P^{21})\hat{f}_0$

$$G^{00}_{\mathrm{t}} = -2\alpha\frac{dp_{0*}}{d\psi},\ G^{01}_{\mathrm{t}} = \frac{2}{3}\alpha(2\frac{dp_{0*}}{d\psi} + 5n_0\frac{dT_0}{d\psi}),\ G^{02}_{\mathrm{t}} = -\frac{8}{3}\alpha n_0\frac{dT_0}{d\psi},$$

$$G^{10}_{\mathrm{t}} = \frac{n_0 q v_0 E_{1\|}}{T_0},\ G^{20}_{\mathrm{t}} = -\frac{2}{3}\alpha(\frac{dp_{0*}}{d\psi} + n_0\frac{dT_0}{d\psi}),\ G^{21}_{\mathrm{t}} = \frac{2}{3}\alpha n_0\frac{dT_0}{d\psi}$$

with $\alpha = \dfrac{1}{qB^2}\mathbf{b}\times\nabla B\cdot\nabla\psi$ and $p_{0*} = p_0 + n_0 q\Phi_0$

From kinetic equation to parallel moment equations

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}} = C(f)$$

$$\int d\gamma$$

$$\mathbf{v}_{\mathrm{D}}\cdot\nabla\bar{f}_0 + \left(\frac{q}{m}\mathbf{v}_\parallel\cdot\mathbf{E}_1 + \mathbf{v}_{\mathrm{D}}\cdot\mathbf{E}_0\right)\frac{\partial \bar{f}_0}{\partial \varepsilon} + \mathbf{v}_\parallel\cdot\nabla f_1 = C(\bar{f}_1)$$

$$\mathbf{v}_{\mathrm{D}} = \tfrac{1}{\Omega}\mathbf{b}\times\left[(v_\parallel^2 + \tfrac{1}{2}v_\perp^2)\tfrac{\nabla B}{B} + \tfrac{q}{m}\nabla\Phi\right] + \tfrac{v_\parallel^2}{\Omega}\tfrac{\mu_0\mathbf{J}_\perp}{B}$$

$$\int d\mathbf{v}\mathsf{P}^{lk}$$

$$\Omega \gg \mathbf{v}\cdot\nabla,\ \frac{q}{m}\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{v}},\ C \gg \frac{\partial}{\partial t}$$
$$(n,T) = (n,T)_0(\psi) + (n,T)_1$$
$$\mathbf{V} = \mathbf{V}_1$$

$$\int d(v,v_\parallel)P^{lk}$$

See NIMROD Team Meeting, 2009 APS

$$\frac{\partial}{\partial t}\mathsf{M} + \frac{1}{T}\frac{\partial T}{\partial t}\hat{\Xi}\mathsf{M} + v_T\hat{\Psi}\nabla\mathsf{M} + v_T\frac{\nabla T}{T}\hat{\Phi}\mathsf{M} + \frac{q}{m v_T}\hat{\Theta}\mathbf{E}\mathsf{M} + \Omega\mathbf{b}\times\mathsf{M} = \frac{1}{\tau}\hat{C}\mathsf{M}$$

$$\mathbf{b}\cdot$$

$$v_T\bar{\Psi}\partial_\parallel M_\parallel - v_T\Upsilon(\partial_\parallel \ln B)M_\parallel = \frac{1}{\tau}CM_\parallel + \bar{G}$$

Perpendicular δ^1-order $(\hat{D}\mathsf{n})^{l(1)} + \mathbf{b}\breve{\times}\mathsf{n}^{l(2)} = C\mathsf{n}^{l(1)} + \mathsf{G}^{l(1)} \Rightarrow \mathsf{n}_\perp^{(2)}$

- Perpendicular vector moments

$$\mathsf{n}_\perp^{1p(2)} = -\Omega^{-1}\mathbf{b} \times [C^{1p1}\mathsf{n}^{11(1)} + \mathsf{g}^{11(1)} - \mathbf{D}_0^{11,0k}\mathsf{n}^{0k(1)} - \mathbf{D}_0^{11,2k} \cdot \mathsf{n}^{2k(1)}]$$

