Recent progress on moment-based closures:

1. Integral parallel closures for arbitrary collisionality
2. Along an inhomogeneous magnetic field

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Five moment equations and closures

- Five moment $(n_a, \mathbf{V}_a, T_a)$ equations for species a

$$d_a n_a + n_a \nabla \cdot \mathbf{V}_a = 0$$

$$\tfrac{3}{2} n_a d_a T_a + n_a T_a \nabla \cdot \mathbf{V}_a + \nabla \cdot \mathbf{h}_a + \nabla \mathbf{V}_a : \boldsymbol{\pi}_a = Q_a$$

$$m_a n_a d_a \mathbf{V}_a - n_a q_a (\mathbf{E} + \mathbf{V}_a \times \mathbf{B}) + \nabla p_a + \nabla \cdot \boldsymbol{\pi}_a = \mathbf{R}_a$$

  where $d_a = \partial_t + \mathbf{V}_a \cdot \nabla$

- Solve the kinetic equation, for $F[\mathsf{N} = \{\mathbf{h}, \boldsymbol{\pi}, \cdots\}]$ in terms of $f^{\mathrm{M}}[\mathrm{M} = \{n, T, \mathbf{V}\}]$

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \partial_{\mathbf{v}} f = C(f)$$

  or equivalently the general moment equations, for N in terms of $\mathsf{G}[\mathrm{M}]$

$$\mathsf{L}\left(\frac{\partial}{\partial t}, \nabla\right)\mathsf{N} + \Omega \mathbf{b} \breve{\times} \mathsf{N} = \mathsf{C}\mathsf{N} + \mathsf{G}$$

- Express $\mathbf{h}_a(\mathsf{N}_a^{11})$, $\boldsymbol{\pi}_a(\mathsf{N}_a^{20})$, Q_a, $\mathbf{R}_a$ in terms of $n_a, \mathbf{V}_a, T_a$
  For high collisionality (Braginskii)

$$\mathbf{R}_{\mathrm{e}} = -(\alpha)(\mathbf{V}_{\mathrm{ei}}) - (\beta)(\nabla T_{\mathrm{e}}), \ \ \mathbf{h}_{\mathrm{e}} = (\beta)(\mathbf{V}_{\mathrm{ei}}) - (\kappa)(\nabla T_{\mathrm{e}})$$

Moment expansion of a distribution function
Tensorial Hermite polynomial

$$\mathsf{p}_a^{lk} = \mathsf{P}^l(\mathbf{c}_a) L_k^{(l+1/2)}(c_a^2) \quad \Rightarrow \quad \mathbf{v}^{l+2k}$$

$$\mathbf{c}_a = \frac{\mathbf{v} - \mathbf{V}_a}{v_{Ta}}, \quad v_{Ta} = \sqrt{\frac{2T_a}{m_a}}$$

- Harmonic tensor (symmetric, traceless/irreducible): spherical harmonics

$$\begin{aligned}
\mathsf{P}^0(\mathbf{c}) &= 1 \\
\mathsf{P}^1(\mathbf{c}) &= \mathbf{c} \\
\mathsf{P}^2(\mathbf{c}) &= \mathbf{cc} - \frac{c^2}{3}\mathsf{I} \quad (\mathsf{I} = \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3 \equiv \mathsf{ii}) \\
\mathsf{P}^3(\mathbf{c}) &= \mathbf{ccc} - \frac{c^2}{5}(\mathbf{c}\mathsf{ii} + \mathsf{i}\mathbf{c}\mathsf{i} + \mathsf{ii}\mathbf{c})
\end{aligned}$$

- (l, k) =(0,0) density (n), (0,1) temperature (T), (1,0) flow velocity $(\mathbf{V})$ (1,1) heat flow $(\mathbf{h})$, and (2,0) viscosity tensor $(\boldsymbol{\pi})$

$$\mathsf{m}^{11} = -\sqrt{\frac{4}{5}}\frac{1}{nv_T T}\mathbf{h} \qquad \mathsf{m}^{20} = \frac{\sqrt{2}}{2p}\boldsymbol{\pi}$$

General moment equations

- Landau (Fokker-Planck) kinetic equation

$$\frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \nabla f_a + \frac{q_a}{m_a}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \partial_{\mathbf{v}} f_a = \sum_b C(f_a, f_b)$$

- Moment expansion: M^{lk} is a symmetric traceless fluid moment

$$f_a(t, \mathbf{x}, \mathbf{v}) = f_a^0 \sum_{lk} \mathsf{M}_a^{lk}(t, \mathbf{x}) \cdot \hat{\mathsf{P}}^{lk}(\mathbf{s}_a)$$

$$\mathsf{N}_a^{lk} \equiv n_a \mathsf{M}_a^{lk}(t, \mathbf{x}) = \int d\mathbf{v} \hat{\mathsf{P}}_a^{lk} f_a(t, \mathbf{x}, \mathbf{v})$$

where $\mathbf{s}_a = \mathbf{v}_a / v_{Ta}$

- $\hat{\mathsf{P}}^{lk}$'s are orthonormal, irreducible, tensorial polynomials and form a complete set

$$\int d\mathbf{v} \hat{\mathsf{P}}^{jp} \hat{\mathsf{P}}^{lk} \cdot \mathsf{M}^{lk} f^{(0)} = \delta_{jl} \delta_{pk} \mathsf{M}^{jp}$$

- General moment equations [Ji Held 2006 2008 PoP]

$$\mathsf{L}_a \mathsf{N}_a + \Omega_a \mathbf{b} \breve{\times} \mathsf{N}_a = \sum_b (\mathsf{A}_{ab} \mathsf{N}_a + \mathsf{B}_{ab} \mathsf{N}_b)$$

Parallel moment equations for closures (in matrix form)

$$[\psi]\partial_z\bar{N}_\| + \{\partial_z \ln B[\Psi_B] + [\Phi](\partial_z \ln T) + \hat{E}_\|[\Theta]\}\bar{N}_\| = [c]\bar{N}_\| + g_\|$$

