Implementation and application of the moment method

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NIMROD Team Meeting

August 13, 2013, Boulder, CO

Closures for fluid moment $(\mathbb{M})$ equations (i) Solve KE with $f = f^{\mathbb{M}} + f^{\mathbb{C}}$ (ii) express $\mathbb{C}$ in terms of $\mathbb{M}$

- Evolution equations of Maxwellian moments $\mathbb{M} = (n_a, \mathbf{V}_a, T_a)$

$$d_a n_a + n_a \nabla \cdot \mathbf{V}_a = 0 \, (d_a \equiv \partial_t + \mathbf{V}_a \cdot \nabla)$$

$$\frac{3}{2} n_a d_a T_a + n_a T_a \nabla \cdot \mathbf{V}_a + \nabla \cdot \mathbf{h}_a + \nabla \mathbf{V}_a : \boldsymbol{\pi}_a = Q_a$$

$$m_a n_a d_a \mathbf{V}_a - n_a q_a (\mathbf{E} + \mathbf{V}_a \times \mathbf{B}) + \nabla p_a + \nabla \cdot \boldsymbol{\pi}_a = \mathbf{R}_a$$

Solve moment equations for n_a ($\mathbb{C}$ =non-Maxwellian moments)

$$\mathsf{Dn}_a + \Omega_a \mathbf{b} \breve{\times} \mathsf{n}_a = \mathsf{Cn}_a + \mathsf{G}_a$$

Express $\mathbf{h}_a(\mathsf{n}_a^{11})$, $\boldsymbol{\pi}_a(\mathsf{n}_a^{20})$, Q_a, $\mathbf{R}_a$ in terms of $n_a, \mathbf{V}_a, T_a$

$$\mathbf{R}_{\mathrm{e}} = -\alpha \mathbf{V}_{\mathrm{ei}} - \beta \nabla T_{\mathrm{e}}, \;\; \mathbf{h}_a = \beta \mathbf{V}_{\mathrm{ei}} - \kappa_a \nabla T_a, \;\; \boldsymbol{\pi}_a = -\eta_a \overline{\overline{\nabla \mathbf{V}}}_a$$

- Neoclassical transport theory $\mathbb{M} = (n_0, T_0)$, $\mathbb{C} = (n_1, T_1, V_{\|1}, h_{\|}, \cdots)$, $f = f_0 + f_1$

Evolution equations of flux-surface averaged density and temperature

$$n_0(t,\psi) = \langle n(t,\mathbf{x}) \rangle, \; T_0(t,\psi) = \langle T(t,\mathbf{x}) \rangle \;\Rightarrow \partial_t n_0 + \cdots, \; \partial_t T_0 + \cdots$$

Express particle and heat fluxes in terms of $\dfrac{dT_0}{d\psi}$ and $\dfrac{dp_0}{d\psi}$

- Evolution equations of Maxwellian moments for toroidal plasmas

$$\mathbb{M} = (n_0, T_0, n_1, T_1, V_{\|1}), \; \mathbb{C} = (h_{\|}, \pi_{\|}, R_{\|}, Q, \cdots), \; f = f_0 + f_1^{\mathrm{M}} + f_1^{\mathrm{N}}$$

General moment equations

- Landau-Fokker-Planck kinetic equation

$$\underbrace{\frac{\partial f_a}{\partial t} + \mathbf{v}\cdot\nabla f_a + \frac{q_a}{m_a}(\mathbf{E}+}_{\mathsf{LN}}\underbrace{\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_a}{\partial \mathbf{v}}}_{-\Omega\mathsf{KN}} = \underbrace{\sum_b C(f_a, f_b)}_{\mathsf{CN}}$$

- Moment expansion: M^{lk}'s, are symmetric traceless fluid moments

$$f_a(t,\mathbf{x},\mathbf{v}) = \hat{f}_a^0 \textstyle\sum_{l,k=0}^{\infty} \mathsf{N}_a^{lk}(t,\mathbf{x}) \cdot \hat{\mathsf{P}}_a^{lk} \ (\mathsf{N}^{00}: \text{density}, \mathsf{N}^{01}: \text{temperature},$$

$$\mathsf{N}^{10}: \text{flow velocity}, \mathsf{N}^{11}: \text{heat flow}, \mathsf{N}^{20}: \text{viscosity, etc.})$$

