Update on moment-based closures
1. Nonlinear collision terms
2. Perfecting Braginskii coefficients

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MINROD Team Meeting

July 29 2009, University of Wisconsin-Madison

Closure vs. transport theory

- Closure theory for 5 moment $(n_a, \mathbf{V}_a, T_a)$ equations

$$d_a n_a + n_a \nabla \cdot \mathbf{V}_a = 0 \, (d_a \equiv \partial_t + \mathbf{V}_a \cdot \nabla)$$

$$\tfrac{3}{2} n_a d_a T_a + n_a T_a \nabla \cdot \mathbf{V}_a + \nabla \cdot \mathbf{q}_a + \nabla \mathbf{V}_a : \boldsymbol{\pi}_a = Q_a$$

$$m_a n_a d_a \mathbf{V}_a - n_a e_a (\mathbf{E} + \mathbf{V}_a \times \mathbf{B}) + \nabla p_a + \nabla \cdot \boldsymbol{\pi}_a = \mathbf{R}_a$$

Solve moment equations for N_a (non-Maxwellian moments)

$$D\mathsf{N}_a + \Omega_a \mathbf{b} \times \mathsf{N}_a = C\mathsf{N}_a + \mathsf{G}_a$$

Express $\mathbf{q}_a(\mathsf{N}_a^{11})$, $\boldsymbol{\pi}_a(\mathsf{N}_a^{20})$, Q_a, $\mathbf{R}_a$ in terms of $n_a, \mathbf{V}_a, T_a$

$$\mathbf{R}_\mathrm{e} = -(\alpha)(\mathbf{V}_\mathrm{ei}) - (\beta)(\nabla T_\mathrm{e}), \;\; \mathbf{q}_\mathrm{e} = (\beta)(\mathbf{V}_\mathrm{ei}) - (\kappa)(\nabla T_\mathrm{e})$$

- Transport theory

Solve momentum balance equation

$$ne\mathbf{E}' + ne\mathbf{V}_\mathrm{ei} \times \mathbf{B} = \mathbf{R}_\mathrm{e} \text{ where } \mathbf{E}' = \mathbf{E} + \mathbf{V}_\mathrm{i} \times \mathbf{B} + (ne)^{-1} \nabla p_\mathrm{e}$$

Express fluxes in terms of thermodynamic drives

$$\mathbf{J} = (\sigma)(\mathbf{E}') - (\alpha')(\nabla T_\mathrm{e}), \;\; \mathbf{q}_\mathrm{e} = (\alpha')(\mathbf{E}') - (\kappa')(\nabla T_\mathrm{e})$$

Transport/closure theories

	Braginskii	Neoclassical	transport	Unified closure
Collisionality	high	high: PS	low	general$^{\star 1}$
Magnetic field strength	general	strong	strong	strong
Magnetic geometry	general	nested	nested	general$^{\star 2}$
Collision operator	Landau	Landau	model	Landau$^{\star 3}$

- Solve general moment equations (not the drift kinetic equation)

$\star 1$ No ordering on collisionality (no subsidiary expansion) $\Rightarrow$ Braginskii

$\star 2$ No flux surface average

$\star 3$ Exact full linearized Coulomb collision operators

Calculation of nonlinear collision terms

$$Df_a = \partial_t f_a + \mathbf{v}\cdot\nabla f_a + \frac{e_a}{m_a}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\partial_{\mathbf{v}} f_a = \sum_b C(f_a, f_b)$$

$$f_a(t,\mathbf{x},\mathbf{v}) = f_a^{(0)} \sum_{lk} \frac{1}{\sqrt{\sigma_k^l}} \mathsf{M}_a^{lk}(t,\mathbf{x}) \cdot \mathsf{P}^{lk}(\mathbf{s}_a)$$

$$C^{(1)} = C(f^0, f^1) + C(f^1, f^0) \quad \Rightarrow \quad (D)\mathsf{M} = (C^{(1)})\mathsf{M} + (C^{(2)})(\mathsf{M},\mathsf{M})$$

Nonlinear terms of collision operator

$$C\left(f_a^{(0)} \mathsf{M}_a^{lk}(t,\mathbf{x})\cdot\mathsf{P}^{lk}(\mathbf{s}_a), f_b^{(0)} \mathsf{M}_b^{nq}(t,\mathbf{x})\cdot\mathsf{P}^{nq}(\mathbf{s}_b)\right)$$

$$= f_a^0 \sum_{u=0}^{\min(2,l,n)} \nu_{abu}^{lk,nq}(v) \sum_{i=0}^{\min(l,n)-2u} d_i^{l-u,n-u} \mathsf{P}^{l+n-2(i+u)}(\hat{\mathbf{v}}) \cdot \overline{\mathsf{M}_a^{lk} \cdot^{i+u} \mathsf{M}_b^{nq}}$$

Moments of collision operator

$$\int d\mathbf{v} \mathsf{P}_a^{jp} C(f_a^{lk}, f_b^{nq}) = \sigma_j C_{ab}^{jp,lk,nq} \overline{\mathsf{M}_a^{lk} \cdot^{\frac{l+n-j}{2}} \mathsf{M}_b^{nq}}$$

Vector moment equations

$$\begin{pmatrix} -x\mathbf{b}\times \mathsf{m}^{11} \\ -x\mathbf{b}\times \mathsf{m}^{12} \\ -x\mathbf{b}\times \mathsf{m}^{13} \\ \vdots \end{pmatrix} = \begin{pmatrix} c^1_{11} & c^1_{12} & c^1_{13} & \cdots \\ c^1_{21} & c^1_{22} & c^1_{23} & \cdots \\ c^1_{31} & c^1_{32} & c^1_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \mathsf{m}^{11} \\ \mathsf{m}^{12} \\ \mathsf{m}^{13} \\ \vdots \end{pmatrix} + \begin{pmatrix} \frac{\sqrt{5}}{2}\hat{\nabla}\ln T + \sqrt{2}a^{101}_{\mathrm{ei}}\frac{\mathbf{V}_{\mathrm{ei}}}{v_{T\mathrm{e}}} \\ \sqrt{2}a^{102}_{\mathrm{ei}}\frac{\mathbf{V}_{\mathrm{ei}}}{v_{T\mathrm{e}}} \\ \sqrt{2}a^{103}_{\mathrm{ei}}\frac{\mathbf{V}_{\mathrm{ei}}}{v_{T\mathrm{e}}} \\ \vdots \end{pmatrix}$$

or $-x\mathbf{b}\times\mathsf{m}^1 = c\mathsf{m}^1 + \mathsf{g}^1$, where $x = |\Omega_\mathrm{e}|\tau_{\mathrm{ei}}$, $\mathbf{V}_{\mathrm{ei}} = \mathbf{V}_\mathrm{e} - \mathbf{V}_\mathrm{i}$ and $c = c^1 = (\frac{1}{Z}C + A)$

$$C = \frac{\tau_{\mathrm{ei}}}{n_\mathrm{e}}C_{\mathrm{ee}} = -\begin{pmatrix} \frac{2\sqrt{2}}{5} & \frac{3}{5}\sqrt{\frac{2}{7}} & \frac{1}{4}\sqrt{\frac{3}{7}} \\ \frac{3}{5}\sqrt{\frac{2}{7}} & \frac{9}{7\sqrt{2}} & \frac{103\sqrt{3}}{280} \\ \frac{1}{4}\sqrt{\frac{3}{7}} & \frac{103\sqrt{3}}{280} & \frac{5657}{3360\sqrt{2}} \end{pmatrix}, \quad (K=3)$$

