# Implementing parallel moment equations in NIMROD

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NIMROD MEETING

August 2, 2018, Logan, UT

## Motivation

- Implementing general moment equations in NIMROD (Integrating over velocity variables).
- Avoiding integrals along field lines in NIMROD.
- In a low collision region, trying to capture some kinetic effects by using fluid models in NIMROD.
- Trying to incorporate time-dependent and non-linear effects in integral closures under large variations of temperature and magnetic fields along field lines.

# General parallel moment equation

- Take gyro-average and linearize

$$v_{\parallel}\frac{\partial \overline{f_a^{\mathrm{N}}}}{\partial \ell} = \overline{C_{\mathrm{L}}(f_a^{\mathrm{N}})} - v_{\parallel}\frac{\partial \overline{f_a^{\mathrm{M}}}}{\partial \ell} + \overline{C_{\mathrm{L}}(f_a^{\mathrm{M}})}$$

- Moment expansion

$$\overline{f_a^{\mathrm{N}}} = \hat{f}_a^{\mathrm{m}} \sum_{lk\neq \mathsf{M}} \hat{P}_a^{lk} n_a^{lk}$$

with $\hat{f}_a^{\mathrm{m}} = \frac{1}{\pi^{3/2} v_{Ta}^3}\exp(-s_a^2)$ and $\mathbf{s}_a = \frac{\mathbf{v}}{v_{Ta}}$

$$\hat{P}_a^{lk} = \frac{1}{\sqrt{\bar{\sigma}_l \lambda_k^l}} s_a^l P_l(\xi) L_k^{(l+1/2)}(s_a^2)$$

where $\xi = v_{\parallel}/v$,$P_l$ is a Legendre polynomial.

- Multiply $\hat{P}^{jp}$and integrate over velocity space

$$v_T \sum_{lk\neq \mathrm{M}} \psi^{jp,lk}\frac{\partial n^{lk}}{\partial \ell} = \frac{1}{\tau}\sum_{lk\neq \mathrm{M}} c^{jp,lk} n^{lk} + \frac{1}{\tau} g^{jp}$$

# Matrix form for parallel moments

- In matrix form,

$$\Psi \frac{\partial [n]}{\partial \ell} + C[n] = [g]$$

  - $[n]$: unknown (initialized as 0)
  - $[g]$: thermodynamic drives (rhs)
  - $[\Psi \frac{\partial}{\partial \eta} - C]$ :operator

$$\left(1 + \Delta t f_\psi \left[\psi\right] \partial_{||} + \Delta t f_c \left[c\right]\right) \left[\Delta n\right]_{\text{pass}} = \Delta t(\left[c\right] \left[n\right]^k + \left[\psi\right] \partial_{||} \left[n\right]^k + \left[g\right])$$

- $\Psi$,$C$ are constant matrices.
- $\frac{\partial}{\partial l} \rightarrow \alpha_i (\mathbf{b} \cdot \nabla) \alpha_j$

# Truncation of the system

- Truncate the system to $N = LK$ moments with $l = 0, 1, ..., L-1$ and

$$k = \begin{cases} 2, 3, ..., K+1, & l = 0 \\ 1, 2, ..., K, & l = 1 \\ 0, 1, ..., K-1, & l > 1 \end{cases}$$

- The unknown $[n]$ has $N = LK$ components, and begins from $n[1] \rightarrow n^{02}$to $n[N] \rightarrow n^{(L-1)(K-1)}$

- (e.g.) $(L, K) = (4, 4)$
    - $n[1] \rightarrow n^{02}$,$n[4] \rightarrow n^{05}$
    - $n[5] \rightarrow n^{11}$,$n[8] \rightarrow n^{14}$
    - $n[9] \rightarrow n^{20}$,$n[12] \rightarrow n^{23}$,...,$n[16] \rightarrow n^{33}$

