Parallel closures in the collisionless limit

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Closure vs. transport theory

- Maxwellian moment $(n_a, \mathbf{V}_a, T_a)$ equations (MMEs)

$$d_a n_a + n_a \nabla \cdot \mathbf{V}_a = 0 \, (d_a \equiv \partial_t + \mathbf{V}_a \cdot \nabla)$$

$$\tfrac{3}{2} n_a d_a T_a + n_a T_a \nabla \cdot \mathbf{V}_a + \nabla \cdot \mathbf{h}_a + \nabla \mathbf{V}_a : \boldsymbol{\pi}_a = Q_a$$

$$m_a n_a d_a \mathbf{V}_a - n_a e_a (\mathbf{E} + \mathbf{V}_a \times \mathbf{B}) + \nabla p_a + \nabla \cdot \boldsymbol{\pi}_a = \mathbf{R}_a$$

  Solve moment equations for n_a (non-Maxwellian moments)

$$D\mathsf{n}_a + \Omega_a \mathbf{b} \times \mathsf{n}_a = C\mathsf{n}_a + \mathsf{G}_a$$

  Express $\mathbf{h}_a(\mathsf{n}_a^{11})$, $\boldsymbol{\pi}_a(\mathsf{n}_a^{20})$, Q_a, $\mathbf{R}_a$ in terms of $n_a, \mathbf{V}_a, T_a$

$$\mathbf{R}_\mathrm{e} = -(\alpha)(\mathbf{V}_\mathrm{ei}) - (\beta)(\nabla T_\mathrm{e}), \;\; \mathbf{h}_\mathrm{e} = (\beta)(\mathbf{V}_\mathrm{ei}) - (\kappa)(\nabla T_\mathrm{e})$$

- Transport theory

  Solve momentum balance equation

$$ne\mathbf{E}' + ne\mathbf{V}_\mathrm{ei} \times \mathbf{B} = \mathbf{R}_\mathrm{e} \text{ where } \mathbf{E}' = \mathbf{E} + \mathbf{V}_\mathrm{i} \times \mathbf{B} + (ne)^{-1} \nabla p_\mathrm{e}$$

  Express fluxes in terms of thermodynamic drives

$$\mathbf{J} = (\sigma)(\mathbf{E}') - (\alpha')(\nabla T_\mathrm{e}), \;\; \mathbf{h}_\mathrm{e} = (\alpha')(\mathbf{E}') - (\kappa')(\nabla T_\mathrm{e})$$

Moment expansion of a distribution function

- Landau-Fokker-Planck kinetic equation

$$\frac{\partial f_a}{\partial t} + \mathbf{v}\cdot\nabla f_a + \frac{q_a}{m_a}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_a}{\partial \mathbf{v}} = \sum_b C(f_a, f_b)$$

- Moment expansion: M^{lk}'s, are symmetric traceless fluid moments

$$f_a(t,\mathbf{x},\mathbf{v}) = f_a^{(0)} \sum\nolimits_{lk} \mathsf{M}_a^{lk}(t,\mathbf{x}) \cdot \hat{\mathsf{P}}_a^{lk}$$

$$f_b(t,\mathbf{x},\mathbf{v}) = f_b^{(0)} \sum\nolimits_{nq} \mathsf{M}_b^{nq}(t,\mathbf{x}) \cdot \hat{\mathsf{P}}_b^{nq}$$

- Moment equations $\int d\mathbf{v}\hat{\mathsf{P}}^{jp} \Rightarrow$

$$\sum_{lk} \hat{D}_a^{jp,lk}(n_a \mathsf{M}_a^{lk}) = \sum_b \sum_{lk,nq} n_a \hat{C}_{ab}^{jp,lk,nq} \overline{\mathsf{M}_a^{lk} \cdot \frac{l+n-j}{2} \; \mathsf{M}_b^{nq}}$$

