Updates on development of moment-based closures

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General moment equations

- Landau (Fokker-Planck) kinetic equation

$$\frac{\partial f_a}{\partial t} + \mathbf{v}\cdot\nabla f_a + \frac{q_a}{m_a}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_a}{\partial \mathbf{v}} = \sum_b C(f_a, f_b) \quad (+X+Y+Z)$$

- Moment expansion

$$f_a(t,\mathbf{x},\mathbf{v}) = f_a^{\mathrm{M}} \sum_{lk} \mathsf{m}_a^{lk}(t,\mathbf{x}) \cdot \hat{\mathsf{p}}_a^{lk} \text{ with } f_a^{\mathrm{M}} = \frac{n_a}{\pi^{3/2} v_{Ta}^3}\exp(-c_a^2)$$

$$\mathsf{n}_a^{lk} \equiv n_a \mathsf{m}_a^{lk}(t,\mathbf{x}) = \int d\mathbf{v}\hat{\mathsf{p}}_a^{lk} f_a(t,\mathbf{x},\mathbf{v})$$

where $\hat{\mathsf{p}}^{lk}$'s are orthonormal polynomials of $\mathbf{c}_a = (\mathbf{v}-\mathbf{V}_a)/v_{Ta}$

- General moment equations: $\int d\mathbf{v}\hat{\mathsf{p}}^{jp}$(kinetic eq.)$\Rightarrow$

$$d_a \mathsf{n}_a^{jp} + \Omega_a \mathbf{b}\breve{\times}\mathsf{n}_a^{jp} + \{\hat{\Xi}^j(d_a \ln T) + \hat{U}_{\mathrm{c}}^j \nabla\cdot\mathbf{V} + \hat{U}_{\mathrm{l}}^j(\underline{\nabla}\mathbf{V})\cdot + \hat{U}_{\mathrm{r}}^j(\nabla\underline{\mathbf{V}})\cdot\}_{pk}\mathsf{n}_a^{jk}$$
$$+\{v_T\hat{\Psi}^{j\pm}\nabla + v_T\hat{\Phi}^{j\pm}\nabla\ln T + v_T^{-1}\hat{\Theta}^{j\pm}\mathbf{a}_a\}_{pk}(\cdot)\mathsf{n}_a^{j\pm1,k} + \hat{U}_{pk}^{j\pm}\nabla\mathbf{V}(:)\mathsf{n}_a^{j\pm2,k}$$
$$= (\hat{C}_{aa}^{jpk} + \hat{A}_{ab}^{jpk})\mathsf{n}_a^{jk} + \hat{B}_{ab}^{jpk}\mathsf{n}_b^{jk} + C_{ab}^{(2)jp}$$

where $d_a \equiv \partial_t + \mathbf{V}_a\cdot\nabla$ and $\mathbf{a}_a \equiv \frac{q_a}{m_a}(\mathbf{E}+\mathbf{V}_a\times\mathbf{B}) - d_a\mathbf{V}_a$

▶ Farewell to the velocity variable $\mathbf{v}$

Several low order moment equations

$\mathsf{p}^{lk} = \mathsf{P}^l(\mathbf{c})L_k^l(c^2)$	$L_k^l = L_k^{(l+\frac{1}{2})}$	n^{lk}	fluid moment equation	indep.
$\mathsf{P}^0 = 1$	$L_0^0 = 1$	n	density (n)	1
	$L_1^0 = \frac{3}{2} - c^2$	0	temperature (T)	1
$\mathsf{P}^1 = \mathbf{c}$	$L_0^1 = 1$	0	flow velocity $(\mathbf{V})$	3
	$L_1^1 = \frac{5}{2} - c^2$	n^{11}	heat flow $(\mathbf{h})$	3
	$L_2^1 = \frac{35}{8} - \frac{7}{2}c^2 + \frac{1}{2}c^4$	n^{12}	heat w. heat flow $(\mathbf{r})$	3
$\mathsf{P}^2 = \mathbf{cc} - \frac{c^2}{3}\mathsf{I}$	$L_0^2 = 1$	n^{20}	viscosity $(\boldsymbol{\pi})$	5
	$L_1^2 = \frac{7}{2} - c^2$	n^{21}	heat viscosity $(\boldsymbol{\theta})$	5

Fluid equations and closures/transport

Maxwellian moment $(n_a, \mathbf{V}_a, T_a)$ equations

(0,0) $d_t n_a + n_a \nabla\cdot\mathbf{V}_a = 0\,(d_t \equiv \partial_t + \mathbf{V}_a\cdot\nabla)$

(0,1) $\frac{3}{2} n_a d_t T_a + n_a T_a \nabla\cdot\mathbf{V}_a + \nabla\cdot\mathbf{h}_a + \nabla\mathbf{V}_a : \boldsymbol{\pi}_a = Q_a$

(1,0) $m_a n_a d_t \mathbf{V}_a - n_a q_a(\mathbf{E} + \mathbf{V}_a \times \mathbf{B}) + \nabla p_a + \nabla\cdot\boldsymbol{\pi}_a = \mathbf{R}_a$

General moment equations $D\mathsf{n} + \Omega\mathbf{b}\breve{\times}\mathsf{n} = C\mathsf{n}$ ($\mathsf{n}^{lk} \to v^{l+2k}$ moment)

$$(1,1)\ d_t\mathbf{h} + \Omega\mathbf{b}\times\mathbf{h} + \frac{7}{5}(\nabla\cdot\mathbf{V})\mathbf{h} + \frac{7}{5}\mathbf{h}\cdot(\nabla\mathbf{V}) + \frac{2}{5}(\nabla\mathbf{V})\cdot\mathbf{h} + \frac{5p}{2m}\nabla T + \frac{T}{m}\nabla\cdot\boldsymbol{\pi} + \frac{7}{2}\frac{\nabla T}{m}\cdot\boldsymbol{\pi} - \mathbf{a}\cdot\boldsymbol{\pi} + \nabla\cdot\boldsymbol{\theta} + \frac{1}{3}\nabla\mathsf{u}^{02} + \nabla\mathbf{V} : \mathsf{u}^{30}$$
$$= C^1_{10}\mathbf{V}_{\mathrm{ei}} + C^1_{11}\mathbf{h} + C^1_{12}\mathbf{r} + \cdots \text{ (}\mathbf{h}\text{ heat flow)}$$