- Notation simplified $n^{lk} = \bar{N}_\|^{lk}$, $dz = d\ell/\lambda_{\rm mfp}$, $\lambda_{\rm mfp} = v_T\tau$

$$\begin{bmatrix} 0 & \psi^0 & & & \\ \tilde{\psi}^0 & 0 & \psi^1 & & \\ & \tilde{\psi}^1 & 0 & \psi^2 & \\ & & \tilde{\psi}^2 & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \partial_z\vec{n}^0 \\ \partial_z\vec{n}^1 \\ \partial_z\vec{n}^2 \\ \partial_z\vec{n}^3 \\ \vdots \end{bmatrix} = \begin{bmatrix} c^0\vec{n}^0 \\ c^1\vec{n}^1 \\ c^2\vec{n}^2 \\ c^3\vec{n}^3 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 \\ \vec{g}^1 \\ \vec{g}^2 \\ 0 \\ \vdots \end{bmatrix}$$

where $\hat{V}_{{\rm ei}\|} = \frac{\mathbf{b}\cdot(\mathbf{V}_{\rm e}-\mathbf{V}_{\rm i})}{v_T}$, $W_\| = \mathbf{bb} : \mathsf{W}$, $(\mathsf{W})_{\alpha\beta} = \partial_\alpha V_\beta + \partial_\beta V_\alpha - \frac{2}{3}\delta_{\alpha\beta}\nabla\cdot\mathbf{V}$,

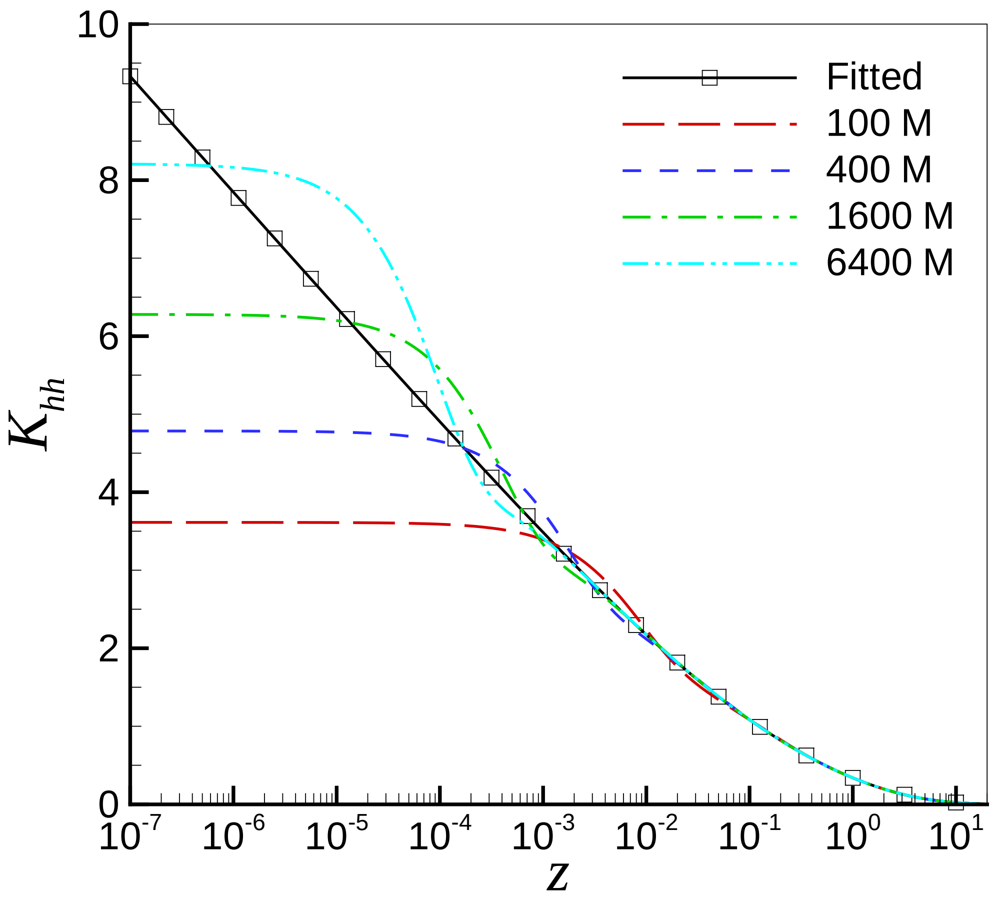
$$\vec{n}^0 = \begin{pmatrix} n^{02} \\ n^{03} \\ n^{04} \\ \vdots \end{pmatrix}, \ \vec{n}^1 = \begin{pmatrix} n^{11} \\ n^{12} \\ n^{13} \\ \vdots \end{pmatrix}, \ \vec{n}^{l\geq 2} = \begin{pmatrix} n^{l0} \\ n^{l1} \\ n^{l2} \\ \vdots \end{pmatrix}$$

