Some notes on $H(\nabla\times)$ and $H(\nabla\cdot)$ conforming finite elements

NIMROD Team Meeting

August 17th, 2017

Description of $H(\nabla\times)$ and $H(\nabla\cdot)$ conforming finite elements provides necessary context for understanding this work. Also, they are cool.

Definitions and appropriate norms:

$H(\nabla\times)$ for $\Omega \in \mathbb{R}^3$

$$H(\nabla\times;\Omega) := \left\{\vec{w} \ : \ \vec{w} \in \left[L^2(\Omega)\right]^3, \nabla \times \vec{w} \in \left[L^2(\Omega)\right]^3\right\}$$
$$\|\vec{w}\|^2_{\nabla\times,\Omega} := |\vec{w}|^2_{0,\Omega} + |\nabla \times \vec{w}|^2_{0,\Omega}$$

$H(\nabla\cdot)$ for $\Omega \in \mathbb{R}^n$

$$H(\nabla\cdot;\Omega) := \left\{\vec{w} \ : \ \vec{w} \in \left[L^2(\Omega)\right]^n, \nabla \cdot \vec{w} \in L^2(\Omega)\right\}$$
$$\|\vec{w}\|^2_{\nabla\cdot,\Omega} := |\vec{w}|^2_{0,\Omega} + |\nabla \cdot \vec{w}|^2_{0,\Omega}$$

We should care about $H(\nabla\times)$ and $H(\nabla\cdot)$ in electromagnetics and MHD because $\vec{E} \in H(\nabla\times)$ and $\vec{B} \in H(\nabla\cdot)$ due to the structure of Maxwell's equations.

The discrete de Rham complex illustrates important properties of the $H(\nabla\times)$ and $H(\nabla\cdot)$ conforming finite element spaces.

Continuous and discrete de Rham sequences:

$$
\begin{array}{ccccccc}
H(\nabla) & \xrightarrow{\nabla} & H(\nabla\times) & \xrightarrow{\nabla\times} & H(\nabla\cdot) & \xrightarrow{\nabla\cdot} & L^2 \\
\downarrow \Pi^{\nabla} & & \downarrow \Pi^{\nabla\times} & & \downarrow \Pi^{\nabla\cdot} & & \downarrow \Pi^{\emptyset} \\
V^{\nabla} & \xrightarrow{\mathbf{G}} & V^{\nabla\times} & \xrightarrow{\mathbf{C}} & V^{\nabla\cdot} & \xrightarrow{\mathbf{D}} & Q
\end{array}
$$

Important properties:

- The interpolation operators (Π) commute with the differential operators.
- The discrete operators have the approprate kernels (direct result of commuting diagram property), i.e. $\mathbf{Gx} \in \ker(\mathbf{C})$ and $\mathbf{Cx} \in \ker(\mathbf{D})$
- $H(\nabla\times)$ conforming elements have continuity of tangential components only

How to construct $H(\nabla\times)$ and $H(\nabla\cdot)$ conforming elements.

Let P_k^{GL} be the 1-D Lagrange cardinal function on the Gauss-Legengre quadrature nodes and P_k^{GLL} be the 1-D Lagrange cardinal function on the Gauss-Legengre-Lobatto nodes.

$H(\nabla\times)$ conforming element on a quadrilateral (Nédélec)

$$\mathcal{N}_{[k+1]} := \left(P_k^{GL} \otimes P_{k+1}^{GLL} \otimes P_{k+1}^{GLL}\right) \times \left(P_{k+1}^{GLL} \otimes P_k^{GL} \otimes P_{k+1}^{GLL}\right) \times \left(P_{k+1}^{GLL} \otimes P_{k+1}^{GLL} \otimes P_k^{GL}\right)$$

$H(\nabla\cdot)$ conforming element on a quadrilateral (Raviart-Thomas)

$$\mathcal{RT}_{[k]} := \left(P_{k+1}^{GLL} \otimes P_k^{GL} \otimes P_k^{GL}\right) \times \left(P_k^{GL} \otimes P_{k+1}^{GLL} \otimes P_k^{GL}\right) \times \left(P_k^{GL} \otimes P_k^{GL} \otimes P_{k+1}^{GLL}\right)$$

$H(\nabla\times)$ and $H(\nabla\cdot)$ conforming elements require specialized coordinate transformations.

For a mapping $x = \mathcal{F}(\hat{x})$, denote $\mathcal{DF}(\hat{x})$ as the Jacobian matrix and $\mathcal{J}(\hat{x})$ as the Jacobian.

Transformation of $\vec{v} \in H(\nabla\times)$

$$\vec{v}(x) = \left[\mathcal{DF}(\hat{x})\right]^{-T} \hat{\vec{v}}(\hat{x})$$

Transformation of $\vec{w} \in H(\nabla\cdot)$

$$\vec{w}(x) = \frac{1}{\mathcal{J}(\hat{x})}\mathcal{DF}(\hat{x})\,\hat{\vec{w}}(\hat{x})$$

The transformation of $H(\nabla\cdot)$ is known as the contravariant Piola transform.