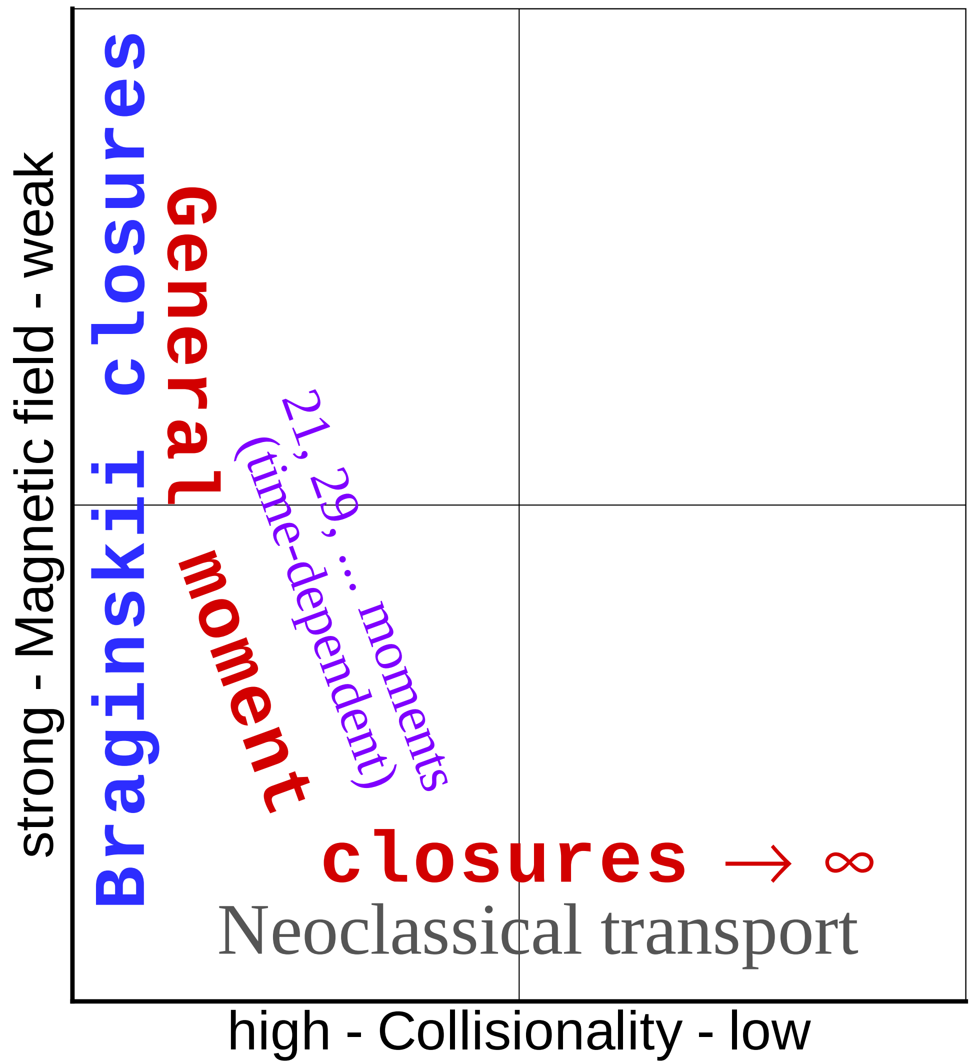
- Drift kinetic equation $\left(\mu = \frac{mv_\perp^2}{2B},\ w = \frac{1}{2}mv^2,\ U = \frac{1}{2}mv^2 + q\Phi\right)$

$$\frac{\partial \bar{f}}{\partial t} + (\mathbf{v}_\parallel + \mathbf{v}_\mathrm{D})\cdot\nabla^{w,\mu}\bar{f} + q(\mathbf{v}_\parallel + \mathbf{v}_\mathrm{D})\cdot\mathbf{E}\frac{\partial \bar{f}}{\partial w} = C(\bar{f})$$

$$\frac{\partial \bar{f}}{\partial t} + (\mathbf{v}_\parallel + \mathbf{v}_\mathrm{D})\cdot\nabla^{U,\mu}\bar{f} - q\mathbf{v}_\parallel\cdot\mathbf{E}^\mathbf{A}\frac{\partial \bar{f}}{\partial U} = C(\bar{f})$$

where $\mathbf{v}_\mathrm{D} = \frac{1}{\Omega}\mathbf{b}\times\left(-\frac{q\mathbf{E}}{m} + \frac{\mu}{m}\nabla B + v_\parallel^2\boldsymbol{\kappa}\right)$