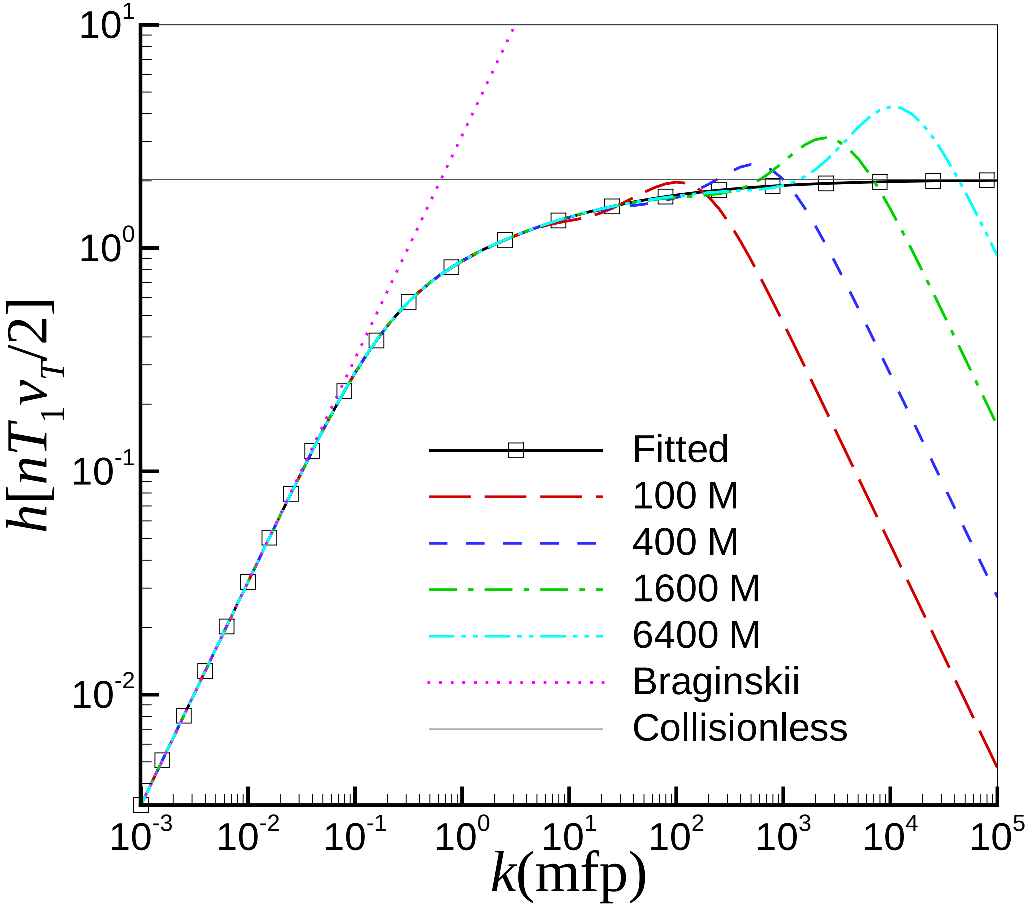
$$\vec{g}^1 = \begin{pmatrix} \sqrt{2}Za_{\rm ei}^{110}n\hat{V}_{{\rm ei}\|} + \frac{\sqrt{5}}{2}\frac{n}{T}\frac{dT}{dz} \\ \sqrt{2}Za_{\rm ei}^{120}n\hat{V}_{{\rm ei}\|} \\ \sqrt{2}Za_{\rm ei}^{130}n\hat{V}_{{\rm ei}\|} \\ \vdots \end{pmatrix}, \ \vec{g}^2 = \begin{pmatrix} -\frac{\sqrt{3}}{2}n\tau_{\rm ee}W_\| \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

Solving the linear ODE system

- Linear system of ordinary differential equations
$$[\Psi]\frac{d[n]}{dz} = [c][n] + [g] \;\Rightarrow\; \frac{d[n]}{dz} = [\Psi^{-1}c][n] + [\Psi^{-1}g]$$
- Use eigenvectors $[W] = [W_1, W_2, \cdots, W_N]$
$$[\Psi^{-1}c][W_A] = k_A[W_A]$$
to transform $[\check{n}] = [W]^{-1}[n]$, $[\check{g}] = [W]^{-1}[\Psi^{-1}g]$ for a diagonal system
$$\frac{d\check{n}_A}{dz} = k_A\check{n}_A + \check{g}_A \;\Rightarrow\; \check{n}_A(y) = \int^y e^{k_A(y-z)}\check{g}_A(z)dz$$
and inverse-transform
$$n_A(y) = \sum_D \int^y \underbrace{\sum_{BC} W_{AB}W^{-1}_{BC}\Psi^{-1}_{CD}}_{\gamma_{AD}} e^{k_B(y-z)} g_D(z)dz$$
- Integral form of closures $n_A(y) = \sum_D \int_{-\infty}^{\infty} K_{AD}(y-z)g_D(z)dz$
where Kernel $K_{AD}(z>0) = -\sum_{\{B|k_B>0\}} \gamma_{AD}(k_B)e^{-k_B z}$ (too many terms)

Convergence test
Heat flow (n^{11}) for $T = T_0(1 + \epsilon_T \sin k_{\rm mfp} z)$, $k_{\rm mfp} = 2\pi\lambda_{\rm mfp}/\lambda_T$





- SSPX, 1600 M, better agreement with temperature measurements [Ji Held Sovinec 2009 PoP]
- SOL, 900 M, capturing kinetic effects and more efficient than PIC [Omotani Dudson 2013 PPCF]

Obtaining fitted kernel functions

- Fitted to the 6400 moment solution in the convergent regime

$$\text{error} \lesssim 1\ \% \quad \text{for} \quad k = 2\pi \frac{\lambda_{\rm mfp}}{\lambda_G} \lesssim 100 \text{ (nearly collisionless)}$$

- In the $z = \dfrac{\ell}{\lambda_{\rm mfp}} \to 0$ limit, let kernels behave asymptotically collisionless

Collisional limit [Braginskii]	Collisionless limit [Ji Held Jhang 2013 PoP]
$h_\parallel = -3.20 \dfrac{nT\tau_{\rm ee}}{m} \partial_\parallel T + 0.703 nT V_{\rm ei\parallel}$	$h_\parallel(z) = \dfrac{9}{5\pi^{3/2}} n_0 v_0 \displaystyle\int_{-\infty}^{\infty} dz' \frac{T_1(z')}{z - z'} - \frac{2}{5} p_0 V_\parallel(z)$
$R_\parallel = -0.703 n \partial_\parallel T - 0.504 \dfrac{mn}{\tau_{\rm ei}} V_{\rm ei\parallel}$	$R_\parallel = 0$
$\pi_\parallel = -0.978 nT\tau_{\rm ee} \dfrac{3}{4} W_\parallel$	$\pi_\parallel(z) = -\dfrac{2}{5} n_0 T_1(z) + \dfrac{4}{5\sqrt{\pi}} \dfrac{p_0}{v_0} \displaystyle\int_{-\infty}^{\infty} dz' \frac{V_\parallel(z')}{z - z'}$

- Tested with the sinusoidal drives $(k_g \ell = 2\pi \dfrac{\ell}{\lambda_g} = 2\pi \dfrac{\lambda_{\rm mfp}}{\lambda_g} z = kz)$

$$\begin{aligned} g(\ell) &= g_0 + g_1 \sin k_g \ell \text{ for odd kernels} \\ g(\ell) &= g_0 + g_1 \cos k_g \ell \text{ for even kernels} \end{aligned}$$