$$(1,2)\ d_t\mathbf{r} + \Omega\mathbf{b}\times\mathbf{r} + \cdots = C^1_{10}\mathbf{V}_{\mathrm{ei}} + C^1_{21}\mathbf{h} + C^1_{22}\mathbf{r} + \cdots \quad \text{(}\mathbf{r}\text{ heat heat flow)}$$

$$(2,0)\ d_t\boldsymbol{\pi} + \Omega\mathbf{b}\breve{\times}\boldsymbol{\pi} + (\nabla\cdot\mathbf{V})\boldsymbol{\pi} + 2\overline{\boldsymbol{\pi}\cdot(\nabla\mathbf{V})} + p\mathsf{W} + \frac{4}{5}\overline{\nabla\mathbf{h}} + \nabla\cdot\mathsf{u}^{30}$$
$$= C^2_{00}\boldsymbol{\pi} + C^2_{01}\boldsymbol{\theta} + \cdots \text{ (}\boldsymbol{\pi}\text{ viscosity)}$$

$$(2,1)\ d_t\boldsymbol{\theta} + \Omega\mathbf{b}\breve{\times}\boldsymbol{\pi} + \cdots = C^2_{10}\boldsymbol{\pi} + C^2_{11}\boldsymbol{\theta} + \cdots \text{ (}\boldsymbol{\theta}\text{ heat viscosity)}$$

where $\mathbf{a} = \frac{q}{m}(\mathbf{E} + \mathbf{V}\times\mathbf{B}) - d_t\mathbf{V}$ and $\mathsf{W} = \nabla\mathbf{V} + (\nabla\mathbf{V})^{\mathrm{T}} - \frac{2}{3}\nabla\cdot\mathbf{V}\mathsf{I}$

Closures: express $\mathbf{h}_a(\mathsf{n}^{11}_a)$, $\boldsymbol{\pi}_a(\mathsf{n}^{20}_a)$, Q_a, $\mathbf{R}_a$ in terms of $n_a, \mathbf{V}_a, T_a$

Electron closures for high collisionality (Braginskii)

$\mathbf{h}_{\mathrm{e}} = (\beta)(\mathbf{V}_{\mathrm{ei}}) - (\kappa)(\nabla T_{\mathrm{e}})$, $\mathbf{R}_{\mathrm{e}} = -\mathbf{R}_{\mathrm{i}} = -(\alpha)(\mathbf{V}_{\mathrm{ei}}) - (\beta)(\nabla T_{\mathrm{e}})$

Transport: relate flux densities $\mathbf{h}_{\mathrm{e}}$, $\mathbf{J}$ to thermodynamic forces ∇T_{e} and $\mathbf{E}$

$\mathbf{h}_{\mathrm{e}} = (\tilde{\alpha})\mathbf{E} - (\tilde{\kappa})(\nabla T_{\mathrm{e}})$, $\mathbf{J} = (\tilde{\sigma})\mathbf{E} - (\tilde{\alpha})(\nabla T_{\mathrm{e}})$

Work accomplished [Co-authors: E.D. Held, C.R. Sovinec, H.Q. Lee, S.-K. Kim, Y.-S. Na, S.S. Kim, G.Y. Park, G.S. Yun, I. Joseph]

- General moment equations: exact calculation of collision operators
  - Linear and nonlinear terms in total- and random-velocity moment expansions
    - “Exact linearized Coulomb collision operator in the moment expansion”, Phys. Plasmas **13**, 102103 (2006).
    - “Landau collision operators and general moment equations for an electron-ion plasma”, Phys. Plasmas **15**, 102101(2008).
    - “Full Coulomb collision operator in the moment expansion”, Phys. Plasmas **16**, 102108 (2009).
    - “Analytical solution of the kinetic equation for a uniform plasma in a magnetic field”, Phys. Rev. E **82**, 016401 (2010).
    - “A framework for moment equations for magnetized plasmas”, Phys. Plasmas **21**, 042102 (2014).

- Closures and transport for high collisionality
  - Large $x = \Omega\tau$ correction for electrons
  - Effects of ion-electron collisions on ion transport
    - “Closure and transport theory for high-collisionality electron-ion plasmas”, Phys. Plasmas **20**, 042114 (2013).
    - “Ion closure theory for high collisionality revisited”, Phys. Plasmas **22**, 062114 (2015).

Work accomplished (cont.)

- Parallel closures and transport for arbitrary collisionality
  - Electrons: for ion charge number $1 \leq Z_{\text{eff}} \leq 100$
  - Ion: for various AZ^2 $(A = m_\text{i}/m_\text{p})$ and $T_\text{i}/T_\text{e} \leq 10$
    - "Moment approach to deriving parallel heat flow for general collisionality", Phys. Plasmas **16**, 022312 (2009).
    - "Moment approach to deriving a unified parallel viscous stress in magnetized plasmas", J. Fusion Energy **28**, 170 (2009).
    - "Linearly exact parallel closures for slab geometry", Phys. Plasmas **20**, 082121 (2013).
    - "Electron parallel closures for arbitrary collisionality", Phys. Plasmas **21**, 122115 (2014).
    - "Electron parallel closures for various ion charge numbers", Phys. Plasmas **23**, 032124 (2016).
    - "Electron heat flow due to magnetic field fluctuations", Plasma Phys. Control. Fusion **58**, 042001 (2016).
    - "Ion parallel closures", Phys. Plasmas **24**, 022127 (2017).
    - "Electron parallel closures for the 3+1$(n, V_\parallel, p_\parallel, p_\perp)$ fluid model", to be submitted.
    - "Electron parallel transport for arbitrary collisionality", to be submitted.