- Moment equations $\int d\mathbf{v}\hat{\mathsf{P}}^{jp} \Rightarrow \mathsf{LN} - \Omega\mathsf{KN} = \mathsf{CN}$

$$\delta^0: \ \Omega\mathsf{KN}_0 = 0 \Rightarrow \mathsf{N}_0: \ n_0(\psi), \ T_0(\psi)$$

$$\delta^1: \ \Omega\mathsf{KN}_1 = (\mathsf{L}-\mathsf{C})\mathsf{N}_0 \Rightarrow \text{Parallel moment equations for } N_{1\|}$$

$$\delta^2: \ \Omega\mathsf{KN}_2 = (\mathsf{L}-\mathsf{C})\mathsf{N}_1 \Rightarrow \text{Perpendicular } \mathsf{N}_{2\perp} \text{ (FLR)}$$

- Parallel moment equations [neoclassical transport, integral (nonlocal) closures]

$$\underline{[\Psi]\lambda\partial_\|[N] + [\Psi_B]\lambda\partial_\| \ln B[N] = [c][N]} + [G\{\partial_\| T, \partial_\| V_\|\}] + \underline{\lambda\frac{B_0\partial_\| \ln B}{B}[g]}$$

For $\mathbf{B} = I\nabla\zeta + \nabla\zeta\times\nabla\psi$, $[g] = \left\{[g_p]\frac{d\ln p}{d\psi} + [g_T]\frac{d\ln T}{d\psi}\right\}\rho_0 I$

Implementing improved Braginskii (high collisionality)

Ji and Held, PoP 20, 042114 (2013)

Fitted formulas are exact within 1% error for arbitrary $r_a = \Omega_a \tau_{aa}$ and ion charge (Z)

$$\mathbf{h}_a = \frac{n_a T_a \tau_{aa}}{m_a}(-\hat{\kappa}^a_{\parallel}\nabla_{\parallel}T_a - \hat{\kappa}^a_{\perp}\nabla_{\perp}T_a -^a \hat{\kappa}^a_{\times}\nabla_{\times}T_a) + \delta_{ae}n_e T_e(\hat{\beta}_{\parallel}\mathbf{V}_{ei\parallel} + \hat{\beta}_{\perp}\mathbf{V}_{ei\perp} + \hat{\beta}_{\times}\mathbf{V}_{ei\times})$$

$$\mathbf{R}_e = \frac{m_e n_e}{\tau_{ei}}(-\hat{\alpha}_{\parallel}\mathbf{V}_{ei\parallel} - \hat{\alpha}_{\perp}\mathbf{V}_{ei\perp} + \hat{\alpha}_{\times}\mathbf{V}_{ei\times}) + n_e(-\hat{\beta}_{\parallel}\nabla_{\parallel}T_e - \hat{\beta}_{\perp}\nabla_{\perp}T_e - \hat{\beta}_{\times}\nabla_{\times}T_e)$$

$$\boldsymbol{\pi}_a = n_a T_a \tau_{aa}(-\hat{\eta}^a_0 \mathsf{W}^a_0 - \hat{\eta}^a_1 \mathsf{W}^a_1 - \hat{\eta}^a_2 \mathsf{W}^a_2 -^a \hat{\eta}^a_3 \mathsf{W}^a_3 -^a \hat{\eta}^a_4 \mathsf{W}^a_4)$$

where $-^e = -1,\ -^i = 1,\ (\mathsf{W})_{\alpha\beta} = \partial_\alpha V_\beta + \partial_\beta V_\alpha - \delta_{\alpha\beta}\frac{2}{3}\nabla\cdot\mathbf{V}$

$$\mathsf{w}_0 = \mathsf{w}_{\parallel\parallel} + \tfrac{1}{2}(\mathsf{w}_{\times\times} + \mathsf{w}_{\perp\perp}),\ \mathsf{w}_1 = \tfrac{1}{2}(\mathsf{w}_{\perp\perp} - \mathsf{w}_{\times\times}),\ \mathsf{w}_2 = 2\mathsf{w}_{\parallel\perp},\ \mathsf{w}_3 = \mathsf{w}_{\times\perp},\ \mathsf{w}_4 = 2\mathsf{w}_{\parallel\times}$$