Integral (nonlocal) parallel closures

- Closure n_A responding to g_B:

$$n_{AB}(z) = \int dz' K_{AB}(z-z') g_B(z'), \quad A, B = h, R, \pi$$

$$h_\parallel(z) = -\frac{1}{2} T v_T \int dz' K_{hh} \frac{n}{T}\frac{dT}{dz'} + T v_T \int dz' K_{hR} Z n \frac{V_{\mathrm{ei}\parallel}}{v_T} + T v_T \int dz' K_{h\pi} (\frac{3}{4} n \tau_{\mathrm{ee}} W_\parallel)$$

$$R_\parallel(z) = -\frac{mn}{\tau_{\mathrm{ei}}} V_{\mathrm{ei}\parallel} + \frac{m v_T}{\tau_{\mathrm{ei}}} \int dz' [-K_{Rh} \frac{n}{2T}\frac{dT}{dz'} + K_{RR} Z n \frac{V_{\mathrm{ei}\parallel}}{v_T} + K_{R\pi} (\frac{3}{4} n \tau_{\mathrm{ee}} W_\parallel)]$$

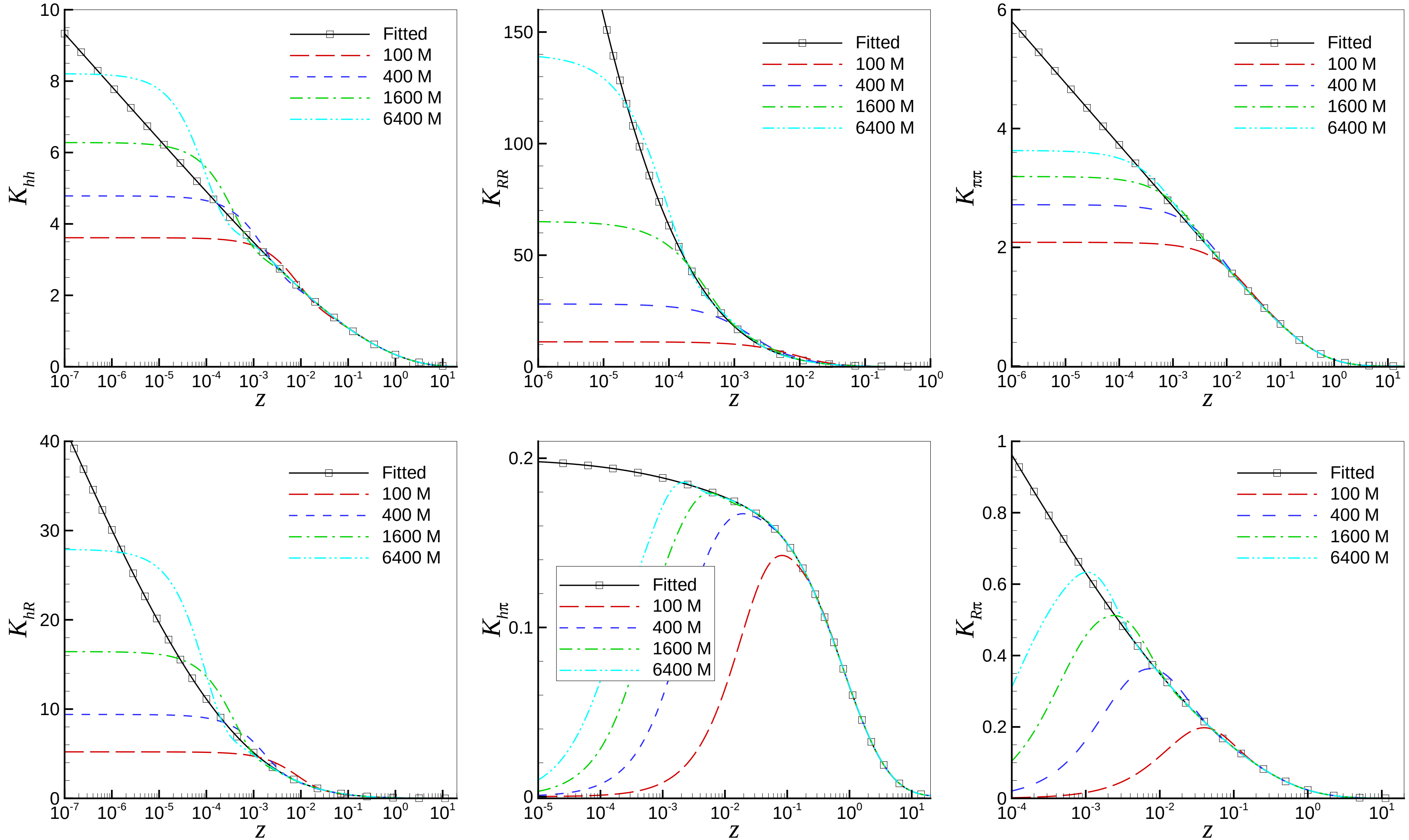
$$\pi_\parallel(z) = T \int dz' K_{\pi h} \frac{n}{T}\frac{dT}{dz'} - 2T \int dz' K_{\pi R} Z n \frac{V_{\mathrm{ei}\parallel}}{v_T} - T \int dz' K_{\pi\pi} (\frac{3}{4} n \tau_{\mathrm{ee}} W_\parallel)$$

