Experimentation with a Vector Potential Formulation

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Thesis
We may be able to learn something useful about NIMROD’s two-fluid advance by testing a vector-potential formulation.

Outline
• Introduction
• Vector-potential formulation
• Ideal MHD results with CYL_SPEC
• NIMROD linear MHD results
• Discussion
Introduction

• Two-fluid NIMROD computations with a pressure gradient and sufficiently large Hall parameter are problematic.

• Given the analysis of the implicit leapfrog, difficulties with linear computations implicate something related to spatial representation.

• Using the vector-potential is complementary in that it removes $\text{div}(B)$ from the possible sources of error.
  
  – Note that artificial numerical induction from electrostatic-$E$ near the limit of resolution is an independent possible error.

• Solving a vector potential is not expected to be a panacea.
  
  – But if it is …
**Formulation:** Replace Faraday’s law for $B$-evolution with a relation for $E$.

- Start with defining relations for potentials with the Coulomb gauge.
  \[ B = \nabla \times A \]
  \[ E = -\frac{\partial}{\partial t} A - \nabla \phi \quad \nabla \cdot A = 0 \]

- Ohm’s law is used to eliminate electric field.
  \[ \frac{\partial}{\partial t} A + \nabla \phi = \mathbf{V} \times \mathbf{B} - \eta \mathbf{J} - \frac{1}{ne} \left( \mathbf{J} \times \mathbf{B} - \nabla p_e \right) \]

- Using Ampere’s law and the definition for $A$,
  \[ \frac{\partial}{\partial t} A + \nabla \phi = \mathbf{V} \times \nabla \times A - \frac{\eta}{\mu_0} \nabla \times \nabla \times A \]
  \[ + \frac{1}{\mu_0 ne} \nabla \cdot \left\{ \left[ \mu_0 p_e + \frac{1}{2} (\nabla \times A)^2 \right] \mathbf{I} - (\nabla \times A)(\nabla \times A) \right\} \]
Invoking the stress tensor allows $C^0$ continuity in the expansion for $A$.

- When using $J \times B$ directly in weak form,

$$\int C^* \cdot J \times B \, dVol = \frac{1}{\mu_0} \int C^* \cdot (\nabla \times B) \times B \, dVol$$

$$= \frac{1}{\mu_0} \int \left( B^2 \nabla \cdot C^* + C^* \cdot \nabla B \cdot B - B \cdot \nabla C^* \cdot B \right) \, dVol + \frac{1}{\mu_0} \oint B \times \left( B \times C^* \right) \cdot dS$$

the underlined term must be converted to a divergence of a dyad and integrated by parts to avoid differentiating $B = \nabla \times A$.

- Working directly from the stress tensor,

$$\frac{1}{\mu_0} \int \left\{ \nabla \cdot \left[ BB - \left( \mu_0 p + B^2 / 2 \right) I \right] \right\} \cdot C^* \, dVol =$$

$$\frac{1}{\mu_0} \int \left[ \left( \mu_0 p + B^2 / 2 \right) I - BB \right] : \nabla C^* \, dVol + \frac{1}{\mu_0} \oint C^* \cdot \left[ BB - \left( \mu_0 p + B^2 / 2 \right) I \right] \cdot dS$$

which has no derivatives on $B$.

- These considerations apply to the Hall term and to flow evolution.
Completing the vector-potential equation requires
gauge and essential conditions.

- Simplify to resistive-MHD for clarity.
- Integrating by parts and using penalty methods to apply the gauge
  (green and blue underlined terms) produces

\[
\int \left[ \mathbf{C}^* \cdot \frac{\partial}{\partial t} \mathbf{A} - \nabla \cdot \mathbf{C}^* \phi + f_A \left( \nabla \cdot \mathbf{C}^* \right) \mathbf{A} \right] d\text{Vol} =
\]

\[
\int \mathbf{C}^* \cdot \left( \nabla \times \mathbf{A} \right) d\text{Vol} - \frac{1}{\mu_0} \int \left( \eta \nabla \times \mathbf{C}^* + \nabla \eta \times \mathbf{C}^* \right) \cdot \nabla \times \mathbf{A} d\text{Vol}
\]

\[
- \oint \left[ \mathbf{C}^* \phi + \frac{\eta}{\mu_0} \left( \nabla \times \mathbf{A} \right) \times \mathbf{C}^* \right] \cdot d\mathbf{S}
\]

\[
\oint \left( \chi^* \phi + f_\phi \chi^* \nabla \cdot \mathbf{A} \right) = 0
\]

for all \( \mathbf{C} \) and \( \chi \) in the function spaces used for (continuous) \( \mathbf{A} \) and
(discontinuous) \( \phi \).