$$A = \frac{\tau_{\mathrm{ei}}}{n_\mathrm{e}}C_{\mathrm{ei}} \approx \frac{\tau_{\mathrm{ei}}}{n_\mathrm{e}}C_{\mathrm{Lorentz}} = a^1_{\mathrm{ei}} = -\begin{pmatrix} \frac{13}{10} & \frac{69}{20\sqrt{7}} & \frac{11}{4}\sqrt{\frac{3}{14}} \\ \frac{69}{20\sqrt{7}} & \frac{433}{280} & \frac{359}{280}\sqrt{\frac{3}{2}} \\ \frac{11}{4}\sqrt{\frac{3}{14}} & \frac{359}{280}\sqrt{\frac{3}{2}} & \frac{2957}{1680} \end{pmatrix}$$

C is diagonally dominant and A is not

$a_{pq} \gg c_{pq}(p \gg q) \quad \Rightarrow \quad c^2 = C^2/Z^2 + (CA + AC)/Z + A^2 \approx A^2(K \gg 1)$

Solving $-x\mathrm{b}\times\mathrm{m}^1 = c\mathrm{m}^1 + \mathrm{g}^1$: geometric method

Defining $\mathsf{m}^1_\times \equiv \mathbf{b}\times\mathsf{m}^1$, $\mathsf{m}^1_\perp \equiv -\mathbf{b}\times(\mathbf{b}\times\mathsf{m}^1)$

$$\begin{bmatrix} -c & x \\ -x & -c \end{bmatrix}\begin{bmatrix} \mathsf{m}^1_\times \\ \mathsf{m}^1_\perp \end{bmatrix} = \begin{bmatrix} \mathsf{g}^1_\times \\ \mathsf{g}^1_\perp \end{bmatrix} \quad\Rightarrow\quad \begin{bmatrix} \mathsf{m}^1_\times \\ \mathsf{m}^1_\perp \end{bmatrix} = \begin{bmatrix} -c & x \\ -x & -c \end{bmatrix}^{-1}\begin{bmatrix} \mathsf{g}^1_\times \\ \mathsf{g}^1_\perp \end{bmatrix}$$

$$\Rightarrow \quad \mathsf{m}^1_\perp = \frac{1}{(c^1)^2 + (x)^2}(x\mathsf{g}^1_\times - c\mathsf{g}^1_\perp) \quad\Rightarrow\quad \mathsf{m}^{1p}_\perp = (x^2+c^2)^{-1}_{pq}(x\mathsf{g}^{1q}_\times - c_{qk}\mathsf{g}^{1k}_\perp)$$

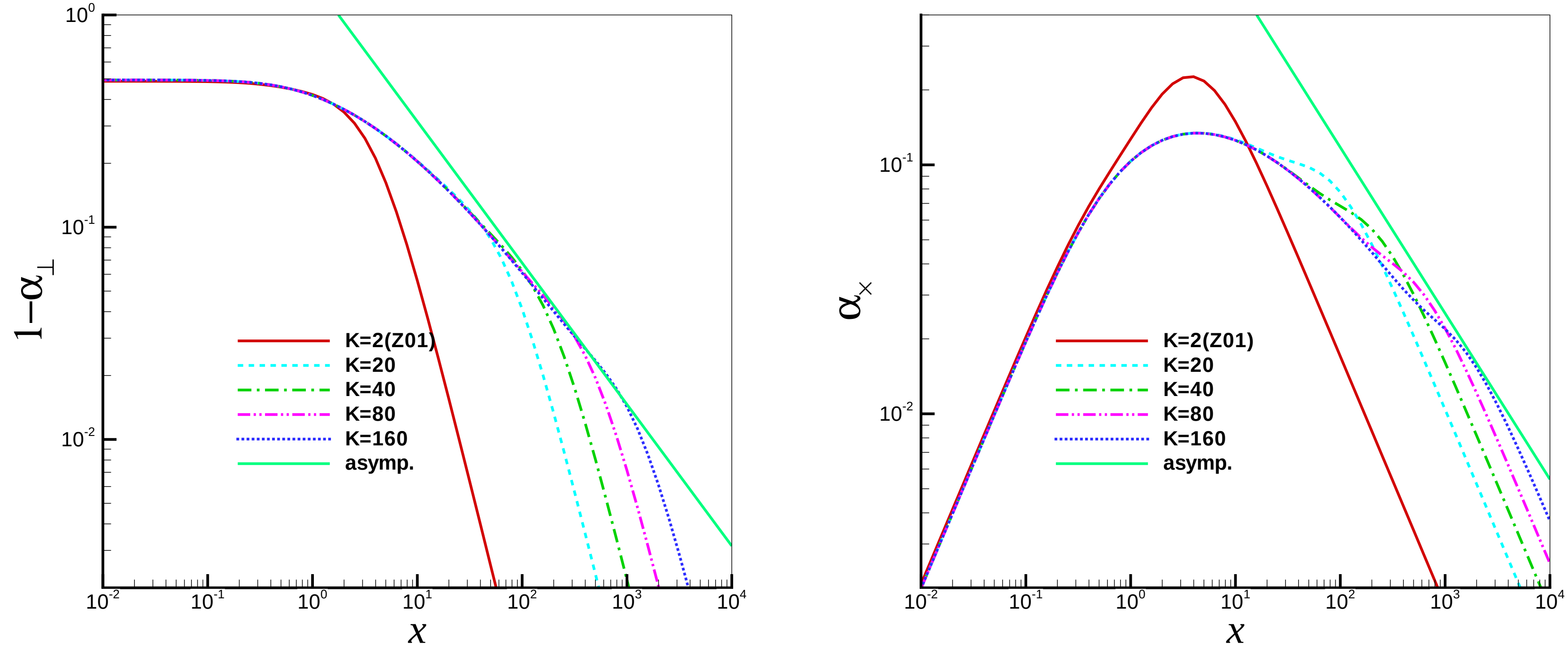
$$\begin{aligned} \mathbf{R}_\mathrm{e} &= \int d\mathbf{v} m_\mathrm{e}\mathbf{v}C(f_\mathrm{e}, f_\mathrm{i}) = \frac{m_\mathrm{e}n_\mathrm{e}}{\tau_\mathrm{ei}}\left(-\mathbf{V}_\mathrm{ei} + \frac{v_{T\mathrm{e}}}{\sqrt{2}}a^{10k}_\mathrm{ei}\mathsf{m}^{1k}_\mathrm{e}\right) \\ \mathbf{q}_\mathrm{e} &= \int d\mathbf{v}\frac{1}{2}m_\mathrm{e}w^2\mathbf{w}f_\mathrm{e} = -\sqrt{\sigma^1_1}n_\mathrm{e}T_\mathrm{e}v_{T\mathrm{e}}\mathsf{m}^{11}_\mathrm{e} \end{aligned}$$