- Viscous stress $\Omega\mathbf{b}\breve{\times}\mathsf{n}^{20(2)} = C^{20k}\mathsf{n}^{2k(1)} + \mathsf{G}^{20(1)} - (\hat{D}\mathsf{n})^{20(1)}$

$$\begin{aligned} \mathsf{n}^{20(2)} &= \mathsf{n}_{\mathrm{CGL}}^{20(2)} - \Omega^{-1}\mathsf{K}^{-1}[\mathsf{G}^{20(1)} - D_{0k}^{2-}\mathsf{n}^{1k(1)} - D_{0k}^{2+}\mathsf{n}^{3k(1)}] \\ &= \mathsf{n}_{\mathrm{CGL}}^{20(2)} - \Omega^{-1}\mathsf{K}^{-1}[-\sqrt{2}n\overline{\nabla\mathbf{V}} - \overline{\mathbf{D}_{01}^{2-}\mathsf{n}^{11(1)}} - \mathbf{D}_{00}^{2+}\mathsf{n}^{30(1)}] \end{aligned}$$

$$\boldsymbol{\sigma} = \int d\mathbf{v}m(\mathbf{www} - \frac{3}{5}\{\mathbf{w}\mathsf{I}\})f = \frac{\sqrt{3}}{2}v_T^3\mathsf{n}^{30}, \quad \mathbf{w} = \mathbf{v} - \mathbf{V}$$

$$\boldsymbol{\pi}^{(2)} = \boldsymbol{\pi}_{\mathrm{CGL}}^{(2)} + \Omega^{-1}\mathsf{K}^{-1}[2p\overline{\nabla\mathbf{V}} + \frac{4}{5}\overline{\nabla\mathbf{h}^{(1)}} + \nabla\cdot\boldsymbol{\sigma}^{(1)}]$$

$$\mathsf{K}^{-1}\nabla\cdot\boldsymbol{\sigma}^{(1)} = \frac{5}{2}\sigma_\parallel^{(1)}(\mathbf{b}\times\boldsymbol{\kappa}\mathbf{b} + \mathbf{bb}\times\boldsymbol{\kappa})$$

Solving parallel moment equations - Fourier method

$$[\Psi]\lambda_C\frac{d[n]}{d\ell} = [c][n] + [g] - [\Upsilon]\lambda_C\frac{d\ln B}{d\ell}[n] \text{ (}\supset\text{ Braginskii, integral closure)}$$

$$[\Psi]\lambda\frac{d[n]}{d\vartheta} = [c][n] + [g] - [\Upsilon]\lambda\frac{d\ln B}{d\vartheta}[n], \quad \lambda = \lambda_C\frac{\mathbf{B}\cdot\nabla\vartheta}{B} \approx \frac{\lambda_C}{qR}$$

- Fourier expansion (axi-symmetric)

$$[n] = [n_{0+}] + \sum_{m=1}^{\infty}([n_{m+}]\cos m\vartheta + [n_{m-}]\sin m\vartheta)$$

- For example $B = \dfrac{B_0}{1+\epsilon\cos\vartheta} \Rightarrow \dfrac{d\ln B}{d\vartheta} \approx \epsilon\sin\vartheta$ (the large aspect ratio approximation)

$$\sin\vartheta\sin m\vartheta = \tfrac{1}{2}[\cos(m-1)\vartheta - \cos(m+1)\vartheta]$$
$$\sin\vartheta\cos m\vartheta = \tfrac{1}{2}[\sin(m+1)\vartheta - \sin(m-1)\vartheta]$$

$$\begin{bmatrix} C & \frac{\epsilon}{2}\lambda\Upsilon & 0 & 0 & 0 & 0 & 0 \\ \frac{\epsilon}{2}\lambda\Upsilon & C & \lambda\Psi & 0 & -\frac{\epsilon}{2}\lambda\Upsilon & 0 & 0 \\ 0 & -\lambda\Psi & C & \frac{\epsilon}{2}\lambda\Upsilon & 0 & 0 & 0 \\ 0 & 0 & \frac{\epsilon}{2}\lambda\Upsilon & C & 2\lambda\Psi & 0 & -\frac{\epsilon}{2}\lambda\Upsilon \\ 0 & -\frac{\epsilon}{2}\lambda\Upsilon & 0 & -2\lambda\Psi & C & \frac{\epsilon}{2}\lambda\Upsilon & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\epsilon\lambda\Upsilon & C & 3\lambda\Psi \\ 0 & 0 & 0 & -\frac{\epsilon}{2}\lambda\Upsilon & 0 & -3\lambda\Psi & C \end{bmatrix} \begin{bmatrix} n_{0+} \\ n_{1-} \\ n_{1+} \\ n_{2-} \\ n_{2+} \\ n_{3-} \\ n_{3+} \end{bmatrix} = - \begin{bmatrix} g_{0+} \\ g_{1-} \\ g_{1+} \\ g_{2-} \\ g_{2+} \\ g_{3-} \\ g_{3+} \end{bmatrix}$$