# Thermodynamic drives

$$\frac{g^{1k}}{\tau} = hsw * \delta_{1k} v_T \frac{\sqrt{5}}{2} \frac{n}{T} \partial_\parallel T + Rsw * \sqrt{2} Z a_{ei}^{1k0} n \frac{V_{ei,\parallel}}{\tau v_T}$$

$$\frac{g^{20}}{\tau} = -\frac{\sqrt{3}}{2} psw * nW_\parallel$$

- Set switch variables to control each drives.

$$\begin{pmatrix} 0... \\ ...0 \\ hsw * \delta_{1k} v_T \frac{\sqrt{5}}{2} \frac{n}{T} \partial_\parallel T + Rsw * \sqrt{2} Z a_{ei}^{110} n \frac{V_{ei,\parallel}}{\tau v_T} \\ Rsw * \sqrt{2} Z a_{ei}^{1(2,\ldots,K)0} n \frac{V_{ei,\parallel}}{\tau v_T} \\ -\frac{\sqrt{3}}{2} psw * nW_\parallel \\ ...0 \end{pmatrix}$$

# $\Psi$matrix

- $v_T \sum_{lk \neq \mathrm{M}} \overline{\Psi}^{l+}_{pk} \partial_\parallel n_\parallel^{l+1,k} + \overline{\Psi}^{l-}_{pk} \partial_\parallel n_\parallel^{l-1,k}$
- $\overline{\Psi}^{l+}_{pk} = \frac{l+1}{\sqrt{(2l+1)(2l+3)}} [\sqrt{l+p+\frac{3}{2}} \delta_{p,k} - \sqrt{p} \delta_{p-1,k}]$
- $\overline{\Psi}^{l-}_{pk} = \frac{l}{\sqrt{(2l-1)(2l+1)}} [\sqrt{l+p+\frac{1}{2}} \delta_{p,k} - \sqrt{p+1} \delta_{p,k-1}]$

$$\begin{pmatrix}
 & & & \overline{\Psi}^{0+}_{21} & \overline{\Psi}^{0+}_{22} & & & & \\
 & & & & \overline{\Psi}^{0+}_{32} & \overline{\Psi}^{0+}_{33} & & & \\
 & & & & & \overline{\Psi}^{0+}_{43} & & & \\
\overline{\Psi}^{1-}_{12} & & & & & & \overline{\Psi}^{1+}_{10} & \overline{\Psi}^{1+}_{11} & \\
\overline{\Psi}^{1-}_{22} & \overline{\Psi}^{1-}_{23} & & & & & & \cdots & \cdots \\
 & \cdots & \cdots & & & & & & \cdots \\
 & & \cdots & \cdots & & & & & \\
 & & & & & & & & \\
 & & & & & & & &
\end{pmatrix}
\begin{matrix} (0,2) \\ (0,3) \\ (0,4) \\ (1,1) \\ (1,2) \\ (1,3) \\ (2,0) \\ (2,1) \\ (2,2) \end{matrix}$$

## Closures for sinusoidal drives

- Parallel closures are related to the general moments by

$$
\begin{aligned}
h_{\|} &= -\frac{\sqrt{5}}{2} v_T T n^{11} \\
R_{\|} &= \frac{m_e v_{T,e}}{\tau_{ei}} \left[ -n_e \frac{V_{ei,\|}}{v_{T,e}} + \frac{1}{\sqrt{2}} \sum_{k=1} a_{ei}^{10k} n^{1k} \right] \\
\pi_{\|} &= \frac{2}{\sqrt{3}} T n^{20}
\end{aligned}
$$

- For sinusoidal drives, $T = T_0 + T_1 \sin\varphi$, $V_{||} = V_0 + V_1 \sin\varphi$, $V_{ei,||} = V_{ei} \cos\varphi$, where $\varphi = 2\pi\ell/\lambda + \varphi_0$.