Closures/transport theory



General moment closures/transport	Neoclassical transport
exact Landau (FP) collision operator	model operator Lorentz+restoring
general (unified) collisionality	high (PS) low (banana)
general magnetic field	flux surfaces (averaged)
collisionless limit: ∞ moments	solving DKE

Parallel closures due to parallel drives $\mathrm{b}\cdot\nabla\bar{f}^{\mathrm{M}}\sim O(\delta^0)$

$$\frac{\partial}{\partial t}(\bar{f}^{\mathrm{M}}+\bar{F})+v_\parallel\mathbf{b}\cdot\nabla(\bar{f}^{\mathrm{M}}+\bar{F})=C(\bar{F})\Rightarrow\bar{F}=\lim_{C\to 0}\frac{1}{v_\parallel\mathbf{b}\cdot\nabla-C}(-v_\parallel\mathbf{b}\cdot\nabla\bar{f}^{\mathrm{M}})$$

$$v_\parallel\partial_\ell(\bar{f}^{\mathrm{M}}+\bar{F})=0\Rightarrow\bar{F}=-\bar{f}^{\mathrm{M}}+g(\ell\text{-indep.})\Rightarrow\bar{F}=g(\ell\text{-indep.})$$

Trivial solutions: $\partial_\ell h_\parallel=0\ \Rightarrow\ h_\parallel(\ell\text{-indep.})$: MMEs should be removed

$$v_\parallel\partial_\ell\bar{F}+C(\bar{F})=\left\{\left(s^2-\frac{3}{2}\right)\frac{2\partial_\ell h_\parallel}{3nT}+v_\parallel\left[\frac{\partial_\ell\pi_\parallel}{nT}-\left(s^2-\frac{5}{2}\right)\frac{1}{T}\frac{\partial T}{\partial\ell}\right]-\frac{4}{3}P_2s^2\partial_\ell V_\parallel\right\}f^{(0)}$$

- Hammett & Perkins (1990): $h_\parallel=\int dv\frac{1}{2}mw^2w_\parallel\neq\int dv\frac{1}{2}mw_\parallel^3$, $\mathbf{w}=\mathbf{v}-\mathbf{V}$
- Chang & Callen (1992): k-space
- Hazeltine (1998): external source

$$\begin{pmatrix}\frac{\tilde{h}_\parallel}{v_TT}\\ \frac{\tilde{\pi}}{T}\end{pmatrix}=\begin{pmatrix}\frac{9}{5\sqrt{\pi}}\frac{|k|}{k}i & -\frac{2}{5}\\ -\frac{2}{5} & \frac{4\sqrt{\pi}}{5}\frac{|k|}{k}i\end{pmatrix}\begin{pmatrix}n\tilde{T}_1\\ \frac{n\tilde{V}_1}{v_T}\end{pmatrix}\Rightarrow\begin{aligned}h_\parallel&=-\frac{9nv_T}{5\pi^{3/2}}\int d\ell'\frac{T_1(\ell')}{\ell-\ell'}-\frac{2}{5}nTV_1\\ \pi_\parallel&=-\frac{2}{5}nT_1+\frac{4p}{5\sqrt{\pi}v_T}\int d\ell'\frac{V_1(\ell')}{\ell-\ell'}\end{aligned}$$

