Symmetric formulation of implicit Hall MHD

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This short note develops a useful way to advance the Hall MHD (HMHD) equations time-implicit by solving symmetric algebraic problems. The basis of this formulation is operator splitting, combined with a new variational approach which only requires $\mathcal{C}^0$ elements for evaluation and solution.

We adopt the implicit leap-frog concepts of Sovinec and attempt to solve the implicit induction equation (Faraday and Ohm combined)

$$\frac{\mathbf{A}^{n+1/2} - \mathbf{A}^{n-1/2}}{\Delta t} = \mathbf{u}^n \times \mathbf{B}^* - \frac{\mathbf{B}^n \bullet \nabla \mathbf{B}^n - \nabla \left[P_e^n + (B^n)^2 / 2\mu_0 \right]}{en^n} \tag{1}$$

An equivalent equation is the curl of Eq. (1)

$$\frac{\mathbf{B}^{n+1/2} - \mathbf{B}^{n-1/2}}{\Delta t} = \nabla \times \left(\mathbf{u}^n \times \mathbf{B}^* - \frac{\mathbf{J}^n \times \mathbf{B}^n - \nabla P_e^n}{en^n} \right) \tag{2}$$

The superscript "*" in these equations indicate a term in which the time centering is not critical, so that we may use the most convenient recent time level and update by predictor-corrector if necessary.

Following Sovinec, we will advance these by first linearizing about a previous estimate for $\mathbf{B}^{n+1/2}$ which can be taken to be $\mathbf{B}^{n-1/2}$. The linearized equations can be written as

$$\delta \mathbf{A} = \delta \mathbf{A}_x - h \delta \mathbf{B} \bullet \nabla \mathbf{B}^{n-1/2} + h \nabla \left(\mathbf{B}^{n-1/2} \bullet \delta \mathbf{B} \right) - \mathbf{H} \bullet \nabla \delta \mathbf{B} \tag{3}$$

$$\delta \mathbf{A}_x \equiv \Delta t \left\{ \mathbf{u}^n \times \mathbf{B}^* - \frac{\mathbf{B}^{n-1/2} \bullet \nabla \mathbf{B}^{n-1/2} - \nabla \left[\mu_0 P_e^n + \left(B^{n-1/2} \right)^2 / 2 \right]}{e \mu_0 n^n} \right\} \tag{4}$$

$$\delta \mathbf{B} = \delta \mathbf{B}_x + \frac{\mathbf{J}^{n-1/2}}{2en^n} \bullet \nabla \delta \mathbf{B} + \left(\mathbf{J}^{n-1/2} \bullet \nabla \frac{1}{2en^n} \right) \delta \mathbf{B} - \nabla \bullet \mathbf{H} \nabla \times \nabla \times \delta \mathbf{A} \tag{5}$$

$$\delta \mathbf{B}_x \equiv \nabla \times \delta \mathbf{A}_x \tag{6}$$

In Eqs. (3-6), we have defined the numerical Hall parameter $h \equiv \Delta t / 2e\mu_0 n^n$ and associated vector field $\mathbf{H} = h\mathbf{B}$.

We will now apply operator splitting to Eqs. (3-6) by separating the "less implicit" terms from the "more implicit" terms. Actually, we keep explicit only those terms which are associated with the Whistler wave

$$\delta\mathbf{A}^{(1)} = \delta\mathbf{A}_x - h\delta\mathbf{B}_x \bullet \nabla\mathbf{B}^{n-1/2} + h\nabla\left(\mathbf{B}^{n-1/2} \bullet \delta\mathbf{B}_x\right) \tag{7}$$

$$\delta\mathbf{A} = \delta\mathbf{A}^{(1)} - \mathbf{H} \bullet \nabla\delta\mathbf{B} \tag{8}$$

$$\delta\mathbf{B}^{(1)} = \delta\mathbf{B}_x + \frac{\mathbf{J}^{n-1/2}}{2en^n} \bullet \nabla\delta\mathbf{B}_x + \left(\mathbf{J}^{n-1/2} \bullet \nabla\frac{1}{2en^n}\right)\delta\mathbf{B}_x \tag{9}$$

$$\delta\mathbf{B} = \delta\mathbf{B}^{(1)} - \nabla \bullet \mathbf{H}\nabla \times \nabla \times \delta\mathbf{A} \tag{10}$$

Written in this way, we obtain a symmetric problem by combining Eq. (8) with Eq. (10)

$$\begin{aligned} \delta\mathbf{B} - \nabla \bullet \mathbf{H}\nabla \times \nabla \times \mathbf{H} \bullet \nabla\delta\mathbf{B} &= \delta\mathbf{B}^{(1)} - \nabla \bullet \mathbf{H}\nabla \times \nabla \times \delta\mathbf{A}^{(1)} \\ &= \delta\mathbf{B}^{(1)} - \nabla \bullet \mathbf{H}\nabla \times \delta\mathbf{B}^{(1)} \end{aligned} \tag{11}$$

The last equivalence is approximate because of the particular way in which terms have been separated, but the error is negligible.

Because the vector potential is now redundant, we can complete the time step by advancing Eq. (9) and then solving for the increment in magnetic field from Eq. (11). This is not as simple as it sounds, however, because Eq. () is a 4th order spatial operator, and a direct finite-element discretization would require introducing $\mathcal{C}^1$ elements with all the attendant complications. To avoid this, we introduce the contrived (one could more optimistically say "invented") variational

$$\begin{aligned} \mathcal{V} \equiv \frac{1}{2}\int d\mathbf{r}\left[|\delta\mathbf{B}|^2 + |\nabla \times \mathbf{V}|^2 + \Lambda\left(\mathbf{V} - \mathbf{H} \bullet \nabla\delta\mathbf{B}\right)^2\right] \\ - \int d\mathbf{r}\left(\delta\mathbf{B} + \nabla \times \mathbf{V}\right) \bullet \delta\mathbf{B}^{(1)} \end{aligned} \tag{12}$$

Assuming homogeneous boundary conditions, the Euler equations are

$$\begin{aligned} \delta\mathbf{B} + \nabla \bullet \mathbf{H}\Lambda\left(\mathbf{V} - \mathbf{H} \bullet \nabla\delta\mathbf{B}\right) &= \delta\mathbf{B}^{(1)} \\ \nabla \times \nabla \times \mathbf{V} + \Lambda\left(\mathbf{V} - \mathbf{H} \bullet \nabla\delta\mathbf{B}\right) &= \nabla \times \delta\mathbf{B}^{(1)} \end{aligned} \tag{13}$$

We are interested in the limit $\Lambda \to \infty$ in which case the above yield $\mathbf{V} \approx \mathbf{H} \bullet \nabla\delta\mathbf{B}$ and

$$\delta\mathbf{B} - \nabla \bullet \mathbf{H} \nabla \times \nabla \times \mathbf{H} \bullet \nabla \delta\mathbf{B} = \delta\mathbf{B}^{(1)} - \nabla \bullet \mathbf{H} \nabla \times \delta\mathbf{B}^{(1)} + \mathcal{O}\left(\Lambda^{-1}\right) \quad (14)$$

A more thourough analysis shows that this holds sufficiently well provided that Λ is much larger than the largest k^2 in the system.

A final remark is to notice that additional terms like resistivity and divergence cleaning could also be incorporated into the variational, if this would be desirable.