- Fitted kernel functions

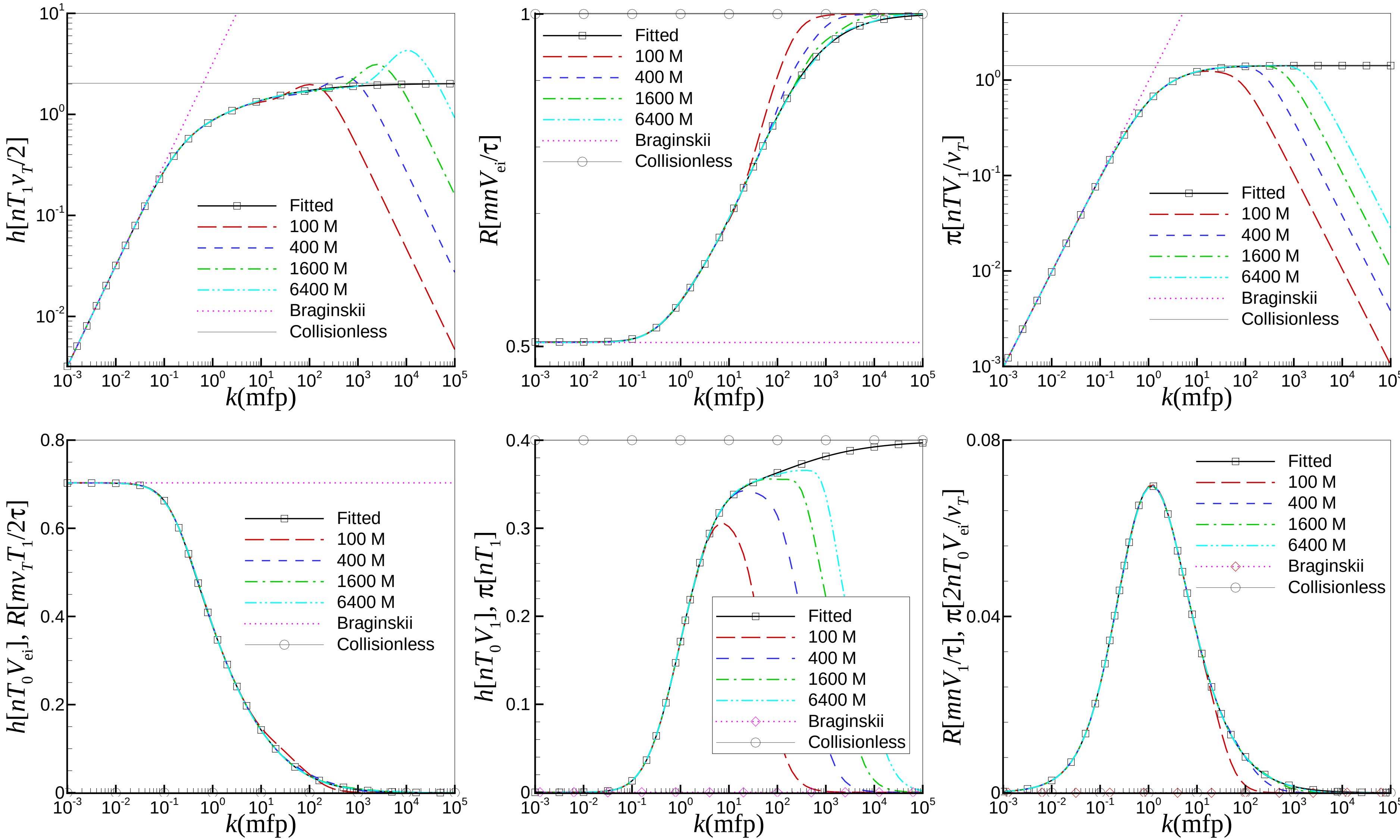
$$K_{AB}(z) = -[d + a \exp(-b z^c)] \ln[1 - \alpha \exp(-\beta z^\gamma)]$$

	a	b	c	d	α	β	γ
K_{hh}	-5.32	0.170	0.646	6.87	1	2.02	0.417
K_{hR}	6.37	5.12	0.160	0.100	1	1	0.583
$K_{h\pi}$	-0.229	2.26	0.594	0.363	0.775	1.49	0.478
K_{RR}	245	8.06	0.147	0.432	1	3.40	0.347
$K_{R\pi}$	-0.226	3.21	0.678	0.696	1	3.40	0.347
$K_{\pi\pi}$	0.724	0.932	0.654	0.195	1	1.60	0.491

Fitted kernel functions



Closures responding to sinusoidal drives



Heat flux along an inhomogeneous magnetic field

$$[\bar{\Psi}]v_T\partial_\parallel[n] + [\bar{\Psi}_B]v_T\partial_\parallel \ln B[n] + [\bar{\Phi}]v_T\partial_\parallel \ln T[n] = \frac{1}{\tau}[c][n] + [\hat{g}]nv_T\partial_\parallel \ln T$$

where $\partial_\parallel = \partial/\partial\ell$ (ℓ arclength along a field line), $\hat{g}^{11} = \sqrt{5}/2$ and $\hat{g}^{lk\neq 11} = 0$

- $dz = \dfrac{d\ell}{\lambda_{\rm mfp}}$, $\quad [\bar{\Psi}]\partial_z[n] + [\bar{\Psi}_B]\partial_z \ln B[n] + [\bar{\Phi}]\partial_z \ln T[n] = [c][n] + [\hat{g}]n\partial_z \ln T$
- $d\theta = 2\pi\dfrac{d\ell}{\lambda_T} = 2\pi\dfrac{\lambda_{\rm mfp}}{\lambda_T}dz = k_{\rm mfp}dz, \quad k_{\rm mfp} = 2\pi\dfrac{\lambda_{\rm mfp}}{\lambda_T}$ (inverse collisionality)

$$[\bar{\Psi}]\partial_\theta[n] + [\bar{\Psi}_B]\partial_\theta \ln B[n] + [\bar{\Phi}]\partial_\theta \ln T[n] = \frac{1}{k_{\rm mfp}}[c][n] + [\hat{g}]n\partial_\theta \ln T$$