Comments on neoclassical transport in the banana regime

- δ^1: $v_\parallel \partial_\parallel^{U,\mu} \bar{f}_{a1} + \mathbf{v}_\mathrm{D} \cdot \nabla^{U,\mu} f_{a0} - \dfrac{q_a v_\parallel E_\parallel^{\mathbf{A}}}{T_a} f_{a0} = C(\bar{f}_{a1}),\ U = \dfrac{1}{2} m v^2 + q\Phi$

  $\mathbf{B} = I(\psi)\nabla\varphi + \nabla\varphi \times \nabla\psi$: $\mathbf{v}_\mathrm{D} \cdot \nabla^{U,\mu} f_{a0} = v_\parallel \partial_\parallel^{U,\mu} k_a$ with $k_a = -\frac{I v_\parallel}{\Omega_a} \frac{\partial f_{a0}}{\partial \psi}$

  Spitzer solution: $C(f_{a\mathrm{S}}) = -\frac{q_a v_\parallel E_\parallel}{T_a} f_{a0}$:high collisionality solution

  $$v_\parallel \partial_\parallel^{U,\mu} (f_{a1} - k_a) = C(f_{a1} - f_{a\mathrm{S}})$$

- Subsidiary expansion $f_{a1} = f_{a1}^{(0)} + f_{a1}^{(1)} + \cdots$ based on low collisionality

  $$v_\parallel \partial_\parallel^{U,\mu} (f_{a1}^{(0)} - k_a) = 0 \ \Rightarrow\ f_{a1}^{(0)} = k_a + g_a(\psi, U, \mu)$$

  Particular solution $\Rightarrow v_\parallel \partial_\parallel^{U,\mu} f_{a1}^{(0)} = S \ \Rightarrow\ f_{a1}^{(0)} = g_a(\{S\})$

  $$v_\parallel \partial_\parallel^{U,\mu} f_{a1}^{(1)} = C(f_{a1}^{(0)} - f_{a\mathrm{S}}) \ \Rightarrow \left\langle \frac{B}{v_\parallel} C_a(g_a + F_a - f_{a\mathrm{S}}) \right\rangle_{\mathrm{FS}} = 0$$

$$\mathbf{B} \cdot \nabla|_{\mathbf{v}} f(\mathbf{x}, U, \mu) = \mathbf{B} \cdot \frac{\partial f}{\partial \mathbf{x}}\bigg|_{U,\mu} + q\mathbf{B}\cdot\nabla\Phi \frac{\partial f}{\partial U}\bigg|_{\mathbf{x},\mu} - (\mu \mathbf{B}\cdot\nabla \ln B + m v_\parallel \mathbf{v}_\perp \cdot \boldsymbol{\kappa}) \frac{\partial f}{\partial \mu}\bigg|_{\mathbf{x},U}$$

$$0 \neq \left\langle B \partial_\parallel^{U,\mu} f_1^{(1)} \right\rangle_{\mathrm{FS}} = \left\langle \frac{B}{v_\parallel} C_a(g_a + k_a - f_{a\mathrm{S}}) \right\rangle_{\mathrm{FS}}$$

Parallel closures on flux surfaces $\mathbf{v}_{\rm D}\cdot\nabla f_0 \sim \mathbf{v}_\parallel\cdot\nabla f_1 \sim O(\delta^1)$

- $v_\parallel \mathbf{b}\cdot\nabla^{w,\mu}\bar{f}_1 + \mathbf{v}_{\rm D}\cdot\nabla\bar{f}_0 + q v_\parallel E_\parallel \frac{\partial \bar{f}_0}{\partial w} = C(\bar{f}_1)$ where $w = \frac{1}{2}mv^2,\ \mu = \frac{mv_\perp^2}{2B}$

$$\bar{f}_1 = \bar{f}_1^{\rm M} + \bar{F}_1,\ f_1^{\rm M} = f_0^{\rm M}(\psi, w)\left[\frac{n_1}{n_0} + \left(\frac{v^2}{v_T^2} - \frac{3}{2}\right)\frac{T_1}{T_0} + \frac{m}{T_0}\mathbf{v}\cdot\mathbf{V}_1\right]$$

$$v_\parallel \mathbf{b}\cdot\nabla^{w,\mu}\bar{F}_1 - C(\bar{F}_1) = -v_\parallel \mathbf{b}\cdot\nabla^{w,\mu}\bar{f}_1^{\rm M} - \mathbf{v}_{\rm D}\cdot\nabla f_0 - q v_\parallel E_\parallel \frac{\partial f_0^{\rm M}}{\partial w}$$