Integral (nonlocal) parallel closures for arbitrary collisionality

- n_A responding to g_B:

$$n_{AB}(\eta) = \int d\eta' K_{AB}(\eta - \eta') g_B(\eta') \;\rightarrow\; n_{AB}(\eta) = \kappa_{AB}(\eta) g_B(\eta)$$

$$h_\parallel(\eta) = -\frac{1}{2} T v_T \int d\eta' K_{hh} \frac{n}{T}\frac{dT}{d\eta'} + T v_T \int d\eta' K_{hR} Z n \frac{V_{\mathrm{ei}\parallel}}{v_T} - T v_T \int d\eta' K_{h\pi} (\frac{3}{4} n \tau_{\mathrm{ee}} W_\parallel)$$

$$R_\parallel(\eta) = -\frac{mn}{\tau_{\mathrm{ei}}} V_{\mathrm{ei}\parallel} + \frac{m v_T}{\tau_{\mathrm{ei}}} \int d\eta' [-K_{Rh} \frac{n}{2T}\frac{dT}{d\eta'} + K_{RR} Z n \frac{V_{\mathrm{ei}\parallel}}{v_T} - K_{R\pi} (\frac{3}{4} n \tau_{\mathrm{ee}} W_\parallel)]$$

$$\pi_\parallel(\eta) = -T \int d\eta' K_{\pi h} \frac{n}{T}\frac{dT}{d\eta'} + 2T \int d\eta' K_{\pi R} Z n \frac{V_{\mathrm{ei}\parallel}}{v_T} - T \int d\eta' K_{\pi\pi} (\frac{3}{4} n \tau_{\mathrm{ee}} W_\parallel)$$

Fitted kernel functions $K_{AB}(\eta) = -[d + a\exp(-b\eta^c)]\ln[1 - \alpha\exp(-\beta\eta^\gamma)]$

- Fourier transform (FT) of integral closures: $\tilde{n}_{AB}(k) = \tilde{K}_{AB}(k)\tilde{g}_B(k)$

$$\tilde{h}_\parallel = -\frac{1}{2} n_0 v_0 \tilde{K}_{hh} ik\tilde{T}_1 + p_0 \tilde{K}_{hR} \tilde{u}_{\mathrm{ei}} - p_0 \tilde{K}_{h\pi} ik\tilde{u}$$

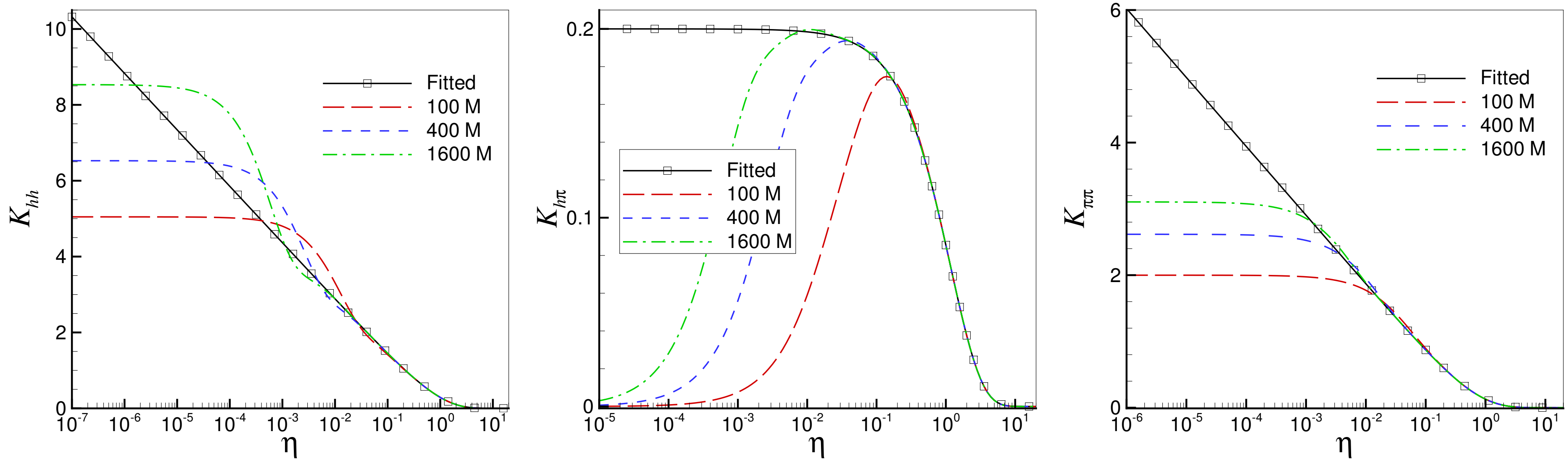
$$\tilde{R}_\parallel = -\frac{1}{2}\frac{m v_0}{\tau_{\mathrm{ee}}}\frac{n_0}{T_0} \tilde{K}_{Rh} ik\tilde{T}_1 - \frac{m n_0}{\tau_{\mathrm{ee}}}\left(1 - \tilde{K}_{RR}\right)\tilde{u}_{\mathrm{ei}} - \frac{m n_0}{\tau_{\mathrm{ee}}} \tilde{K}_{R\pi} ik\tilde{u}$$

$$\tilde{\pi}_\parallel = -n_0 \tilde{K}_{\pi h} ik\tilde{T}_1 + 2\frac{p_0}{v_0} \tilde{K}_{\pi R} \tilde{u}_{\mathrm{ei}} - \frac{p_0}{v_0} \tilde{K}_{\pi\pi} ik\tilde{u}$$

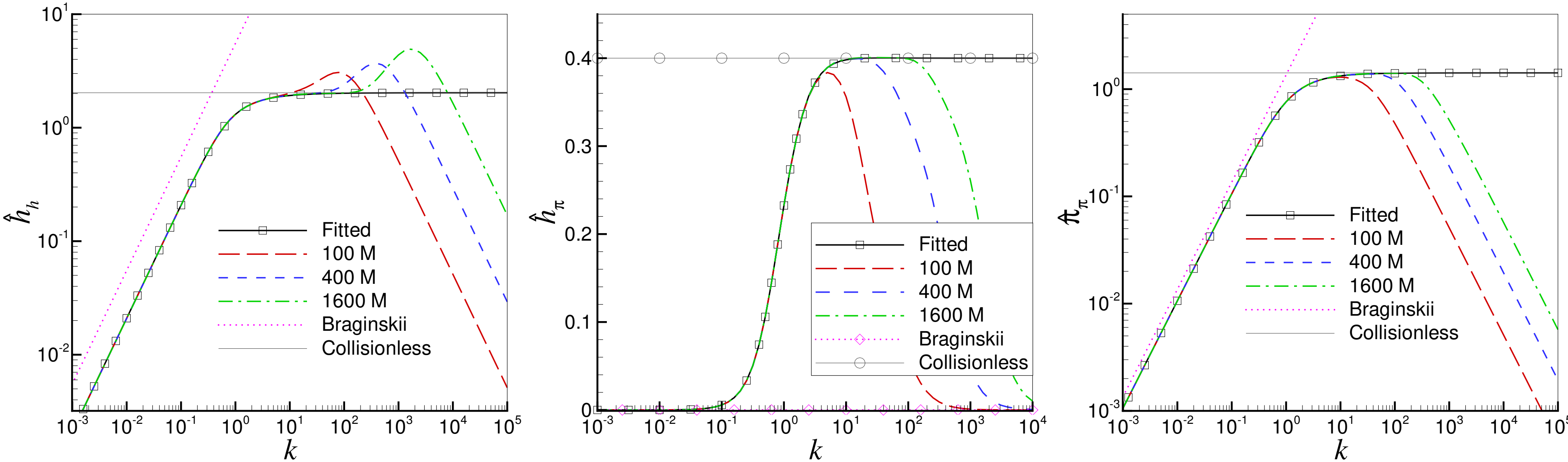
Fitted kernels in k space $\tilde{K}_{AB} = \dfrac{a k^\alpha}{1 + d_1 k^\delta + d_2 k^{2\delta} + d_3 k^{3\delta} + d_4 k^{4\delta} + d_5 k^{5\delta} + d_6 k^{6\delta}}$