$$\begin{aligned} \mathbf{R}_\mathrm{e} &= -\alpha_\parallel\mathbf{V}_{\mathrm{ei}\parallel} - \alpha_\perp\mathbf{V}_{\mathrm{ei}\perp} + \alpha_\times\mathbf{V}_{\mathrm{ei}\times} - \beta_\parallel\nabla_\parallel T_\mathrm{e} - \beta_\perp\nabla_\perp T_\mathrm{e} - \beta_\times\nabla_\times T_\mathrm{e} \\ \mathbf{q}_\mathrm{e} &= \beta_\parallel T_\mathrm{e}\mathbf{V}_{\mathrm{ei}\parallel} + \beta_\perp T_\mathrm{e}\mathbf{V}_{\mathrm{ei}\perp} + \beta_\times T_\mathrm{e}\mathbf{V}_{\mathrm{ei}\times} - \kappa^\mathrm{e}_\parallel\nabla_\parallel T_\mathrm{e} - \kappa^\mathrm{e}_\perp\nabla_\perp T_\mathrm{e} - \kappa^\mathrm{e}_\times\nabla_\times T_\mathrm{e} \end{aligned}$$

where $\alpha_A = \hat{\alpha}_A\frac{m_\mathrm{e}n_\mathrm{e}}{\tau_\mathrm{ei}}$, $\beta_A = \hat{\beta}_A n_\mathrm{e}$, $\kappa^\mathrm{e}_A = \hat{\kappa}^\mathrm{e}_A\frac{n_\mathrm{e}T_\mathrm{e}\tau_\mathrm{ei}}{m_\mathrm{e}}$ $(A = \parallel, \times, \perp)$

Convergence $\mathbf{R} = -\alpha\mathbf{V} - \beta\nabla T,\ \mathbf{q} = \beta T\mathbf{V} - \kappa\nabla T$

$$
\begin{aligned}
1 - \hat{\alpha}_\perp &= -a_{\text{ei}}^{10p}(x^2+c^2)^{-1}_{pq}c_{qk}a_{\text{ei}}^{1k0} \sim -a_{\text{ei}}^{10p}c_{pk}a_{\text{ei}}^{1k0}x^{-2} \to \infty x^{-2} \\
\hat{\alpha}_\times &= x a_{\text{ei}}^{10p}(x^2+c^2)^{-1}_{pk}a_{\text{ei}}^{1k0} \sim a_{\text{ei}}^{10p}a_{\text{ei}}^{1p0}x^{-1} \to \infty x^{-1}
\end{aligned}
$$



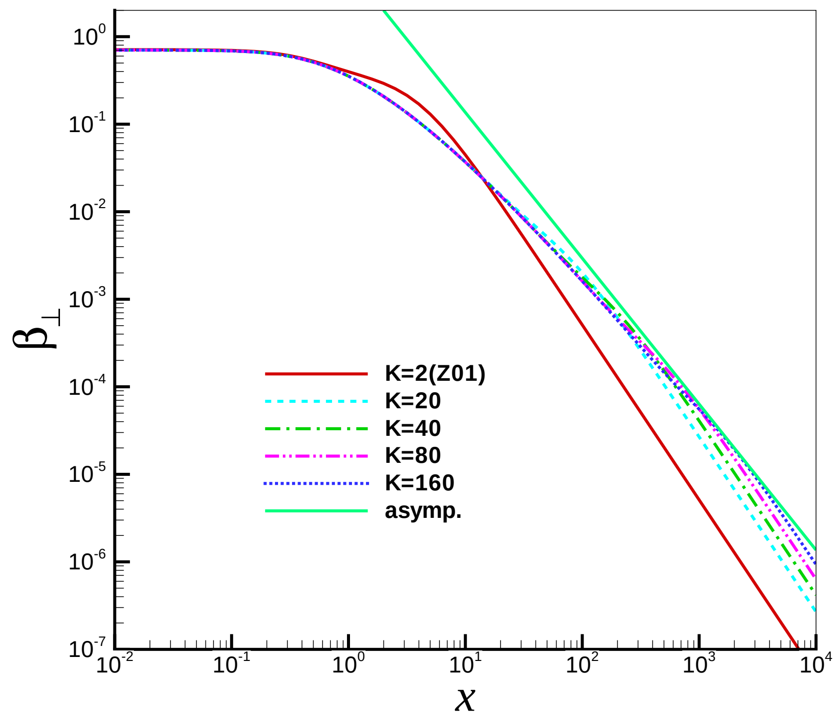
$x \to \infty \quad \Rightarrow \quad K \to \infty \quad \Rightarrow \quad c^2 \sim A^2,\ ca \sim Aa$

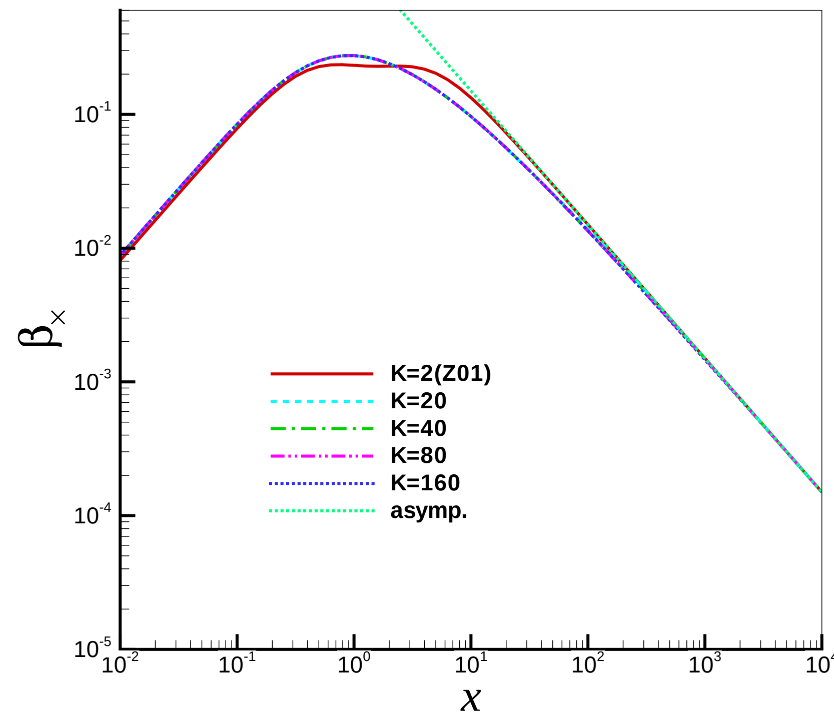
$\Rightarrow$ Lorentz operator gives asymptotic behavior of transport coefficients