Eliminate MMEs

$$\mathbf{v}_\parallel\cdot\nabla F - C(F) = \left[\left(s^2 - \frac{3}{2}\right)\frac{1}{T}\frac{2}{3}\partial_\parallel h_\parallel + \frac{v_T s_\parallel}{T}\partial_\parallel \pi_\parallel \right.$$
$$\left. + g^{02}\hat{P}^{02} + g^{11}\hat{P}^{11} + g^{20}\hat{P}^{20} + g^{21}\hat{P}^{21}\right]\frac{f_0}{n}$$

$$\begin{pmatrix} \frac{\tilde{h}_\parallel}{v_T T} \\ \frac{\tilde{\pi}}{T} \end{pmatrix} = \begin{pmatrix} \frac{9}{5\sqrt{\pi}}\frac{|k|}{ik} & \frac{3}{10} & \frac{33}{40} & -\frac{3}{4} \\ -\frac{2}{5} & \frac{3\sqrt{\pi}}{5}\frac{|k|}{ik} & \frac{\sqrt{\pi}}{10}\frac{|k|}{ik} & -\frac{\sqrt{\pi}}{2}\frac{|k|}{ik} \end{pmatrix} \begin{pmatrix} \frac{\tilde{g}^{11}}{v_T ik} \\ \frac{\tilde{g}^{20}}{v_T ik} \\ \frac{\tilde{g}^{02}}{v_T ik} \\ \frac{\tilde{g}^{21}}{v_T ik} \end{pmatrix}$$

$$\hat{g} \sim \mathbf{b}\times\nabla B\cdot\nabla T_0(\psi) = -I\frac{\partial B}{\partial \ell}\frac{dT_0}{d\psi} \Rightarrow h_\parallel \sim \int \frac{B(\ell')}{\ell - \ell'}d\ell' \frac{dT_0}{d\psi}$$

Future work

- Find closures in the moderate collisionality regime
- Interpolate closures to the collisionless regime
- Develop 21 or 29 moment equations (dynamic closures) in NIMROD
- Develop moment equations and closures for impurities and neutrals

Notes added after the presentation:

Ping Zhu pointed out that the annihilation operator of $\mathbf{B}\cdot\nabla$ on p. 6 is not the flux surface average but an orbital average. In the original work, the orbital average is taken to remove $f_1^{(1)}$. But, for passing particles in the axi-symmetric geometry, both averages are the same. Furthermore, the same arguments can be applied to the orbit integral (ϑ-integral in the axi-symmetric geometry). Note that $0 = \int_0^{2\pi} d\vartheta \left.\frac{\partial f_1^{(1)}}{\partial \vartheta}\right|_{\mathbf{v}}$ but $0 \neq \int_0^{2\pi} d\vartheta \left.\frac{\partial f_1^{(1)}}{\partial \vartheta}\right|_{U,\mu}$ from

$$\left.\frac{\partial f_1^{(1)}}{\partial \vartheta}\right|_{\mathbf{v}} = \left.\frac{\partial f_1^{(1)}}{\partial \vartheta}\right|_{U,\mu} + \left.\frac{\partial U}{\partial \vartheta}\right|_{\mathbf{v}} \left.\frac{\partial f_1^{(1)}}{\partial U}\right|_{\mathbf{x},\mu} + \left.\frac{\partial \mu}{\partial \vartheta}\right|_{\mathbf{v}} \left.\frac{\partial f_1^{(1)}}{\partial \mu}\right|_{\mathbf{x},U}.$$