$$\text{electron}:\ \hat{\alpha}_{\perp} = 1 - \frac{1.46Z^{\frac{2}{3}}r + \alpha_0(1-\hat{\alpha}_{\parallel})}{r^{\frac{5}{3}} + \alpha_2 r^{\frac{4}{3}} + \alpha_1 r + \alpha_0},\ \hat{\alpha}_{\times} = \frac{Z^{\frac{2}{3}}r(2.53r + a_0/a_5)}{r^{\frac{8}{3}} + a_4 r^{\frac{7}{3}} + a_3 r^2 + a_2 r^{\frac{5}{3}} + a_1 r + a_0},$$

$$\hat{\beta}_{\perp} = \frac{6.33Z^{\frac{5}{3}}r + \beta_0\hat{\beta}_{\parallel}}{r^{\frac{8}{3}} + \beta_4 r^{\frac{7}{3}} + \beta_3 r^2 + \beta_2 r^{\frac{5}{3}} + \beta_1 r + \beta_0},\ \hat{\beta}_{\times} = \frac{Zr(\frac{3}{2}r + b_0/b_5)}{r^3 + b_4 r^{\frac{7}{3}} + b_3 r^2 + b_2 r^{\frac{5}{3}} + b_1 r + b_0},$$

$$\hat{\kappa}^e_{\perp} = \frac{(\frac{13}{4}Z + \sqrt{2})r + \kappa_0\hat{\kappa}^e_{\parallel}}{r^3 + \kappa_4 r^{\frac{7}{3}} + \kappa_3 r^2 + \kappa_2 r^{\frac{5}{3}} + \kappa_1 r + \kappa_0},\ \hat{\kappa}^e_{\times} = \frac{r(\frac{5}{2}r + k_0/k_5)}{r^3 + k_4 r^{\frac{7}{3}} + k_3 r^2 + k_2 r^{\frac{5}{3}} + k_1 r + k_0},$$

$$\hat{\eta}^e_2 = \frac{(\frac{6}{5}Z + \frac{3}{5}\sqrt{2})r + h'_0\hat{\eta}^e_0}{r^3 + h'_4 r^{\frac{7}{3}} + h'_3 r^2 + h'_2 r^{\frac{5}{3}} + h'_1 r + h'_0},\ \hat{\eta}^e_4 = \frac{r(r + h_0/h_5)}{r^3 + h_4 r^{\frac{7}{3}} + h_3 r^2 + h_2 r^{\frac{5}{3}} + h_1 r + h_0},$$

All coefficients are functions of Z and given in the paper, $\parallel = \perp\ (r=0)$

$$\text{ion}:\ \hat{\kappa}^i_{\parallel} = (5.586 + 101.7\zeta + 289.1\zeta^2)/\Delta^{i1}_{\parallel},\ \zeta = Z^{-1}\sqrt{m_e T_i / m_i T_e}$$