Convergence $\mathbf{R} = -\alpha \mathbf{V} - \beta \nabla T$, $\mathbf{q} = \beta T \mathbf{V} - \kappa \nabla T$

$$
\begin{aligned}
\hat{\beta}_\perp &= \sqrt{2\sigma_1^1} a_{\mathrm{ei}}^{10p} (x^2 + c^2)^{-1}_{pq} c_{q1} \sim \sqrt{2\sigma_1^1} a_{\mathrm{ei}}^{10p} c_{p1} x^{-2} \to \infty x^{-2} \\
\hat{\beta}_\times &= -\sqrt{2\sigma_1^1} x a_{\mathrm{ei}}^{10p} (x^2 + c^2)^{-1}_{p1} \sim -\sqrt{2\sigma_1^1} a_{\mathrm{ei}}^{101} x^{-1}
\end{aligned}
$$

where $\sigma_1^1 = \frac{5}{4}$



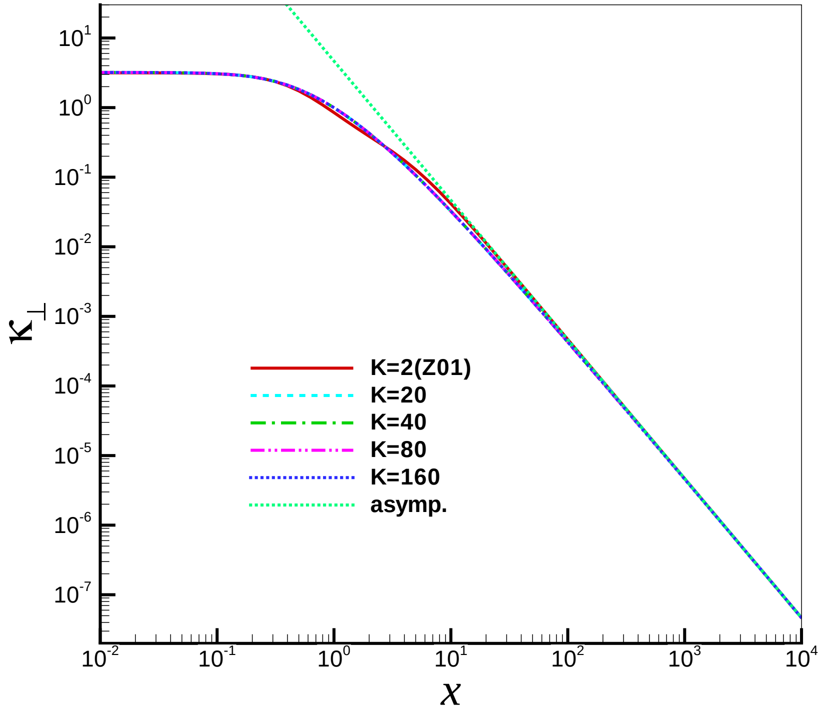


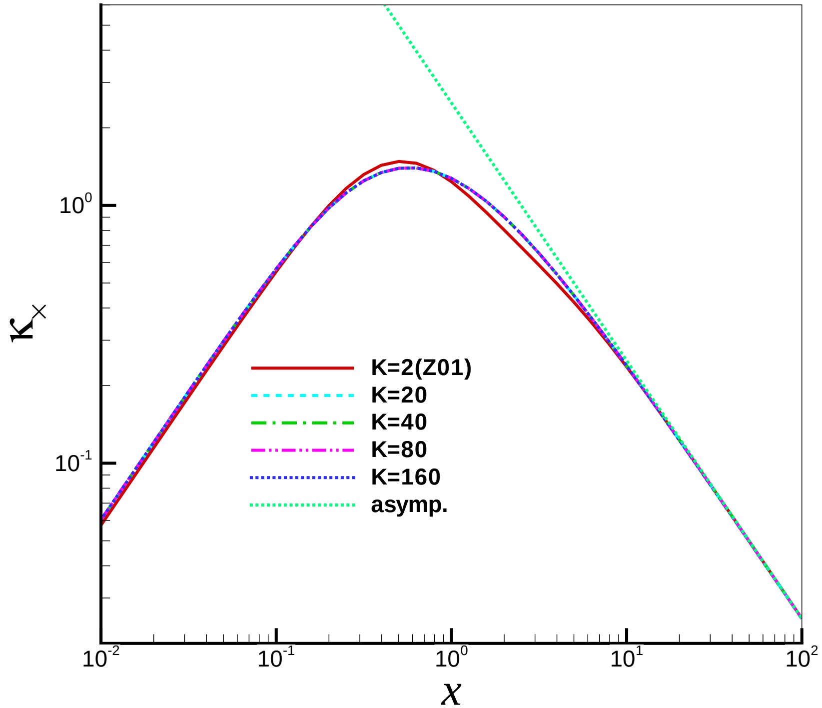
$\beta_\perp$ needs $K \to \infty \Rightarrow$ asymptotic behavior from Lorentz operator

$\beta_\times$ converges for $K = 20$ or less

Convergence $\mathbf{R} = -\alpha\mathbf{V} - \beta\nabla T$, $\mathbf{q} = \beta T\mathbf{V} - \kappa\nabla T$

$$\begin{aligned} \hat{\kappa}_\perp &= -2\sigma_1^1 (x^2 + c^2)^{-1}_{1q} c_{q1} \sim -2\sigma_1^1 c_{11} x^{-2} \\ \hat{\kappa}_\times &= 2\sigma_1^1 x (x^2 + c^2)^{-1}_{11} \sim 2\sigma_1^1 x^{-1} \end{aligned}$$





$\kappa_\perp$ and $\kappa_\times$ converge for $K = 20$ or less

$K = 20$ gives 21st degree polynomials (22 terms) of x^2

$\Rightarrow$ find fitting polynomials (6 terms)

Lorentz gas: geometric method

$$\partial_t f + \mathbf{v}\cdot\nabla f + \frac{e_\mathrm{e}}{m_\mathrm{e}}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\partial_\mathbf{v} f = C_\mathrm{Lorentz}(f) \;\Leftarrow\; f = f_0 + \hat{\mathbf{v}}\cdot\mathbf{f}_1$$

where $f_0 = \dfrac{n}{\pi^{3/2}v_T^3}e^{-s^2}$, $s = \dfrac{v}{v_{T\mathrm{e}}}$ and $C_\mathrm{Lorentz}(\hat{\mathbf{v}}\cdot\mathbf{f}_1) = -\nu_\mathrm{ei}s^{-3}\hat{\mathbf{v}}\cdot\mathbf{f}_1$

$$\mathbf{v}\cdot\nabla f_0 + \frac{q}{m}\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{v}}f_0 + \frac{q}{m}\mathbf{v}\times\mathbf{B}\cdot\frac{\partial}{\partial\mathbf{v}}\hat{\mathbf{v}}\cdot\mathbf{f}_1 = -\nu_\mathrm{ei}s^{-3}\hat{\mathbf{v}}\cdot\mathbf{f}_1$$

$$-y\mathbf{b}\times\mathbf{f}_1 = -s^{-3}\mathbf{f}_1 + \mathbf{g}$$

where $\mathbf{g} = \lambda s\left(\dfrac{5}{2}-s^2\right)\dfrac{1}{T}\nabla T f_0 - \lambda\dfrac{e}{T}\left(\mathbf{E}+\dfrac{1}{ne}\nabla p\right)sf_0$, $\lambda = \dfrac{v_T}{\nu_\mathrm{ei}}$, $y = -\dfrac{\Omega_\mathrm{e}}{\nu_\mathrm{ei}}$, and $\nu_\mathrm{ei} = \dfrac{3\sqrt{\pi}}{4\tau_\mathrm{ei}}$

$$\mathbf{f}_\perp = -\frac{s^6}{1+y^2s^6}\left(y\mathbf{g}_\times + s^{-3}\mathbf{g}_\perp\right)$$