Solving parallel moment equations - finite difference method

- Finite difference method $[\Psi]\lambda\frac{d[n]}{d\vartheta} = [c][n] + [g] - [\Upsilon]\lambda\frac{d\ln B}{d\vartheta}[n]$

$$[\Psi]\frac{\partial}{\partial\theta}[n] \longrightarrow \begin{bmatrix} & \Psi & & & & -\Psi \\ -\Psi & & \Psi & & & \\ & -\Psi & & \Psi & & \\ & & \ddots & & \ddots & \end{bmatrix} \begin{bmatrix} n^1 \\ n^2 \\ n^3 \\ \vdots \end{bmatrix}$$

- Modified midpoint rule

$$\frac{\lambda_{i+1/2}}{\Delta\theta}[\Psi][n_{i+1}-n_i]+[\Psi_B](\lambda\frac{\partial}{\partial\theta}\ln B)_{i+1/2}\frac{1}{2}[n_{i+1}+n_i] = [c]\frac{1}{2}[n_{i+1}+n_i]+[\tau_{\mathrm{C}}\hat{G}_{i+1/2}]$$

$$\begin{aligned} \begin{matrix} L_i \\ R_i \end{matrix} &= 1 \pm (\frac{\Delta\theta}{2}\frac{\partial\ln B}{\partial\theta})_{i+1/2}[\Psi^{-1}\Psi_B] \mp (\frac{\Delta\theta}{2\lambda})_{i+1/2}[\Psi^{-1}c] \\ S_i &= (\frac{\Delta\theta\tau_{\mathrm{C}}}{\lambda})_{i+1/2}[\Psi^{-1}\hat{G}_{i+1/2}] \\ U_i &= L_i^{-1}R_i, \quad U_{N\cdots i} = U_N U_{N-1}\cdots U_i \end{aligned}$$

- $[L_i][n_{i+1}] = [R_i][n_i] + [S_i]$ (matrix dimension = number of moments) recursively

$$(1-U_{N\cdots 1})[n_1] = U_{N\cdots 2}[L_1^{-1}][S_1]+U_{N\cdots 3}[L_2^{-1}][S_2]+\cdots+U_N[L_{N-1}^{-1}][S_{N-1}]+[L_N^{-1}][S_N]$$

Parallel moments in the collisionless limit

- Lower collisionality requires more moments

$$[\Psi]\lambda\frac{d[n]}{d\vartheta} = [c][n] + [g] - [\Upsilon]\lambda\frac{d\ln B}{d\vartheta}[n]$$

- Moment equations with no collision operator are solvable

$$\frac{d[n]}{d\vartheta} + [\Psi^{-1}\Upsilon]\frac{d\ln B}{d\vartheta}[n] = \lambda^{-1}[\Psi^{-1}g]$$

  - The eigenvalues of $[\Psi^{-1}\Upsilon]$ are integers: $[\Psi^{-1}\Upsilon][W_A] = i_A[W_A]$

$$[W] = [W_1, W_2, \cdots, W_N], \quad [\check{n}] = [W^{-1}n], \quad [\check{g}] = [W^{-1}\Psi^{-1}g]$$

$$\frac{\partial\check{n}_A}{\partial\theta} + i_A\frac{\partial\ln B}{\partial\theta}\check{n}_A = \breve{g}_A$$

$$B^{-i_A}\frac{\partial}{\partial\theta}(\check{n}_A B^{i_A}) = \breve{g}_A$$

Future work

- Implement and test code in NIMROD