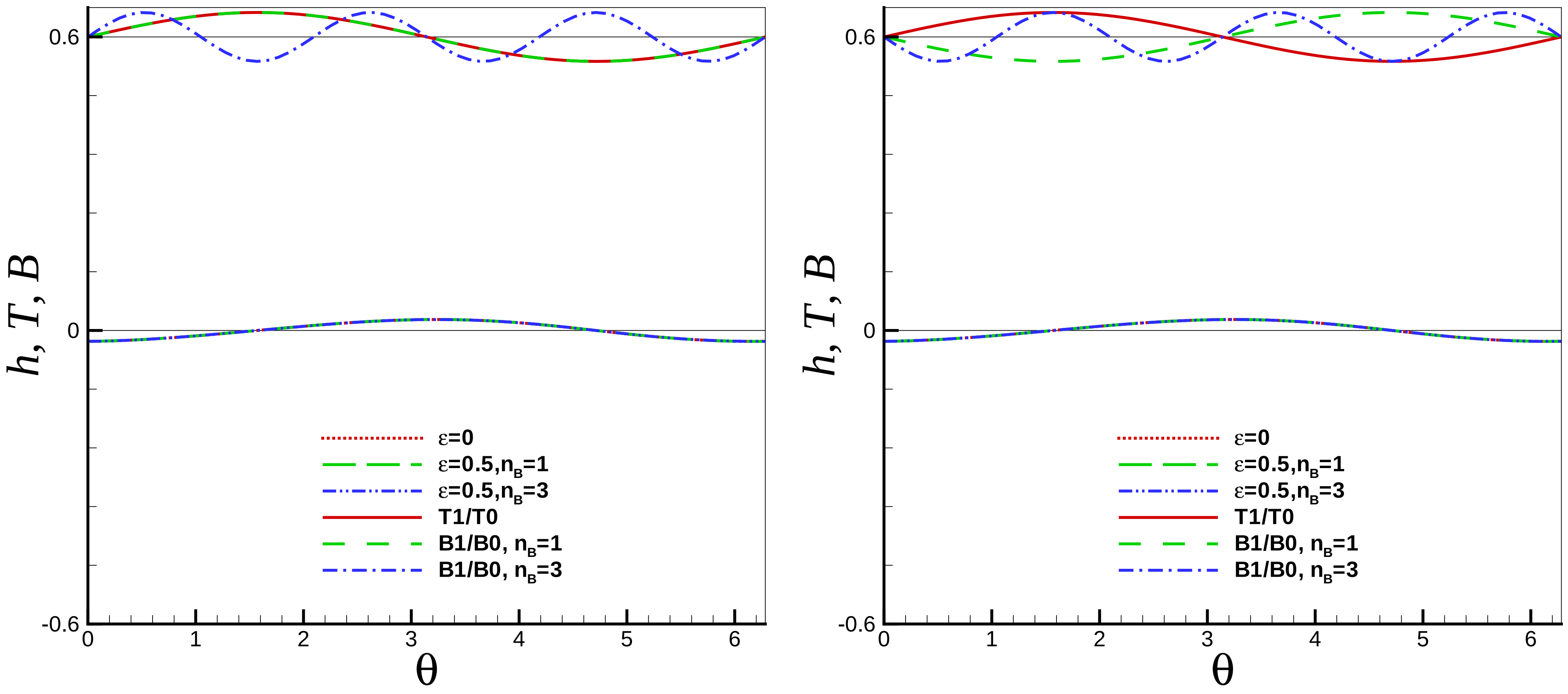
- Temperature

$$T(\ell) = T_0\left[1 + \epsilon_T \sin\left(\frac{2\pi}{\lambda_T}\ell\right)\right] = T_0[1 + \epsilon_T \sin\theta]$$

- Magnetic field $n_B = \dfrac{\lambda_T}{\lambda_B}$

$$B(\ell) = T_0\left[1 + \epsilon \sin\left(\frac{2\pi}{\lambda_B}\ell + \phi\right)\right] = T_0[1 + \epsilon\sin(n_B\theta + \phi)]$$

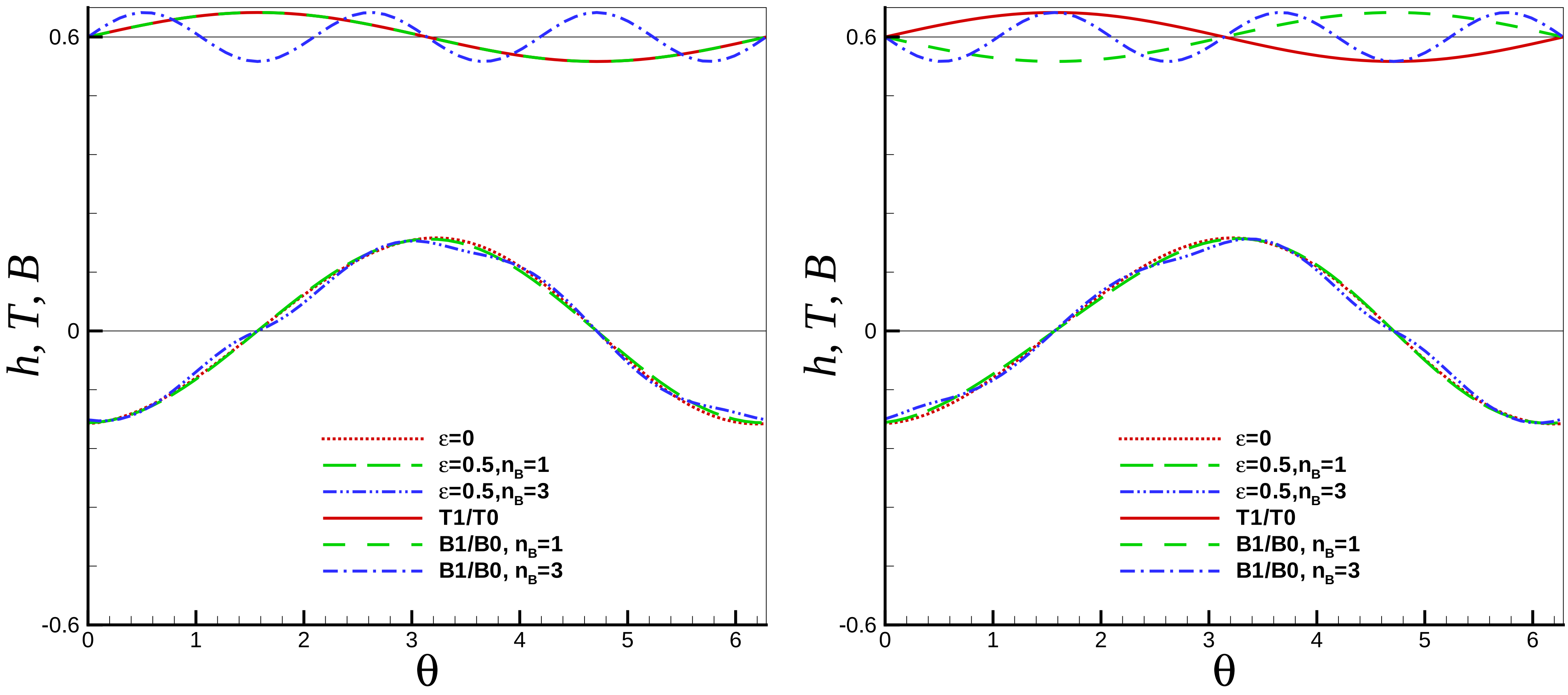
Heat flow for $T = T_0(1 + \epsilon_T \sin\theta)$, $k_{\mathrm{mfp}} = 0.01$