Asymptotic behaviors

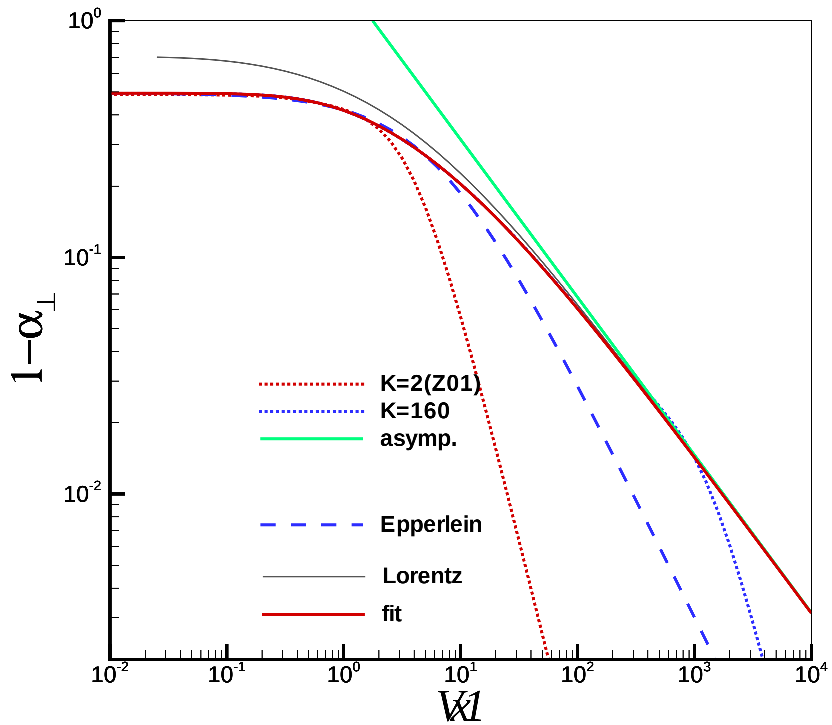
$$1-\hat{\alpha}_\perp \sim x^{-2/3},\ \hat{\alpha}_\times \sim x^{-2/3},\ \hat{\beta}_\perp \sim x^{-5/3},\ \hat{\beta}_\times \sim x^{-1},\ \hat{\kappa}_\perp \sim x^{-2},\ \hat{\kappa}_\times \sim x^{-1}$$

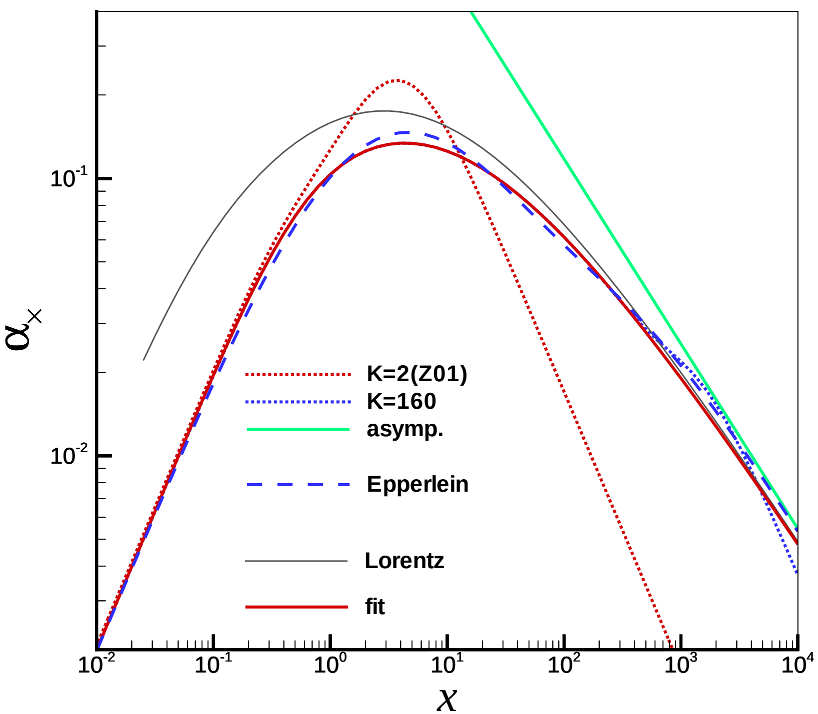
Fitting functions

	Braginskii	Epperlein & Haines (1986)	this work
$1-\hat{\alpha}_\perp$	$\frac{\alpha_1' x^2+\alpha_0'}{x^4+\delta_1 x^2+\delta_0}$	$\frac{\alpha_1' x+\alpha_0'}{x^2+a_1' x+a_0'}$	$\frac{\alpha_1' x+\alpha_0'}{x^{\frac{5}{3}}+a_2' x^{\frac{4}{3}}+a_1' x+a_0'}$
$\hat{\alpha}_\times$	$\frac{x(\alpha_1'' x^2+\alpha_0'')}{x^4+\delta_1 x^2+\delta_0}$	$\frac{x(\alpha_1'' x+\alpha_0'')}{(x^3+a_2'' x^2+a_1'' x+a_0'')^{8/9}}$	$\frac{x(\alpha_1'' x+\alpha_0'')}{x^{\frac{8}{3}}+a_4'' x^{\frac{7}{3}}+a_3'' x^{\frac{6}{3}}+a_2'' x^{\frac{5}{3}}+a_1'' x+a_0''}$
$\hat{\beta}_\perp$	$\frac{\beta_1' x^2+\beta_0'}{x^4+\delta_1 x^2+\delta_0}$	$\frac{\beta_1' x+\beta_0'}{(x^3+b_2' x^2+b_1' x+b_0')^{8/9}}$	$\frac{\beta_1' x+\beta_0'}{x^{\frac{8}{3}}+b_4' x^{\frac{7}{3}}+b_3' x^{\frac{6}{3}}+b_2' x^{\frac{5}{3}}+b_1' x+b_0'}$
$\hat{\beta}_\times$	$\frac{x(\beta_1'' x^2+\beta_0'')}{x^4+\delta_1 x^2+\delta_0}$	$\frac{x(\beta_1'' x+\beta_0'')}{x^3+b_2'' x^2+b_1'' x+b_0''}$	$\frac{x(\beta_1'' x+\beta_0'')}{x^3+b_4'' x^{\frac{7}{3}}+b_3'' x^{\frac{6}{3}}+b_2'' x^{\frac{5}{3}}+b_1'' x+b_0''}$
$\hat{\kappa}_\perp$	$\frac{\gamma_1' x^2+\gamma_0'}{x^4+\delta_1 x^2+\delta_0}$	$\frac{\gamma_1' x+\gamma_0'}{x^3+c_2' x^2+c_1' x+c_0'}$	$\frac{\gamma_1' x+\gamma_0'}{x^3+c_4' x^{\frac{7}{3}}+c_3' x^{\frac{6}{3}}+c_2' x^{\frac{5}{3}}+c_1' x+c_0'}$
$\hat{\kappa}_\times$	$\frac{x(\gamma_1'' x^2+\gamma_0'')}{x^4+\delta_1 x^2+\delta_0}$	$\frac{x(\gamma_1'' x+\gamma_0'')}{x^3+c_2'' x^2+c_1'' x+c_0''}$	$\frac{x(\gamma_1'' x+\gamma_0'')}{x^3+c_4'' x^{\frac{7}{3}}+c_3'' x^{\frac{6}{3}}+c_2'' x^{\frac{5}{3}}+c_1'' x+c_0''}$
Error		less than 15 %	less than 1%
$Z=$	$1,2,3,4,\infty$	$1-8,10,12,14,20,30,60,\infty$	arbitrary (function of Z)