Example: ion closures for $AZ^2 = 1$ and $T_\mathrm{i}/T_\mathrm{e} = 4$

Kernels



Closures for sinusoidal drives



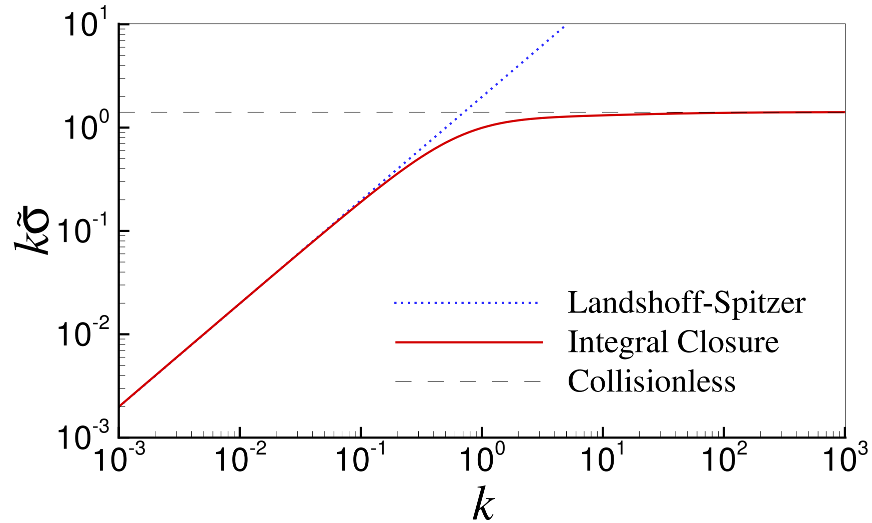
Momentum balance equation with closures ⇒ transport

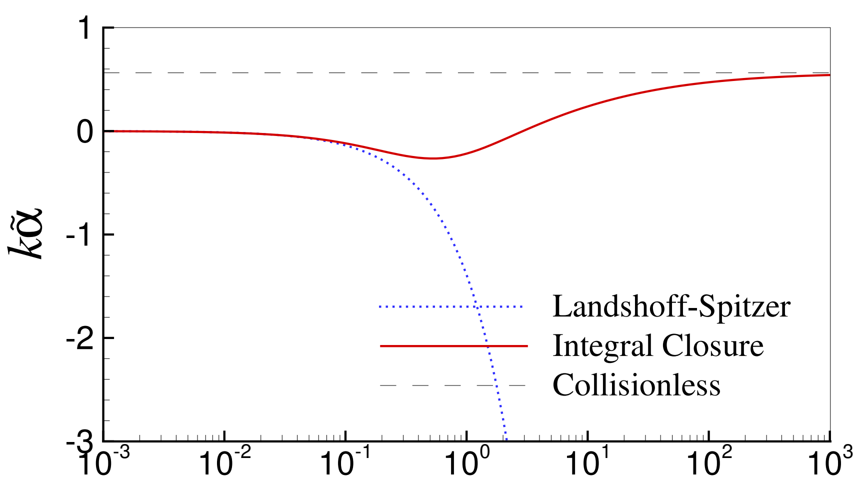
- Parallel momentum balance equation

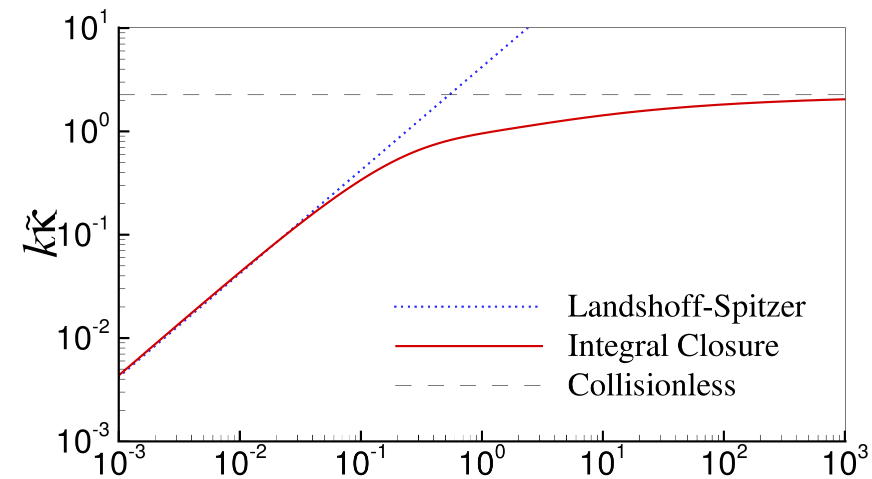
$$neE_{\|} + \partial_{\|} p + \partial_{\|}\pi_{\|} = R_{\|} \overset{\mathrm{FT}}{\Leftrightarrow} ne\tilde{E}_{\|} + \frac{ik}{\lambda_{\mathrm{C}}}\tilde{p} + \frac{ik}{\lambda_{\mathrm{C}}}\tilde{\pi}_{\|} = \tilde{R}_{\|}$$

- Transport relations in k space (for single harmonic drives)

$$\begin{aligned} \tilde{J}_{1\|} &= \frac{n_0 e^2 \tau_{\mathrm{ei}}}{m}\tilde{\sigma}_{\|}\tilde{E}'_{1\|} - \frac{n_0 e \tau_{\mathrm{ei}}}{m}\tilde{\alpha}_{\|} ik\tilde{T}_1 \\ \tilde{h}_{1\|} &= \frac{n_0 e T_0 \tau_{\mathrm{ei}}}{m}\tilde{\alpha}_{\|}\tilde{E}'_{1\|} - \frac{n_0 T_0 \tau_{\mathrm{ei}}}{m}\tilde{\kappa}_{\|} ik\tilde{T}_1 \end{aligned}$$