$$B = B_0(1 + \varepsilon \sin\theta)$$
$$B = B_0(1 + \varepsilon \sin 3\theta)$$

$$B = B_0(1 - \varepsilon \sin\theta)$$
$$B = B_0(1 - \varepsilon \sin 3\theta)$$

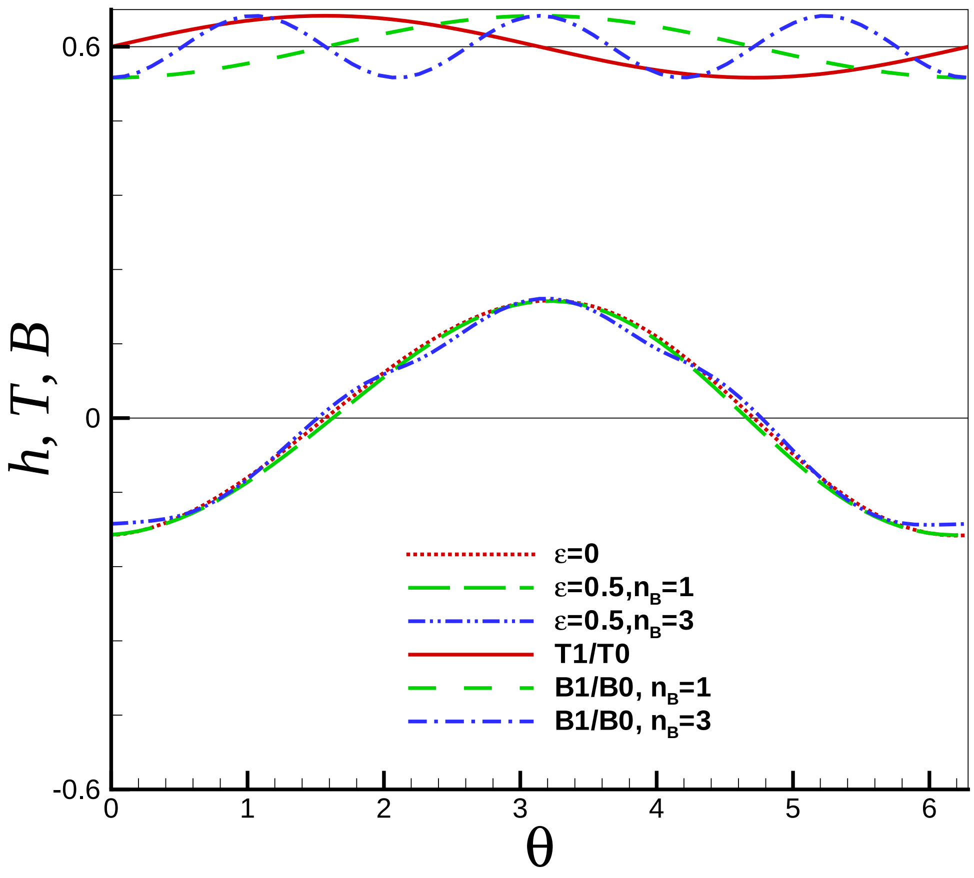
Heat flow for $T = T_0(1 + \epsilon_T \sin\theta)$, $k_{\rm mfp} = 0.1$

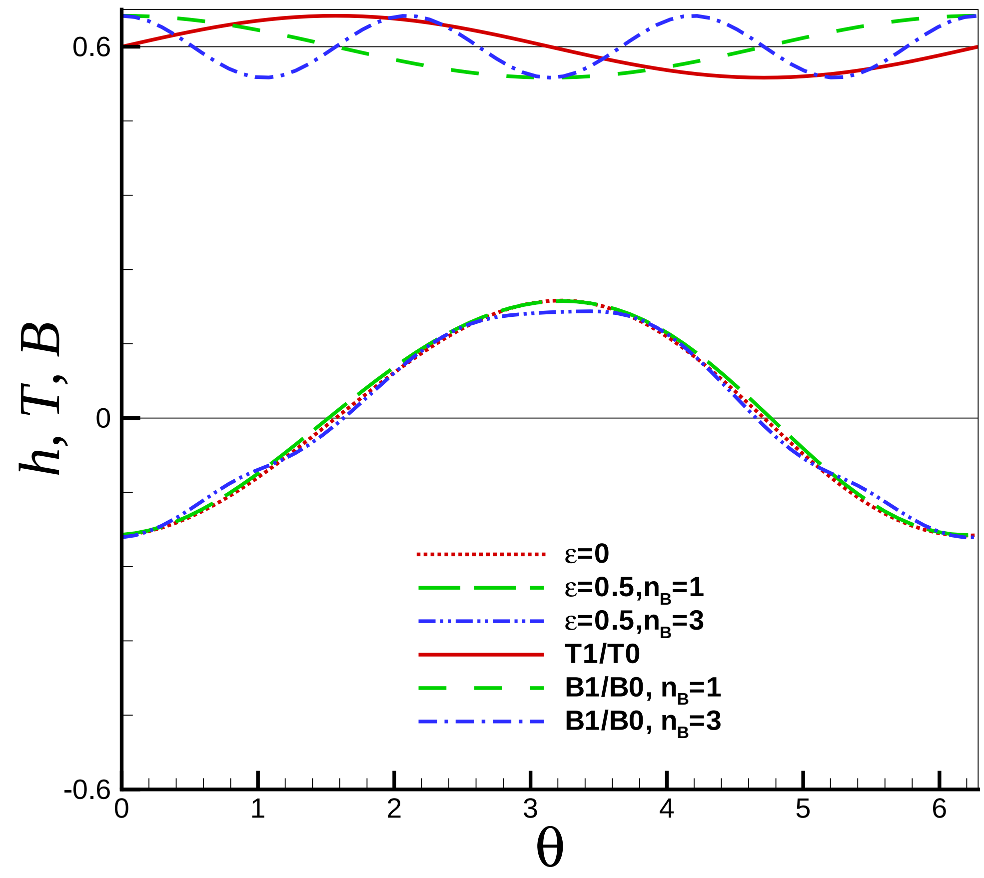


$$B = B_0(1 + \varepsilon \sin\theta)$$
$$B = B_0(1 + \varepsilon \sin 3\theta)$$