Fit for $Z = 1$, $\mathbf{R} = -\alpha \mathbf{V} - \beta \nabla T$, $\mathbf{q} = \beta T \mathbf{V} - \kappa \nabla T$

$$
\begin{aligned}
1 - \hat{\alpha}_\perp &= -a_{\mathrm{ei}}^{10p}(x^2 + c^2)^{-1}_{pq} c_{qk} a_{\mathrm{ei}}^{1k0} \sim x^{-5/3} \not\approx x^{-2} \\
\hat{\alpha}_\times &= x a_{\mathrm{ei}}^{10p}(x^2 + c^2)^{-1}_{pk} a_{\mathrm{ei}}^{1k0} \sim x^{-2/3} \not\approx x^{-1}
\end{aligned}
$$

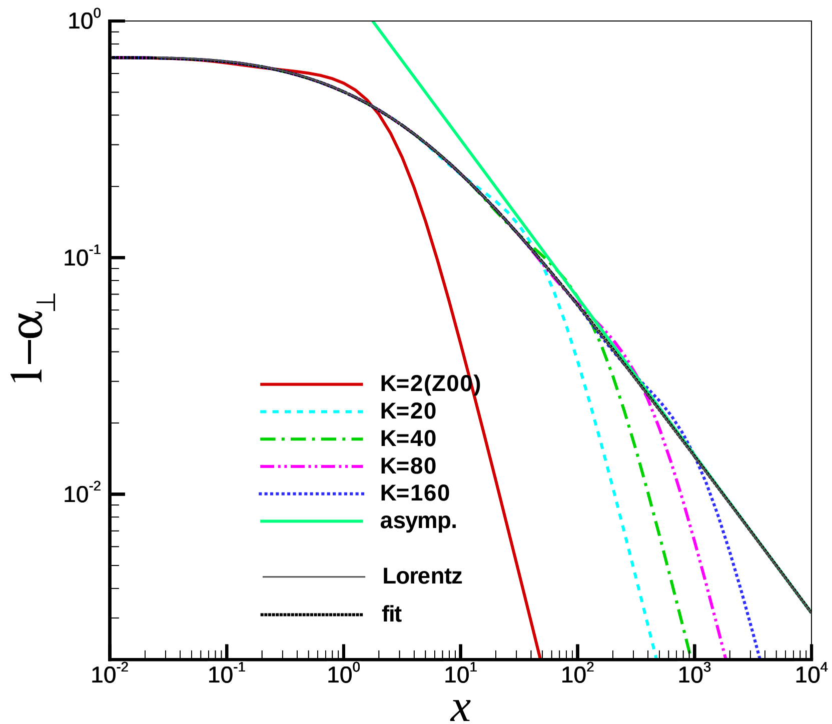


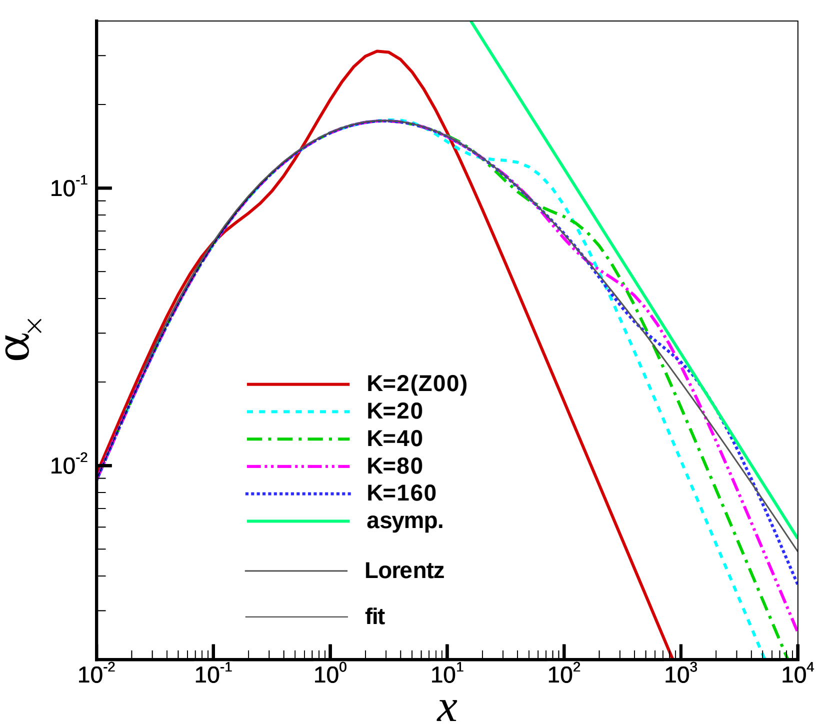


$\Rightarrow Z = 1$ Lorentz gas is not a good approximation for a finite x

Fit for $Z = 100$, $\mathbf{R} = -\alpha\mathbf{V} - \beta\nabla T$, $\mathbf{q} = \beta T\mathbf{V} - \kappa\nabla T$

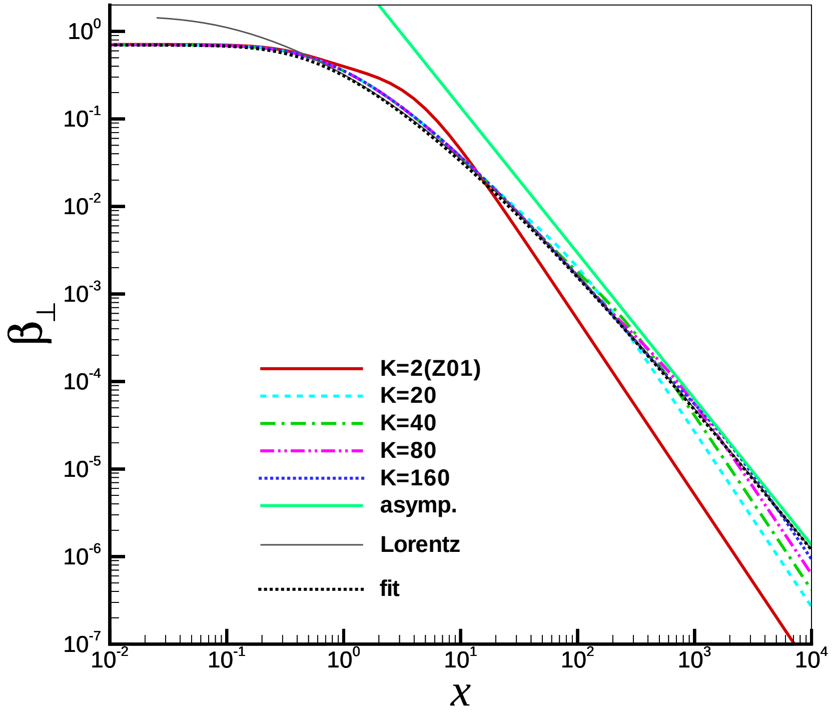
$$
\begin{aligned}
1 - \hat{\alpha}_\perp &= -a_{\rm ei}^{10p}(x^2 + c^2)^{-1}_{pq} c_{qk} a_{\rm ei}^{1k0} \sim x^{-5/3} \not\approx x^{-2} \\
\hat{\alpha}_\times &= x a_{\rm ei}^{10p}(x^2 + c^2)^{-1}_{pk} a_{\rm ei}^{1k0} \sim x^{-2/3} \not\approx x^{-1}
\end{aligned}
$$