Resistivity and fast magnetic reconnection

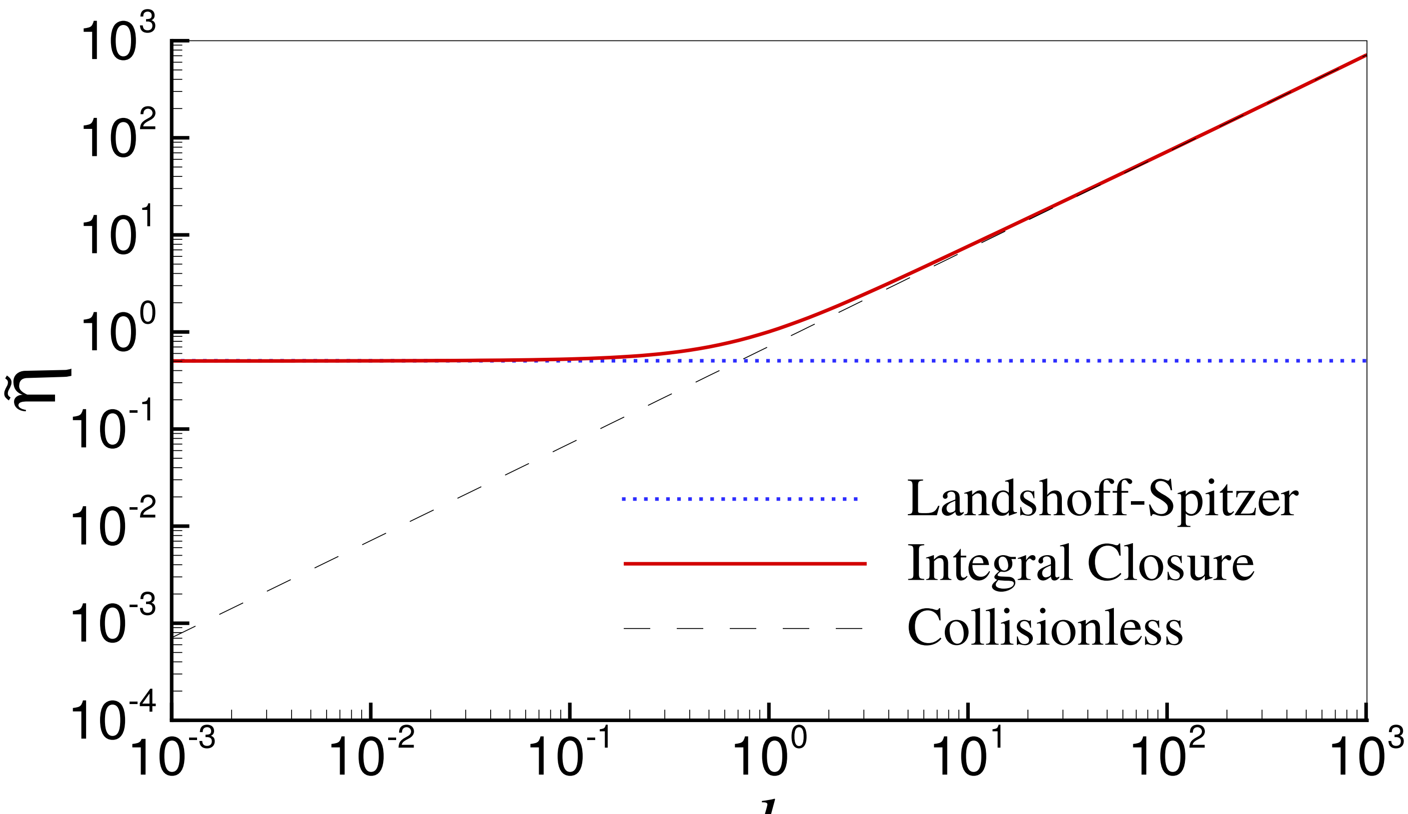
- Resistive Ohm's law and Faraday's law

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J} \text{ and } \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

- Magnetic field diffusion equation and magnetic reconnection time

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B} \Rightarrow \tau_{\text{rec}} = \frac{|\nabla^{-1}|^2}{\eta/\mu_0}$$

- Fast reconnection timescale may be predicted by correct resistivity



Kinetic equations for a partially ionized plasma

$$
\begin{aligned}
\frac{\partial f_\mathrm{e}}{\partial t} + \mathbf{v}\cdot\frac{\partial f_\mathrm{e}}{\partial \mathbf{r}} + \mathbf{a}_\mathrm{e}\cdot\frac{\partial f_\mathrm{e}}{\partial \mathbf{v}} &= C(f_\mathrm{e}) - Y_\mathrm{e}(\mathrm{e}+\mathrm{i}\rightarrow \mathrm{n}) + Z_\mathrm{e}(\mathrm{e}+\mathrm{n}\rightarrow \mathrm{e}+\mathrm{i}+\mathrm{e}) \\
\frac{\partial f_\mathrm{i}}{\partial t} + \mathbf{v}\cdot\frac{\partial f_\mathrm{i}}{\partial \mathbf{r}} + \mathbf{a}_\mathrm{i}\cdot\frac{\partial f_\mathrm{i}}{\partial \mathbf{v}} &= C(f_\mathrm{i}) + C_\mathrm{x}(f_\mathrm{i}, f_\mathrm{n}) - Y_\mathrm{i} + Z_\mathrm{n} \\
\frac{\partial f_\mathrm{n}}{\partial t} + \mathbf{v}\cdot\frac{\partial f_\mathrm{n}}{\partial \mathbf{r}} + \mathbf{a}_\mathrm{n}\cdot\frac{\partial f_\mathrm{n}}{\partial \mathbf{v}} &= C_\mathrm{x}(f_\mathrm{n}, f_\mathrm{i}) + Y_\mathrm{i} - Z_\mathrm{n}
\end{aligned}
$$