$$\Delta^{i1}_{\parallel} = 1 + 13.50\zeta + 36.46\zeta^2,\ \text{etc.}$$

Implementing 2-fluid 21 moment equations in NIMROD

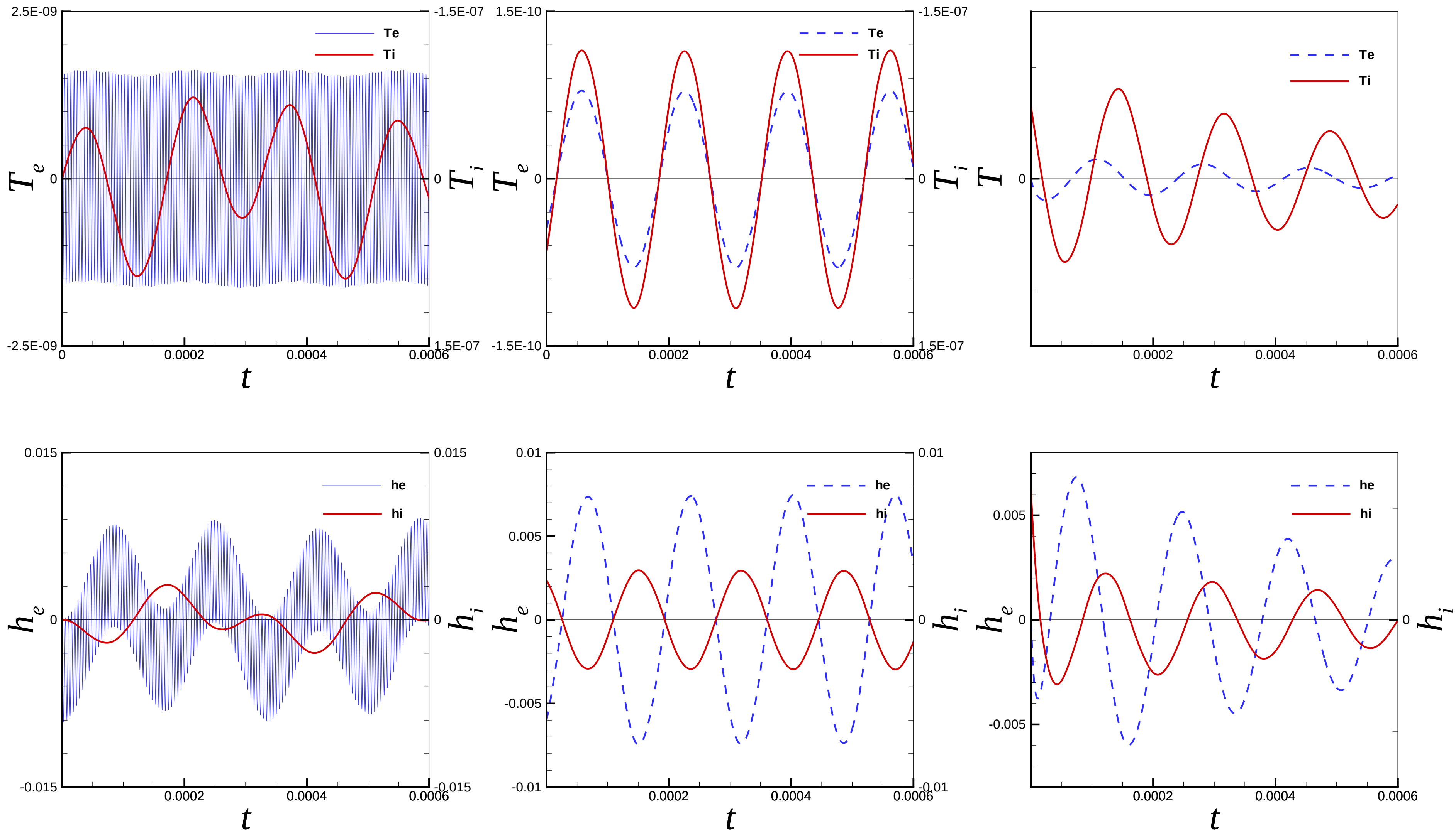
[here $d_t = \partial_t + \mathbf{V}\cdot\nabla,\ \mathbf{a} = \frac{q}{m}(\mathbf{E}+\mathbf{V}\times\mathbf{B}) - d_t\mathbf{V},\ z_{\mathrm{e}} = Z,\ z_{\mathrm{i}} = 0$]