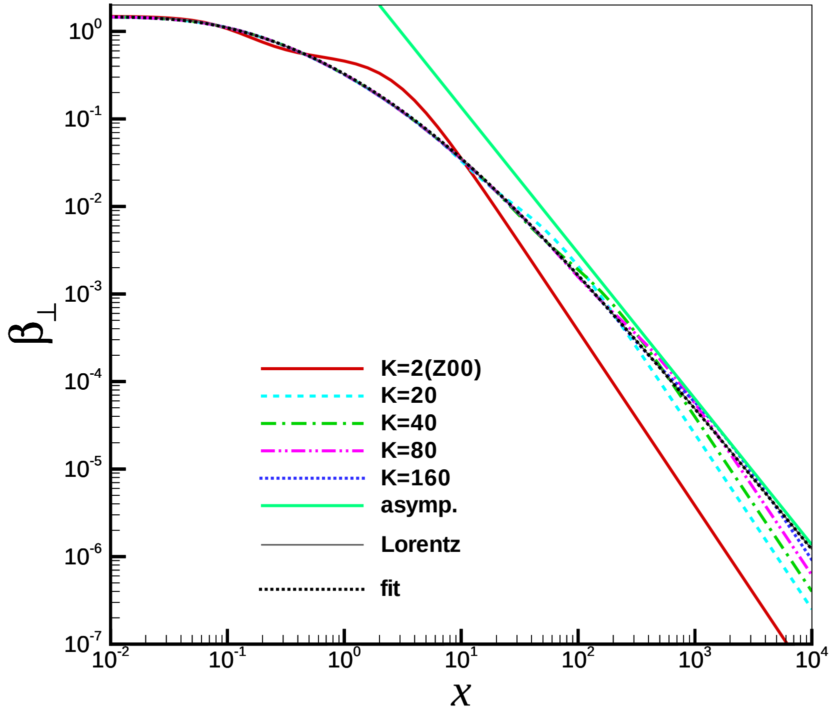




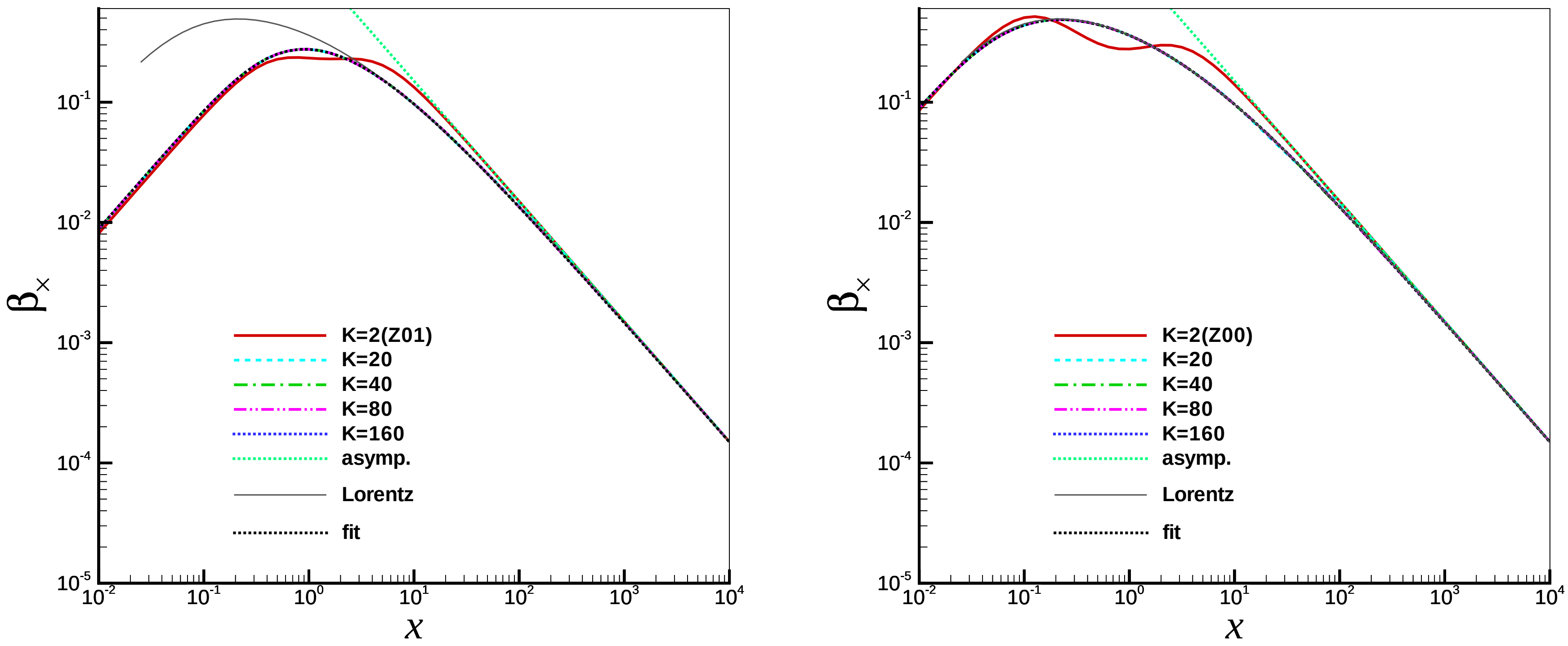
$\Rightarrow Z = 100 \quad \approx \quad$ Lorentz gas