Ionization: $$Z_\mathrm{e} = \int d\mathbf{v}'\sigma_\mathrm{z}|\mathbf{v}-\mathbf{v}'|[f_\mathrm{e}(\mathbf{v}_\mathrm{f})f_\mathrm{n}(\mathbf{v}_\mathrm{f}') - f_\mathrm{e}(\mathbf{v})f_\mathrm{n}(\mathbf{v}')] + \frac{\delta f_\mathrm{e}^\mathrm{ejected}(\mathbf{v})}{\delta t}$$

$$Z_\mathrm{n} = \int d\mathbf{v}'\sigma_\mathrm{z}|\mathbf{v}-\mathbf{v}'|f_\mathrm{e}(\mathbf{v}')f_\mathrm{n}(\mathbf{v}) \approx n_\mathrm{e}\left<\sigma_\mathrm{z}v_\mathrm{e}\right> f_\mathrm{n}(\mathbf{v})$$

Recombination: $$Y_\mathrm{e} = f_\mathrm{e}(\mathbf{v})\int d\mathbf{v}'\sigma_\mathrm{r}|\mathbf{v}-\mathbf{v}'|f_\mathrm{i}(\mathbf{v}')$$

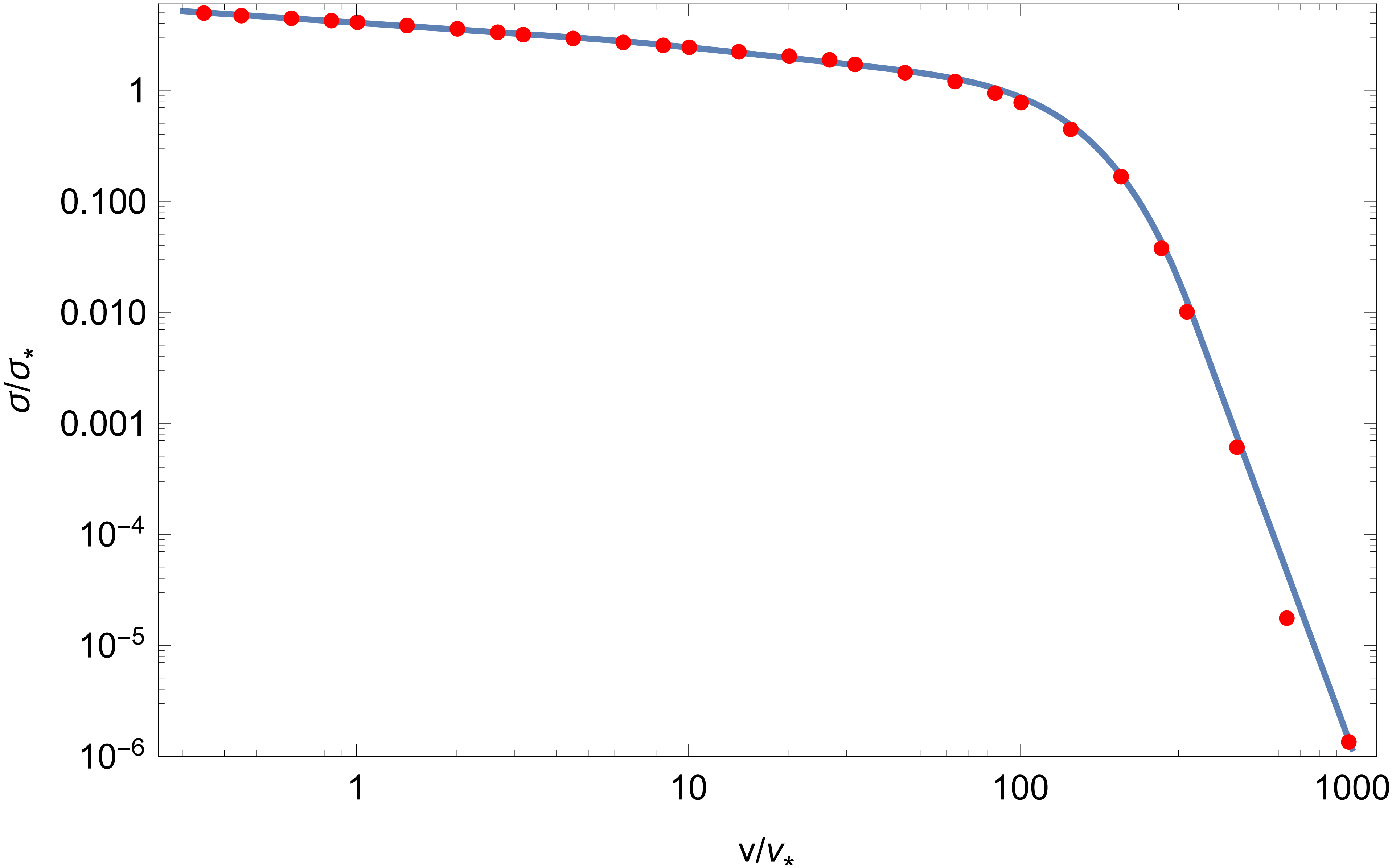
$$Y_\mathrm{i} = f_\mathrm{i}(\mathbf{v})\int d\mathbf{v}'\sigma_\mathrm{r}|\mathbf{v}-\mathbf{v}'|f_\mathrm{e}(\mathbf{v}') \approx n_\mathrm{e}\left<\sigma_\mathrm{r}v_\mathrm{e}\right> f_\mathrm{i}(\mathbf{v})$$

McWhirter 1965 $$\left<\sigma_\mathrm{r}v_\mathrm{e}\right> = 7.57\times 10^{-18}\sqrt{\frac{13.6\mathrm{eV}}{T_\mathrm{e}}}\mathrm{m}^3/\mathrm{s}\ (\text{not } 0.7\times 10^{-19})$$

Voronov 1997 $$\left<\sigma_\mathrm{z}v_\mathrm{e}\right> = 2.91\times 10^{-14}\frac{(13.6\mathrm{eV}/T_\mathrm{e})^{0.39}}{0.232+13.6\mathrm{eV}/T_\mathrm{e}}\exp\left(-\frac{13.6\mathrm{eV}}{T_\mathrm{e}}\right)\mathrm{m}^3/\mathrm{s}$$

Cross section for the charge exchange operator

Atomic data for fusion vol. 1, C. F. Barnett, ORNL 1990



$\mathrm{v}_\star = 1.38 \times 10^4 \mathrm{m/s}$ and $\sigma_\star = 10^{-19} \mathrm{m}^2$