- $$\underline{\partial_t \mathbf{h}} + \mathbf{V}\cdot\nabla\mathbf{h}\underline{+\Omega\mathbf{b}\times\mathbf{h}} + \frac{7}{5}(\nabla\cdot\mathbf{V})\mathbf{h} + \frac{7}{5}\mathbf{h}\cdot(\nabla\mathbf{V}) + \frac{2}{5}(\nabla\mathbf{V})\cdot\mathbf{h}$$
  $$\underline{+\frac{5p}{2m}\nabla T} + \frac{T}{m}\nabla\cdot\boldsymbol{\pi} + \frac{7}{2}\frac{\nabla T}{m}\cdot\boldsymbol{\pi} - \mathbf{a}\cdot\boldsymbol{\pi} + \nabla\cdot\boldsymbol{\theta} + \frac{1}{3}\nabla\mathsf{u}^{02} + \nabla\mathbf{V}:\mathsf{u}^{30}$$
  $$= \underline{\frac{1}{2\tau_{aa}}}\left\{3z_a n_{\mathrm{e}} T_{\mathrm{e}} \mathbf{V}_{\mathrm{ei}}\underline{-\left(\frac{2\sqrt{2}}{5} + z_a\frac{13}{10}\right)(2\mathbf{h})} + \left(\frac{6\sqrt{2}}{35} + z\frac{69}{70}\right)\frac{\mathbf{r}}{v_T^2}\right\}$$
  ($\mathbf{h}$: heat flow implemented)
- $$d_t\mathbf{r} + 14\frac{d_tT}{m}\mathbf{h} + \Omega\mathbf{b}\times\mathbf{r} + \frac{28}{5}\frac{T}{m}(\nabla\cdot\mathbf{V}\mathbf{h} + \mathbf{h}\cdot\nabla\mathbf{V} + \nabla\mathbf{V}\cdot\mathbf{h}) + \frac{9}{5}\nabla\cdot\mathbf{V}\mathbf{r} + \frac{9}{5}\mathbf{r}\cdot\nabla\mathbf{V} + \frac{4}{5}\nabla\mathbf{V}\cdot\mathbf{r}$$
  $$- 4\mathbf{a}\cdot\boldsymbol{\theta} + 14\frac{T\nabla T}{m^2}\cdot\boldsymbol{\pi} + 4\frac{T}{m}\nabla\cdot\boldsymbol{\theta} + 18\frac{\nabla T}{m}\cdot\boldsymbol{\theta}$$
  $$- \frac{7}{3}\mathbf{a}\mathsf{u}^{02} + \frac{7}{3}\frac{T}{m}\nabla\mathsf{u}^{02} + 7\frac{\nabla T}{m}\mathsf{u}^{02} - \nabla\mathsf{u}^{03} + \nabla\cdot\mathsf{u}^{22} + \frac{4T}{m}\nabla\mathbf{V}:\mathsf{u}^{30} - 2\nabla\mathbf{V}:\mathsf{u}^{31}$$
  $$= \frac{1}{\tau_{aa}}\left\{-\frac{15}{4}z_a v_{T\mathrm{e}}^2 n_{\mathrm{e}} T_{\mathrm{e}}\mathbf{V}_{\mathrm{ei}} - \left(\frac{3\sqrt{2}}{10} + z_a\frac{69}{40}\right)v_T^2(-2\mathbf{h}) - \left(\frac{9\sqrt{2}}{14} + z_a\frac{433}{280}\right)\mathbf{r}\right\}$$
  ($\mathbf{r}$: heat-weighted heat flow)
- $$d_t\boldsymbol{\pi} + \Omega\mathbf{b}\breve{\times}\boldsymbol{\pi} + (\nabla\cdot\mathbf{V})\boldsymbol{\pi} + 2\overline{\boldsymbol{\pi}\cdot(\nabla\mathbf{V})} + \frac{4}{5}\overline{\nabla\mathbf{h}} + 2p\overline{\nabla\mathbf{V}} + \nabla\cdot\mathsf{u}^{30}$$
  $$= \frac{1}{\tau_{aa}}\left\{-\left(\frac{3\sqrt{2}}{5} + z_a\frac{6}{5}\right)\boldsymbol{\pi} - \left(\frac{9\sqrt{2}}{70} + z_a\frac{18}{35}\right)\frac{-2\boldsymbol{\theta}}{v_T^2}\right\}$$
  ($\boldsymbol{\pi}$: viscosity)
- $$d_t\boldsymbol{\theta} + \frac{7}{2}\frac{d_tT}{m}\boldsymbol{\pi} + \Omega\mathbf{b}\breve{\times}\boldsymbol{\theta} + \frac{T}{m}(\nabla\cdot\mathbf{V}\boldsymbol{\pi} + 2\overline{\boldsymbol{\pi}\cdot\nabla\mathbf{V}} + 2\overline{\nabla\mathbf{V}\cdot\boldsymbol{\pi}}) + \frac{9}{7}\nabla\cdot\mathbf{V}\boldsymbol{\theta} + \frac{18}{7}\overline{\boldsymbol{\theta}\cdot\nabla\mathbf{V}} + \frac{4}{7}\overline{\nabla\mathbf{V}\cdot\boldsymbol{\theta}}$$
  $$- \frac{14}{5}\overline{\mathbf{a}\mathbf{h}} + \frac{28}{5}\frac{1}{m}\overline{\nabla T\mathbf{h}} + \frac{2}{5}\overline{\nabla\mathbf{r}} + \frac{14}{5}\frac{T}{m}\overline{\nabla\mathbf{h}} + \frac{14}{15}\overline{\nabla\mathbf{V}}u^{02}$$
  $$- \mathbf{a}\cdot\mathsf{u}^{30} + \frac{T}{m}\nabla\cdot\mathsf{u}^{30} + \frac{9}{2}\frac{\nabla T}{m}\cdot\mathsf{u}^{30} - \frac{1}{2}\nabla\cdot\mathsf{u}^{31} + \nabla\mathbf{V}:\mathsf{u}^{40}$$
  $$= \frac{1}{\tau_{aa}}\left\{-\left(\frac{9}{10\sqrt{2}} + z_a\frac{9}{5}\right)v_T^2\boldsymbol{\pi} - \left(\frac{41}{28\sqrt{2}} + z_a\frac{51}{35}\right)(-2\boldsymbol{\theta}_a)\right\}$$
  ($\boldsymbol{\theta}$: heat-weighted viscosity)