$$
\begin{aligned}
h_{\|}(\ell) &= -\frac{1}{2} n T_1 v_T \hat{h}_h \cos\varphi + n T_0 V_{ei} \hat{h}_R \cos\varphi - n T_0 V_1 \hat{h}_\pi \sin\varphi \\
R_{\|}(\ell) &= -n T_1 \frac{2\pi}{\lambda} \hat{R}_h \cos\varphi - \frac{m n V_{ei}}{\tau_{ei}} \hat{R}_R \cos\varphi - n m V_1 \frac{2\pi v_T}{\lambda} \hat{R}_\pi \sin\varphi \\
\pi_{\|}(\ell) &= -n T_1 \hat{\pi}_h \sin\varphi + 2 n T_0 \frac{V_{ei}}{v_T} \hat{\pi}_R \sin\varphi - n T_0 \frac{V_1}{v_T} \hat{\pi}_\pi \cos\varphi
\end{aligned}
$$

- Calculated closures are compared with theoretical values.

# NIMROD Fluid equations and closures

- Fluid closures are calculated by moment equations.

$$
\begin{aligned}
\rho D_t \mathbf{V} &= \mathbf{J}\times\mathbf{B} - \nabla p - \nabla\cdot\boldsymbol{\pi} \\
\frac{n}{\gamma-1} D_t T &= -p\nabla\cdot\mathbf{V} - \boldsymbol{\pi} : \nabla V - \nabla\cdot\mathbf{h} + Q \\
\mathbf{E} &= -\mathbf{V}\times\mathbf{B} + \frac{1}{ne}\mathbf{J}\times\mathbf{B} + \frac{m_\mathrm{e}}{ne^2}\left[\frac{\partial\mathbf{J}}{\partial t} + \nabla\cdot(\mathbf{JV}+\mathbf{VJ}) - \frac{e}{m_\mathrm{e}}(\nabla p)\right] \\
&\quad -\frac{1}{ne}\nabla\cdot\boldsymbol{\pi} + \eta\mathbf{J}
\end{aligned}
$$

- Collisional momentum exchange

$$
\begin{aligned}
\eta J_{||} &\rightarrow \frac{R_{||}}{n_\mathrm{e} e} \\
\frac{R_{||}}{n_\mathrm{e} e} &= \frac{m_\mathrm{e} v_{T,\mathrm{e}}}{n_\mathrm{e} e \tau_\mathrm{ei}}\left[-n_\mathrm{e}\hat{V}_{\mathrm{ei},||} + \frac{1}{\sqrt{2}}\sum_{k=1} a_\mathrm{ei}^{10k} n^{1k}\right]
\end{aligned}
$$

# Algorithm

- By using whole $N = LK$ moments in a single vector, we can use the matrix form of $\Psi$.

$$(1 + \Delta t f_\psi [\psi] \partial_\| + \Delta t f_c [c]) [\Delta n]_{\text{pass}} = \Delta t([c] [n]^k + [\psi] \partial_\| [n]^k + [g])$$