Moments of the charge exchange operator

$$C_{\rm x}(f_{\rm n}, f_{\rm i}) = \int d\mathbf{v}'\sigma_X|\mathbf{v}-\mathbf{v}'|[f_{\rm n}(\mathbf{v}')f_{\rm i}(\mathbf{v}) - f_{\rm n}(\mathbf{v})f_{\rm i}(\mathbf{v}')]$$

$$C_{\rm x}(f_{\rm i}, f_{\rm n}) = \int d\mathbf{v}'\sigma_X|\mathbf{v}-\mathbf{v}'|[f_{\rm i}(\mathbf{v}')f_{\rm n}(\mathbf{v}) - f_{\rm i}(\mathbf{v})f_{\rm n}(\mathbf{v}')]$$

$$\int d\mathbf{v}'\sigma_X|\mathbf{v}-\mathbf{v}'|f_b^{lk}(\mathbf{v}') = n_b\sigma_\star \mathsf{P}^l(\hat{\mathbf{v}})\cdot \mathsf{M}_b^{lk} S_{ab}^{lk},\ \hat{w}_a = \frac{\frac{1}{2}m_a v^2}{T_a} = s_a^2$$

$$S_{ab}^{lk} = \frac{1}{\pi^{1/2}}\underbrace{\int_0^\infty d\hat{w}'\hat{w}'^{(l+1)/2}L_k^l(\hat{w}')e^{-\hat{w}'}}_{\rm Gauss-Laguerre}\underbrace{\int_{-1}^1 d\xi' P_l(\xi')\hat{\sigma}_X|\mathbf{v}-\mathbf{v}'|}_{\rm Gauss-Legendre}$$

$$\sigma_l A_{{\rm x}ab}^{lpk}\mathsf{M}_a^{lk} = -\int d\mathbf{v}\mathsf{P}_a^{lp}(\mathbf{s}_a)\int d\mathbf{v}'\sigma_X|\mathbf{v}-\mathbf{v}'|[f_a^{lk}(\mathbf{v})f_b^0(\mathbf{v}') - f_a^{lk}(\mathbf{v}')f_b^0(\mathbf{v})]$$

$$= -\sigma_l n_b\sigma_\star \mathsf{N}_a^{lk}\frac{2}{\sqrt{\pi}}\int d\hat{w}\hat{w}^{(1+l)/2}L_p^l(\hat{w})e^{-\hat{w}}$$

$$\times [\hat{w}^{l/2}L_k^l(\hat{w})S_{ab}^{00} - S_{aa}^{lk}(T_b^a)^{3/2}e^{\hat{w}(1-T_b^a)}]$$

$$\sigma_l B_{{\rm x}ab}^{lpk}\mathsf{M}_a^{lk} = -\int d\mathbf{v}\mathsf{P}_a^{lp}(\mathbf{s}_a)\int d\mathbf{v}'\sigma_X|\mathbf{v}-\mathbf{v}'|[f_a^0(\mathbf{v})f_b^{lk}(\mathbf{v}') - f_a^0(\mathbf{v}')f_b^{lk}(\mathbf{v})]$$

$$= -\sigma_l n_a\sigma_\star \mathsf{N}_b^{lk}\frac{2}{\sqrt{\pi}}\int d\hat{w}\hat{w}^{(1+l)/2}L_p^l(\hat{w})e^{-\hat{w}}$$

$$\times [S_{ab}^{lk} - S_{aa}^{00}(T_b^a)^{3/2}e^{\hat{w}(1-T_b^a)}(\hat{w}T_b^a)^{l/2}L_k^l(\hat{w}T_b^a)]$$

Moment equations and closures

$$\text{Recombination:} \quad \int d\mathbf{v} \mathsf{P}_a^{lp} \nu_{\rm r} f_{\rm i}^0 \mathsf{P}_{\rm i}^{lk} \cdot \mathsf{M}_{\rm i}^{lk} = \sigma_l Y_{a{\rm i}}^{lpk} n_{\rm i} \mathsf{M}_{\rm i}^{lk}$$

$$\text{Ionization:} \quad \int d\mathbf{v} \mathsf{P}_a^{lp} \nu_{\rm z} f_{\rm n}^0 \mathsf{P}_{\rm n}^{lk} \cdot \mathsf{M}_{\rm n}^{lk} = \sigma_l Z_{a{\rm n}}^{lpk} n_{\rm n} \mathsf{M}_{\rm n}^{lk}$$

- Moment equations

$$\begin{pmatrix} \mathsf{c}_{\rm ii} & \mathsf{c}_{\rm in} \\ \mathsf{c}_{\rm ni} & \mathsf{c}_{\rm nn} \end{pmatrix} \begin{pmatrix} \mathsf{N}_{\rm i} \\ \mathsf{N}_{\rm n} \end{pmatrix} = - \begin{pmatrix} \mathsf{g}_{\rm i} \\ \mathsf{g}_{\rm n} \end{pmatrix}$$

$$\begin{aligned} \mathsf{c}_{\rm ii} &= \frac{\tau_{\rm ii}}{n_{\rm i}}(C_{\rm i} + A_{\rm xin} - Y_{\rm ii}) \\ \mathsf{c}_{\rm in} &= \frac{\tau_{\rm ii}}{n_{\rm n}}(B_{\rm xin} + Z_{\rm in}) \\ \mathsf{c}_{\rm ni} &= \frac{\tau_{\rm ii}}{n_{\rm i}}(B_{\rm xni} + Y_{\rm ni}) \\ \mathsf{c}_{\rm nn} &= \frac{\tau_{\rm ii}}{n_{\rm n}}(A_{\rm xni} - Z_{\rm nn}) \end{aligned}$$

- Heat flux density

$$\mathbf{h}_a = -\kappa_{a{\rm i}} \nabla T_{\rm i} - \kappa_{a{\rm n}} \nabla T_{\rm n} + \alpha_{a{\rm i}} \mathbf{V}_{\rm i} + \alpha_{a{\rm n}} \mathbf{V}_{\rm n}$$

Collision frequency and thermal conductivity

- Collision frequency (in units of s^{-1})
- For $n_\mathrm{i} = n_\mathrm{e} = f(2 \times 10^{20}\mathrm{m}^{-3})$ and $n_\mathrm{n} = (1-f)(2 \times 10^{20}\mathrm{m}^{-3})$
- For $f = 0.01/0.5/0.99$ and $T_\mathrm{i} = T_\mathrm{e} = T_\mathrm{n}$