⋆ 1 year plan: Implement linearized 13 moment $(\mathbf{h}, \boldsymbol{\pi})$ equations $\rightarrow$ nonlinear terms

Apply to sound waves (validation) $\rightarrow$ Landau damping?

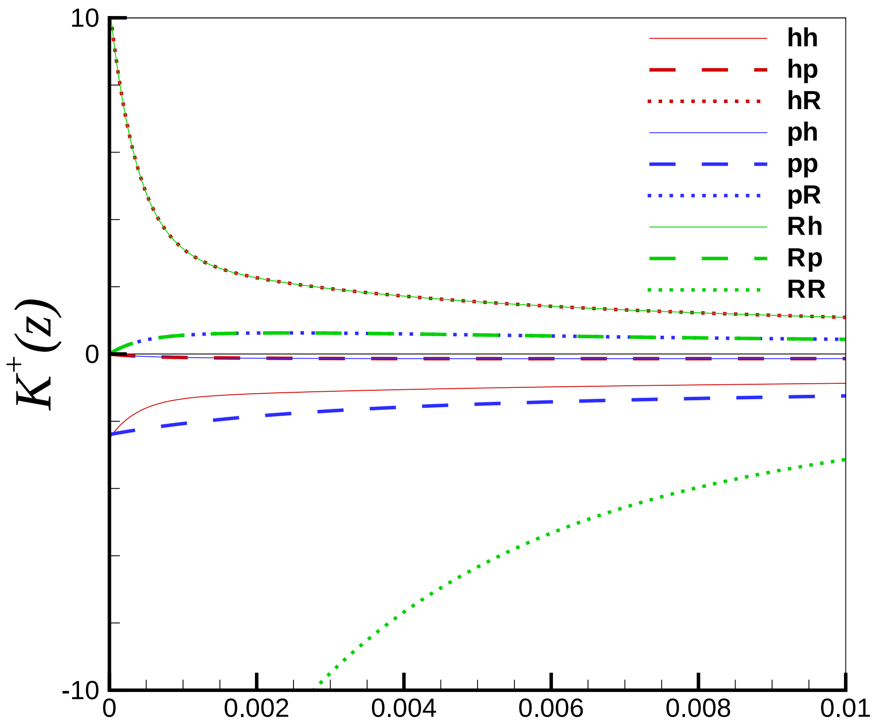
Eigen modes for $\{n_1, u_1, T_{1\mathrm{e}}, T_{1\mathrm{i}}, h_{1\mathrm{e}}, h_{1\mathrm{i}}\}$ and collision effect



3 modes mixed | Adjusted | Collision effect

Parallel closures for arbitrary collisionality in slab geometry

- 1600 moment solution



- Collisionless limit

$$h_\parallel(z) = \frac{9}{5\pi^{3/2}} n_0 v_0 \int_{-\infty}^{\infty} dz' \frac{T_1(z')}{z-z'} - \frac{2}{5} p_0 V_\parallel(z)$$

$$\pi_\parallel(z) = -\frac{2}{5} n_0 T_1(z) + \frac{4}{5\sqrt{\pi}} \frac{p_0}{v_0} \int_{-\infty}^{\infty} dz' \frac{V_\parallel(z')}{z-z'}$$