- The red-underlined surface term is dropped when essential
  conditions are applied on tangential components of \( \mathbf{A} \).

- Physical conditions on the normal component of \( \mathbf{A} \) are not obvious.
**Eigenvalues:** Ideal-MHD spectra from CYL_SPEC indicate more spectral pollution with the vector potential.

- Results for my usual, uniform equilibrium case with $m=1$, $k_z=2$, $\beta=1$.
- Fields are continuous except the scalar for divergence/gauge control.
- $B$ and $A$ formulations use hyperbolic control (latter is Lorenz gauge).

- The vector-potential computations have many numerical modes, including $\omega=0$ modes, outside the physical bands.
CYL_SPEC results on an kink-unstable equilibrium also show better performance with the B formulation.

- Computed growth rates with 6 quintic elements with an element border at the $q=1$ surface differ by less than 0.1%.
- With the B formulation, only the kink is unstable.
- With the A formulation, there are a number of overstable modes.
**NIMROD with A:** Modification for linear MHD was done in steps to check the development.

- Data structures for vector and scalar potentials and the time-advance were added first without removing the $B$ advance.
  - Flow was independent of the computed $A$.
  - Direct comparison of the advanced $B$ and $\text{curl}(A)$ verified the time-advance for $A$.
- Modification of the flow advance replaced $\mathbf{J} \times \mathbf{B}$ with the divergence of the Maxwell stress.
  - Computations are less robust, even when advancing $\mathbf{B}$.
  - In some cases, the semi-implicit operator needs an isotropic contribution with at least a small coefficient.
  - The behavior may reflect the parallel-force instability identified by Brackbill and Barnes (JCP 35, 426).
- Eliminating the separate magnetic-field advance was the last step.
Results on a $\beta=0$ tearing mode at $S=10^6$, $Pm=10^{-3}$ are okay with some caveats.

- If the $f_A \int \left[ \left( \nabla \cdot C^* \right) \left( \nabla \cdot A \right) \right] dVol$ stabilization term is not used, the polynomial degree of the discontinuous scalar must be the same as that of $A$.
- Using $A$ of greater degree than $V$ and $p$ crashes.
- Varying the weight for the scalar effectively changes the gauge.
  - Growth rates of the two computations shown at right differ by 0.06%.

Contours of $A_y$ (left) and $B_z$ (right) with $f_\phi = 10$ (top) and $f_\phi = 10^{-5}$ (bottom).
Obtaining results for a cylindrical internal kink case has been problematic.

- The original case has $\beta=0.5\%$, but fast-growing noise also appears at $\beta=0$.
- A small component ($10^{-6}-10^{-5}$) of the isotropic semi-implicit operator is needed, but it is easy to suppress the physical mode, too.
- Increasing the polynomial degree from 4 to 6 and relying on the $f_A$ stabilization term reproduces the physical mode on a packed $24\times24$ mesh.
  - The growth rate is still too large.
  - The eigenfunction is noisy.
Discussion

• Why the tearing and internal-kink cases behave differently is not clear.
  • Both use packed polar meshes.
  • Dissipation parameters are similar.
• Using a discontinuous scalar of 1 polynomial degree below that of the vector is similar to 2D incompressible-fluid computations.
  • An out-of-plane component with Fourier representation is a different feature of our computations.
• Tests show that using $\mathbf{A}$ to project a separate $\mathbf{B}$ expansion is more problematic than using $\text{curl}(\mathbf{A})$ directly in the force computation.
• No development for the Hall-MHD system has been performed.
• Resistive-MHD results with vector potential make NIMROD’s $\mathbf{B}$ formulation look good.