T_i	1 eV	10 eV	100 eV	1 keV
$-\hat{C}_\mathrm{i}^{111}$	4.3E6/1.7E8/3.2E8	1.9E5/8.0E6/1.5E7	7700/3.3E5/6.5E6	296/1.3E4/2.6E4
$-\hat{A}_\mathrm{xin}^{111}$	2.1E6/1.1E6/2.1E4	5.2E6/2.6E6/5.2E4	1.2E7/5.8E6/1.2E5	2.1E7/1.1E7/2.1E5
$\hat{Y}_\mathrm{ii}^{111}$	56/2800/5500	18/882/1750	5.6/280/550	1.8/88/170
$\hat{Z}_\mathrm{in}^{111}$	0.014/0.72/1.4	1.1E4/5.3E5/1.0E6	6.4E4/3.2E6/6.3E6	4.4E4/2.2E6/4.3E6

- Thermal conductivity (in arbitrary units) for 2×2 calculation $\kappa_\mathrm{ii} = 0.282$

$$\mathbf{h}_\mathrm{i} = -\kappa_\mathrm{ii}\nabla T_\mathrm{i} - \kappa_\mathrm{in}\nabla T_\mathrm{n}$$
$$\mathbf{h}_\mathrm{n} = -\kappa_\mathrm{ni}\nabla T_\mathrm{i} - \kappa_\mathrm{nn}\nabla T_\mathrm{n}$$

T_i	1 eV	10 eV	100 eV	1 keV
κ_ii	0.282/0.282/0.282	0.282/0.282/0.282	0.282/0.282/0.282	0.282/0.282/0.282
κ_in	0.282/0.282/0.282	0.282/0.282/0.282	0.282/0.282/0.282	0.282/0.282/0.282
κ_ni	27.9/0.283/.00360	23.1/0.234/.00244	18.0/0.182/.00185	23.2/0.234/.00237
κ_nn	86.6/46.2/43.7	24.0/0.964/0.708	18.0/0.193/.0122	23.2/0.234/.00265

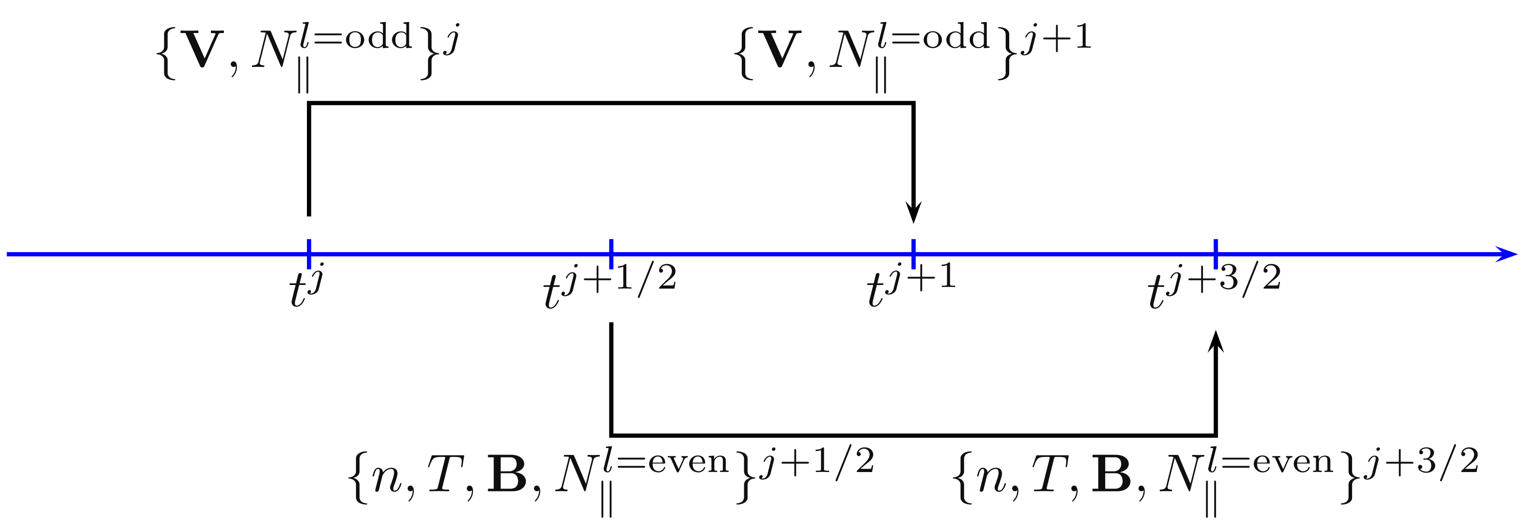
Parallel moment equations in NIMROD (Hankyu Lee)

- Total velocity moment equations (no $\nabla\mathbf{V}$ coupled terms)

$$\partial_t \bar{N}_{\|}^{lp} + \sum_k \Big[\bar{\Xi}^l_{pk} (\partial_t \ln T) \bar{N}_{\|}^{lk} + v_T (\bar{\Psi}^{l+}_{pk} \partial_{\|}^{l+} + \bar{\Phi}^{l+}_{pk} \partial_{\|} \ln T + \frac{q}{2T} \bar{\Theta}^{l+}_{pk} E_{\|}) \bar{N}_{\|}^{l+1,k}$$

$$+ v_T (\bar{\Psi}^{l-}_{pk} \partial_{\|}^{l-} + \bar{\Phi}^{l-}_{pk} \partial_{\|} \ln T + \frac{q}{2T} \bar{\Theta}^{l-}_{pk} E_{\|}) \bar{N}_{\|}^{l-1,k} \Big] = \sum C^{lpk} \bar{N}_{\|}^{lk}$$

- NIMROD's flow velocity $(\mathbf{V})$ is staggered 1/2 step from number density (n), temperature (T), and magnetic field $(\mathbf{B})$ [Sovinec *et al*, International Sherwood Fusion Theory Conference 2008]



Future work

- Closures for e-i-n plasmas
  - Complete closures for ions and neutrals
  - Compute the cross section for the electron impact ionization (atomic physics)
  - Compute electron closures with neutrals
- Applications of integral closures
  - Radial transport due to magnetic field fluctuation
  - Neoclassical toroidal viscosity
  - Generalize to an inhomogeneous magnetic field
- Parallel closures for nonlinear and non-adiabatic phenomena
  - Implement parallel moment equations in NIMROD

Thanks!

Enjoy the Moments

and the NIMROD open!