$$B = B_0(1 - \varepsilon \sin\theta)$$
$$B = B_0(1 - \varepsilon \sin 3\theta)$$

Heat flow for $T = T_0(1 + \epsilon_T \sin\theta)$, $k_{\mathrm{mfp}} = 0.1$



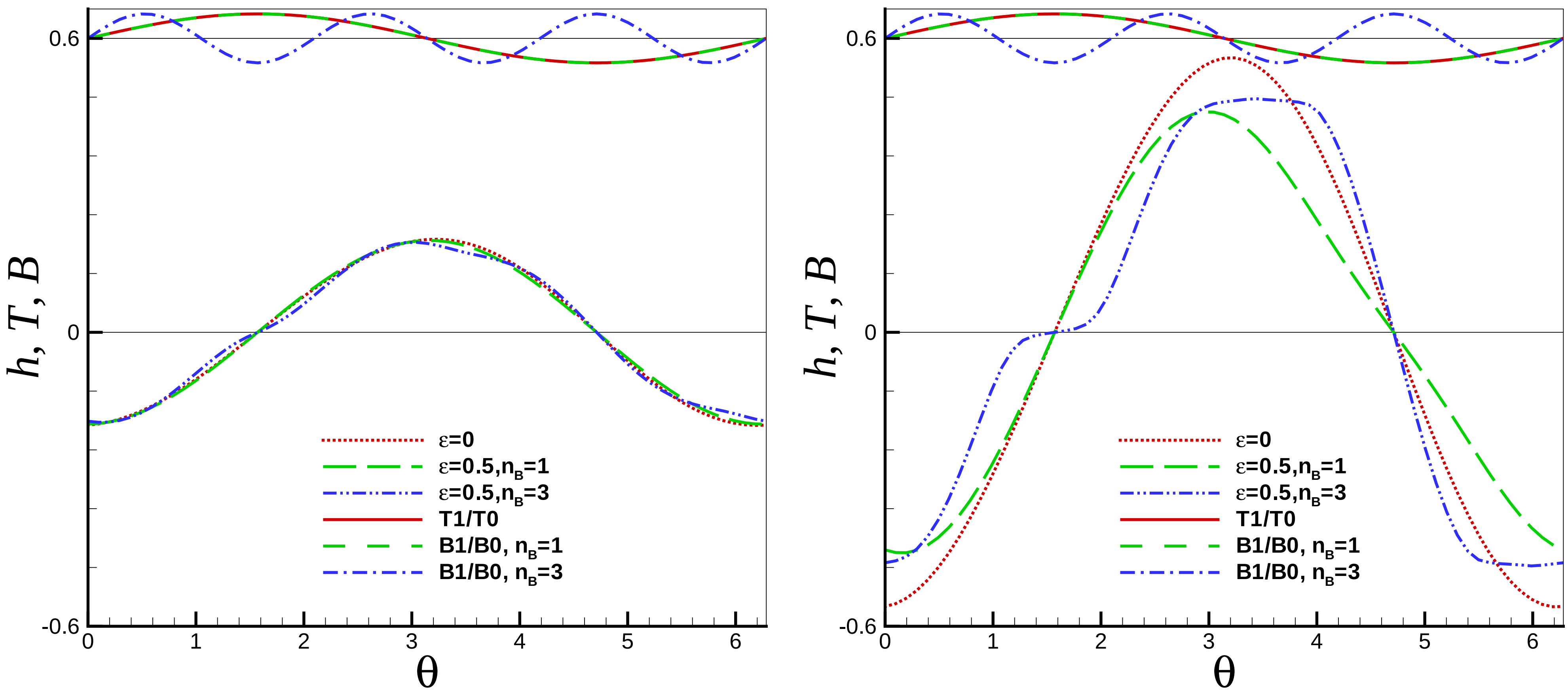


$$B = B_0[1 + \varepsilon \sin(\theta - \frac{\pi}{2})]$$
$$B = B_0[1 + \varepsilon \sin(3\theta - \frac{\pi}{2})]$$

$$B = B_0[1 + \varepsilon \sin(\theta + \frac{\pi}{2})]$$
$$B = B_0[1 + \varepsilon \sin(3\theta + \frac{\pi}{2})]$$

Heat flow for $T = T_0(1 + \epsilon_T \sin\theta)$, $k_{\text{mfp}} = 0.1$ vs. $k_{\text{mfp}} = 1$

For $B = B_0(1 + \varepsilon \sin\theta)$ and $B = B_0(1 + \varepsilon \sin 3\theta)$



$$[\bar{\Psi}]\partial_\theta[n] + [\bar{\Psi}_B]\partial_\theta \ln B[n] = \frac{1}{k_{\text{mfp}}}[c][n] + [n\partial_\theta \ln T]$$

Summary and future work

- Parallel closures for arbitrary collisionality
  - Braginskii's theory in the high collision limit
  - Collisionless limit
- Limitations of integral closures: $\{\partial_z \ln B[\Psi_B] + [\Phi](\partial_z \ln T) + \hat{E}_\parallel[\Theta]\}\bar{N}_\parallel$ ignored
  - Slab geometry approximation
  - Small temperature variation
- Generalize to (getting and analyzing results)
  - Arbitrary magnetic geometry
  - Finite temperature variation
  - Neoclassical transport theory (code written)
- Include
  - Ions
  - Ions and impurities
  - Neutrals