$\Downarrow$

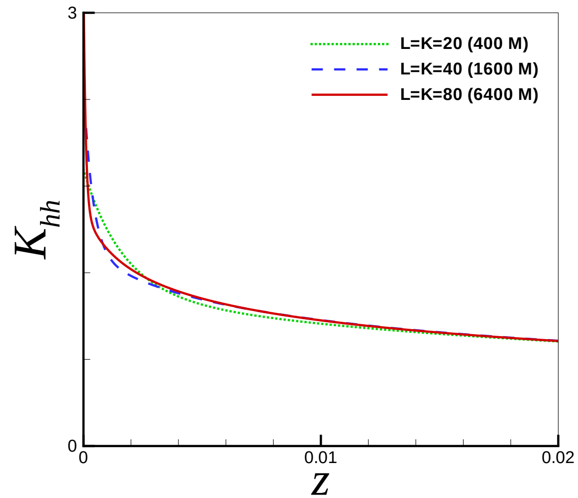
$\Rightarrow \star$ Find simple fitted formulas for kernel functions

$$h_{\mathrm{e}\parallel}(y) = \int [K_{hh}(y-z)\partial_\parallel T + K_{hp}(y-z)\partial_\parallel V_\parallel + K_{hR}(y-z)(V_{\mathrm{e}\parallel} - V_{\mathrm{i}\parallel})]$$

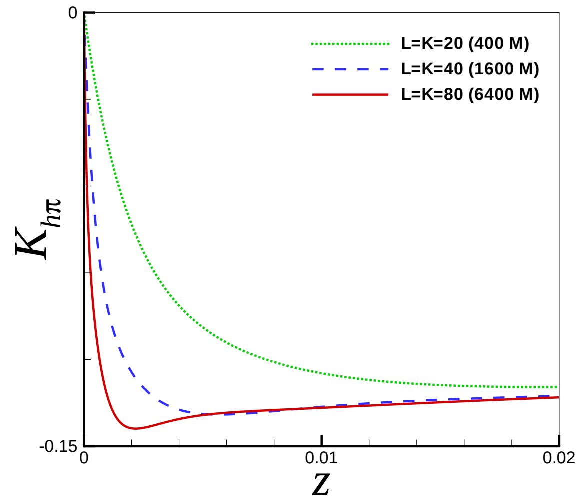
$$\pi_{\mathrm{e}\parallel}(y) = \int [K_{ph}(y-z)\partial_\parallel T + K_{pp}(y-z)\partial_\parallel V_\parallel + K_{pR}(y-z)(V_{\mathrm{e}\parallel} - V_{\mathrm{i}\parallel})]$$

$$R_{\mathrm{e}\parallel}(y) = \int [K_{Rh}(y-z)\partial_\parallel T + K_{Rp}(y-z)\partial_\parallel V_\parallel + K_{RR}(y-z)(V_{\mathrm{e}\parallel} - V_{\mathrm{i}\parallel})] \times \frac{m_{\mathrm{e}} v_{T\mathrm{e}}}{\tau_{\mathrm{ei}}} + \frac{m_{\mathrm{e}} n_{\mathrm{e}}}{\tau_{\mathrm{ei}}} V_{\mathrm{ei}\parallel}$$

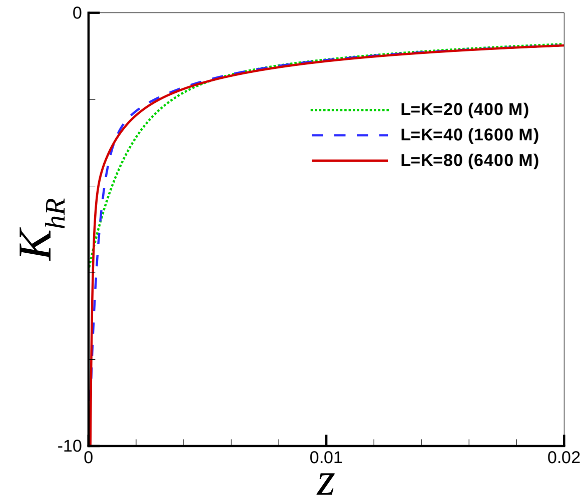
Convergence [$z = \ell/\lambda_{\rm mfp} \to 0$: collisionless, area: collisional]
h: heat flow-$\partial_\| T$, π: viscosity-$\partial_\| V_\|$, R: friction-$V_{{\rm ei}\|}$