Fit $\beta_\perp$ for $Z = 1,\ 100$

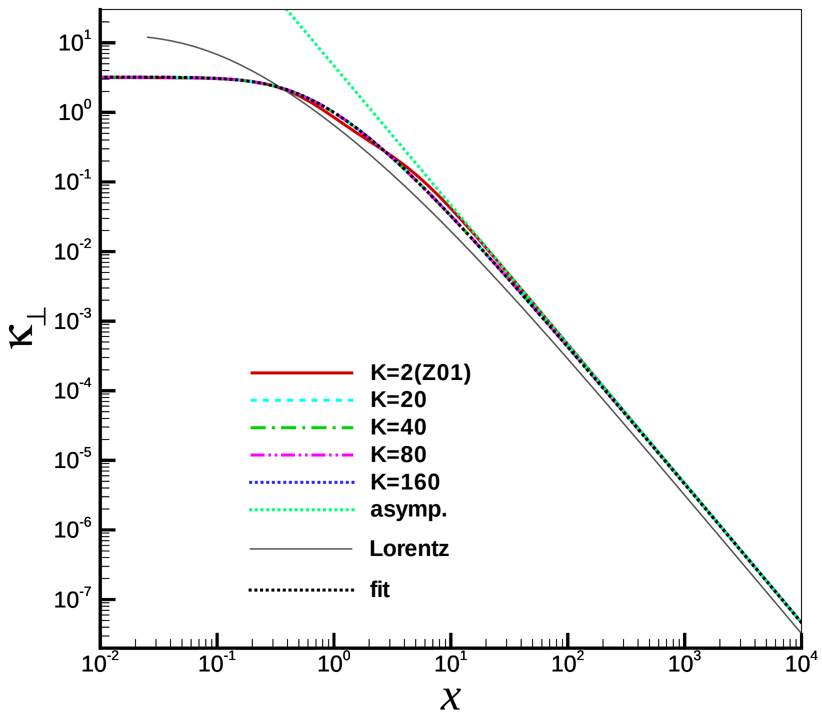


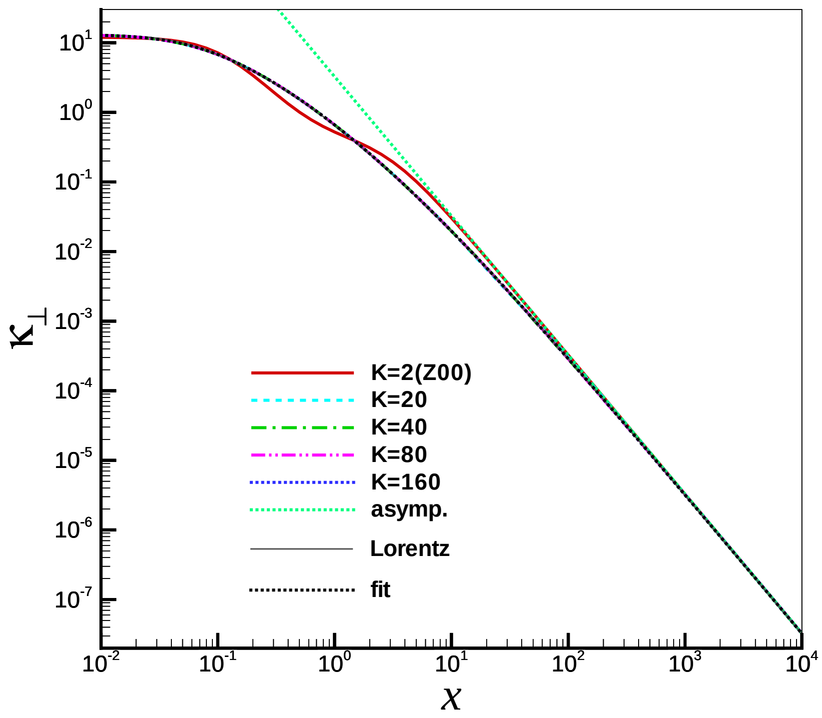


Fit $\beta_\times$ for $Z = 1,\ 100$

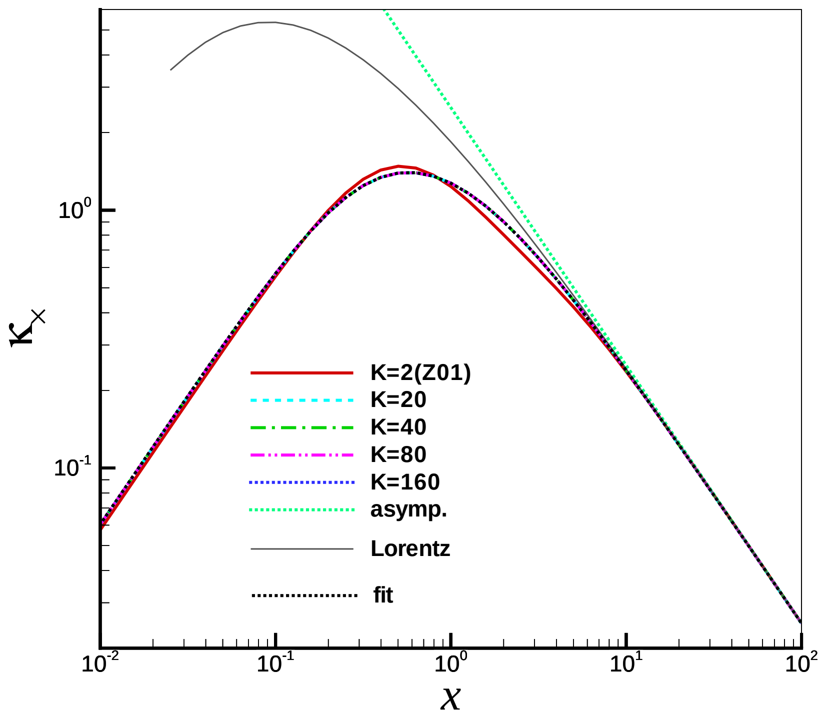


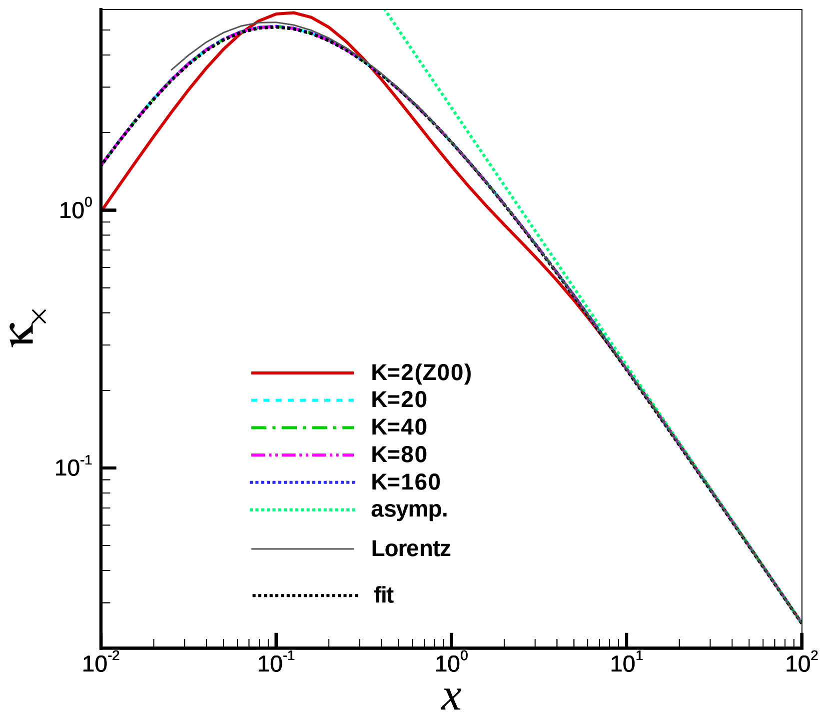
Fit $\kappa_\perp$ for $Z = 1,\ 100$





Fit $\kappa_\times$ for $Z = 1,\ 100$





Future work

- Unified closures for general collisionality/magnetic geometry
  - Braginskii's theory in the high collision limit
  - More accurate neoclassical transport theory
  - Transport theory without flux surface average
- Find fitting functions for kernels
- Compare with neoclassical transport theory
  - Flux surfaces
  - Axi-symmetric geometry
- Apply to interesting fusion devices