The semi-characteristic method for the Vlasov equation

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Thanks to

- Luis Chacón
- Guangye Chen
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- JCP 230, 7018 (‘11); 231, 5374 (‘12); 233, 1 (‘12)
(1988 – 2012) Adventures in evolving background \( \mathcal{D} \)
- Wandering the dessert
- The “w” formulation
- Left as an exercise for the student

Another noise reduction philosophy

The semi-characteristic method
- Formulation
- Initial results
198? GK $\mathcal{F}$ method used for ES turbulence
Could be generalized to EM, …?
Moment equations $+$ closure (bad design decision)
“Evolving background” $\mathcal{F}$ method
\[
\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x}\cdot n_\alpha \mathbf{u}_\alpha = 0
\]

\[
m_\alpha n_\alpha \left( \frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \frac{\partial \mathbf{u}_\alpha}{\partial x} \right) = q_\alpha n_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B})
\]

\[
- k_B T_\alpha \frac{\partial n_\alpha}{\partial x} - \frac{\partial}{\partial x}\cdot \left< \pi \right>_\alpha
\]

\[
f_\alpha = \overline{f}_\alpha + \tilde{f}_\alpha
\]

\[
\overline{f}_\alpha \equiv n_\alpha \left( \frac{1}{\sqrt{2\pi} v_{T\alpha}} \right)^3 e^{-w_\alpha^2 / 2v_{T\alpha}^2}
\]

\[
\left< \pi \right>_\alpha \equiv m_\alpha \int d\mathbf{w}_\alpha \mathbf{w}_\alpha \tilde{f}_\alpha h_\alpha
\]
First attempts < 1996

- Orbit equations
- Weight based on full particle velocity $v$
- Various constraint schemes
- Pretty much an unmitigated bust (sorry, Carl!)
- Wander, find water, fluid calculations, …
The “w” formulation

• **2006**
  – 28 October, Philadelphia, PA
  – **14-15 August**, Seattle, WA
  – 27 January Boulder, CO

• Introduce particular velocity as fundamental particle quantity
  – Eliminated circular equations
  – Symmetry
  – Energy conservation
Apply Cloud in Cell rule

\[
\frac{\dot{h}_i}{1 - \tilde{h}_i} = \int dz_{sw}(w, w_i) s(x - x_i) \left[ \nabla \cdot u - \frac{1}{v_T^2} \left( w : \frac{\partial u}{\partial x} + w \cdot \tilde{A} \right) \right]
\]

\[
\frac{\dot{h}_i}{1 - \tilde{h}_i} = \sum_g S(x_g - x_i) \left( \frac{\partial}{\partial x_g} \cdot u - \frac{w_i w_i}{v_T^2} \cdot \frac{\partial u}{\partial x_g} \right)
\]

\[
\tilde{A}_g = - \sum_i W_i \tilde{g}_i (w_i w_i - v_T^2 1) \cdot \frac{\partial}{\partial x_g} S(x_g - x_i) / n \Delta_g
\]

- Now symmetric and closure eliminated from weight advance
Symmetry Leads to Energy Integral

\[
\frac{d}{dt} \left\{ -M \bar{v}_T^2 \sum_i W_i \left[ \log(1 - \tilde{h}_i) + \tilde{h}_i \right] \right\} = -M \sum_g \Delta g n_g u_g \cdot \tilde{A}_g
\]

\[
\approx \frac{d}{dt} \left( -\frac{M \bar{v}_T^2}{2} \sum_i W_i |\tilde{h}_i|^2 \right)
\]

\[
\int dx \left( \frac{Mn\mu^2}{2} + Mn\varphi + \frac{B^2}{2\mu_0} + \frac{P_e}{\Gamma_e - 1} + M \bar{v}_T^2 n \log n \right)
\]

\[
- M \bar{v}_T^2 \sum_i W_i \left[ \log(1 - \tilde{h}_i) + \tilde{h}_i \right] = \mathcal{E} = \text{const.}
\]

Dimits (Sherwood 2007) has given similar result

A large oasis appears

- ORNL contract with Chacón & Chen
- Use JFNK implicit methods

Langmuir wave       Ion-acoustic wave

Figure 3: Time history of moment energy (Trichel) vs. time for $\Omega_{pe}$: (a) 0.1, (b) 0.6, (c) 0.8.
Soon to run dry

2-stream instability

Figure 5: Phase-space evolution of the electron two-stream instability. Numbers are time in units of $\omega_p^{-1}$. 
2-stream crashes late in time

- Issue is synchronization
  - Particles give low-order moments \( n, u \)
  - Moments de-synchronize
    - Numerical dissipation (viscosity)
- We should synchronize

A correction electric field is used to maintain synchronization between the low-order moments of the moment and particle representations.
Synchronization problem (left as an exercise for the student)

- Have 2 representations of low-order moments
- Fluid is quiet, but not accurate
- Particle is accurate, but noisy
- Simple (linear feedback) remedies not successful
- Kalman filtering problem?
Another approach – perturbed orbits

• Instead of tracking perturbed distribution along orbits, track orbit perturbations


Direct Implicit Large Time-Step Particle Simulation of Plasmas

A. BRUCE LANGDON, BRUCE I. COHEN, AND ALEX FRIEDMAN
Semi-characteristic method

\[ \frac{\partial f}{\partial t} + [H, f] = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial t} + [H - F(f, t), f] = 0 \]

\[ \frac{dP}{dt} = -\frac{\partial H}{\partial Q} \]
\[ \frac{dQ}{dt} = \frac{\partial H}{\partial P} \]

Equations have same solution, but different characteristics
Semi-characteristic method

- Suppose we have an “underlying” equilibrium
  \( \bar{H}, \bar{f} \Rightarrow [\bar{H}, \bar{f}] = 0 \)

- Then \( \bar{H}(\bar{f}) \) &
  \( \bar{H}[P_0(P,Q,t), Q_0(P,Q,t)] = \bar{H}[f(P,Q,t)] \)

- So choose
  \( F = \bar{H}[P_0(P,Q,t), Q_0(P,Q,t)] \)
Semi-characteristic method

- Then a miracle occurs
- Phase space flow of equilibrium is removed
- Since shot noise constant in time, it can be subtracted
Semi-characteristic method

\[
\begin{align*}
\delta \dot{P} &= -\partial_Q H (P, Q, t, \delta \Phi) - \partial_\tau \delta P \\
\delta \dot{Q} &= \partial_P H (P, Q, t, \delta \Phi) - \partial_\tau \delta Q
\end{align*}
\]

- Particle load is stepped in eq fields (\(\tau\)) to generate “f-string”
- Markers are moved by standard method
- Then advective mapping applied
  - Allows dissipation (required to solve hyperbolic problem)
  - Can be made energy-momentum conservative
- Perturbed fields found from sources (\(\delta P, \delta Q\))
- All standard PIC tricks can be applied
- All advective tricks can be applied
Example, Landau damping of small amplitude Langmuir wave
Summary

• A cautionary tale
  – Using moments + closure requires synchronization
  – Algorithm for low collisionality largely unknown
  – Other issues solved by particular velocity

• Semi-characteristic method developed
  – PIC shot noise removed
  – Initial experiments successful
  – Remaining issues
    • Non-periodic phase space (advection)
    • Beam-beam instabilities (controlled by randomness)
    • Lyapunov exponents > 0 (is it an issue with SC?)

• Happy to work with anyone on these issues