Tested in a single time shot by taking off mass matrix term

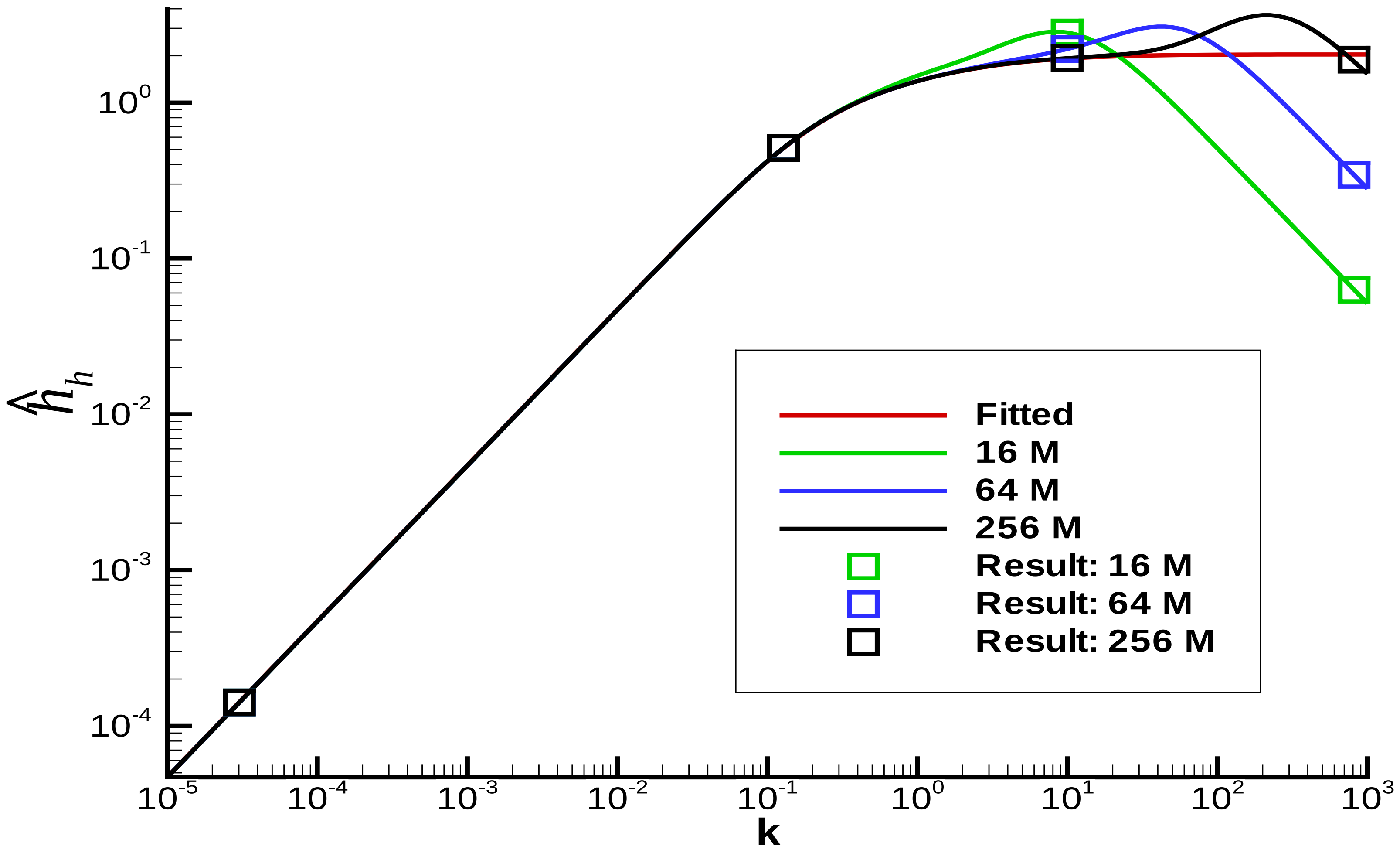
- Alternatively, an algorithm for separated vectors of odd and even $l$ moments is possible (for memory complexity).

$$\begin{aligned} \partial_t [n]^{\text{even}} &= [c] [n]^{\text{even}} + [\psi] [n]^{\text{odd}} + [g(n, T, \nabla \mathbf{V})] \\ \partial_t [n]^{\text{odd}} &= [c] [n]^{\text{odd}} + [\psi] [n]^{\text{even}} + [g(n, T, \mathbf{V}_{\text{ei}})] \end{aligned}$$