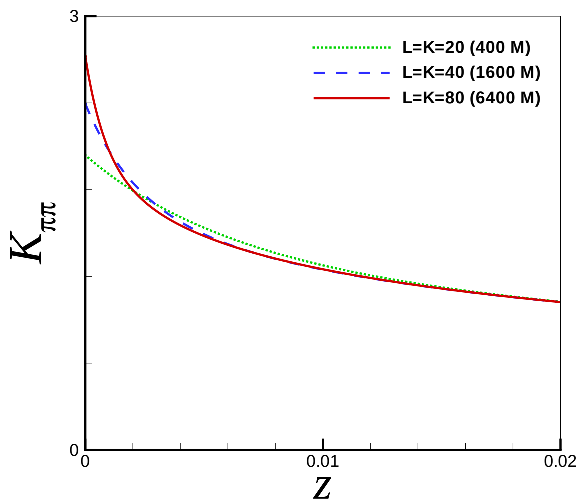
$\propto \ln z$, area



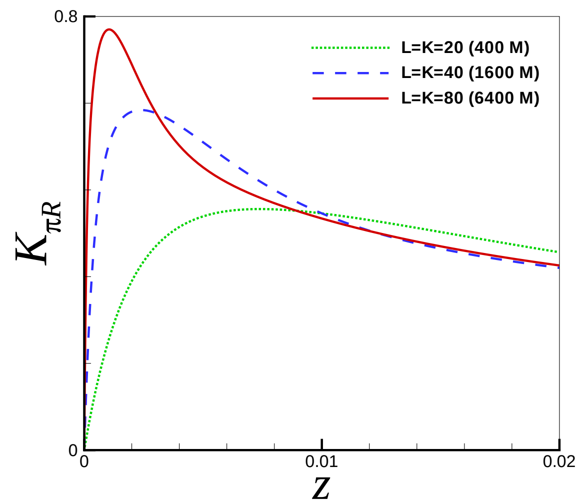
$\propto \theta(z)$



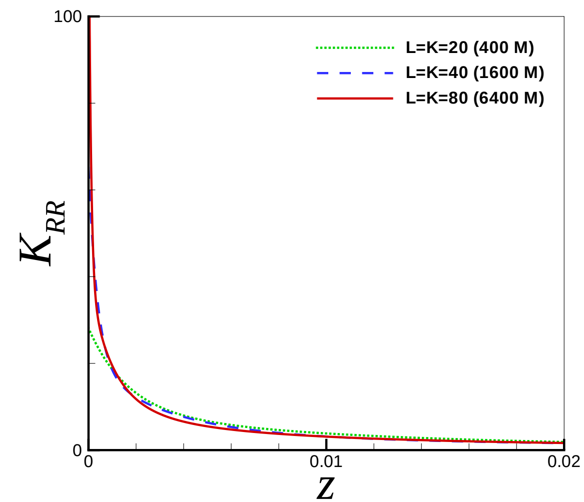
$\propto$ (?) $\ln z$, area



$\propto \ln z$, area



$\propto$ (?)$\theta(z)$



$\propto$ (?) $\ln z$, area

Future work

- Parallel closures for arbitrary magnetic geometry (no slab approximation)
  - Free streaming term $\mathbf{v}_{\|} \cdot \nabla f$ $(\mathbf{b} \cdot \nabla v_{\|} \neq 0)$
  - Neoclassical theory being computed using a finite difference method
  - Fourier series expansion
    - removes the singularities coming from the periodic differential operator
- Time-dependent parallel moment closures
- Transport with neutrals
  - Collisionality
- Moment approach to multiple ion species
  - Impurity transport
  - Exact Coulomb collision operators
  - Collisionality
  - Magnetic field geometry