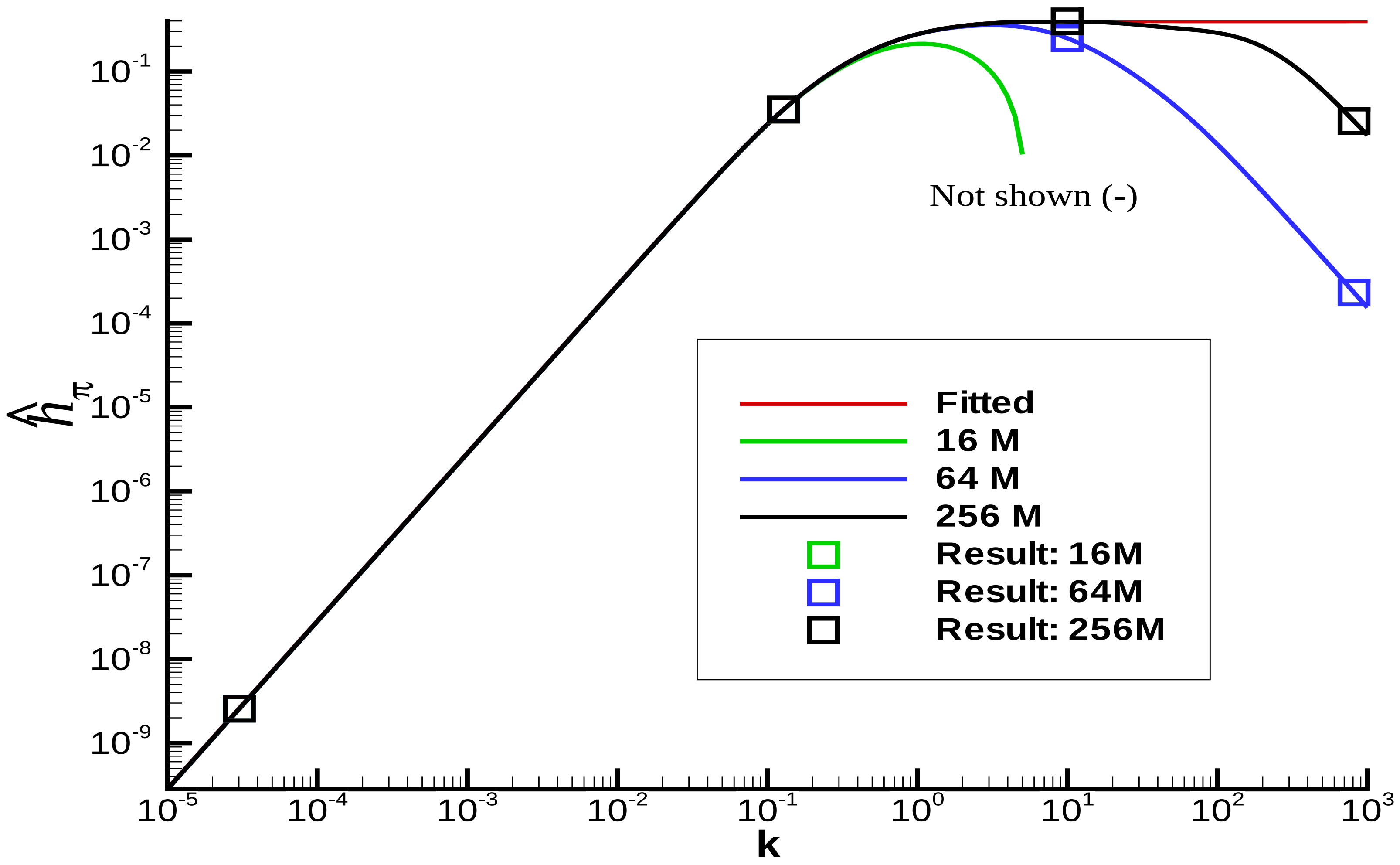
Predictor-corrector time advance $t^k \rightarrow t^{k+1}$

$$(1 + \Delta t f_{[n]} [c]) [\Delta n]_{\text{pass}} = \Delta t([c] [n]^k + [\psi] [n]^{k+1/2} + [g])$$

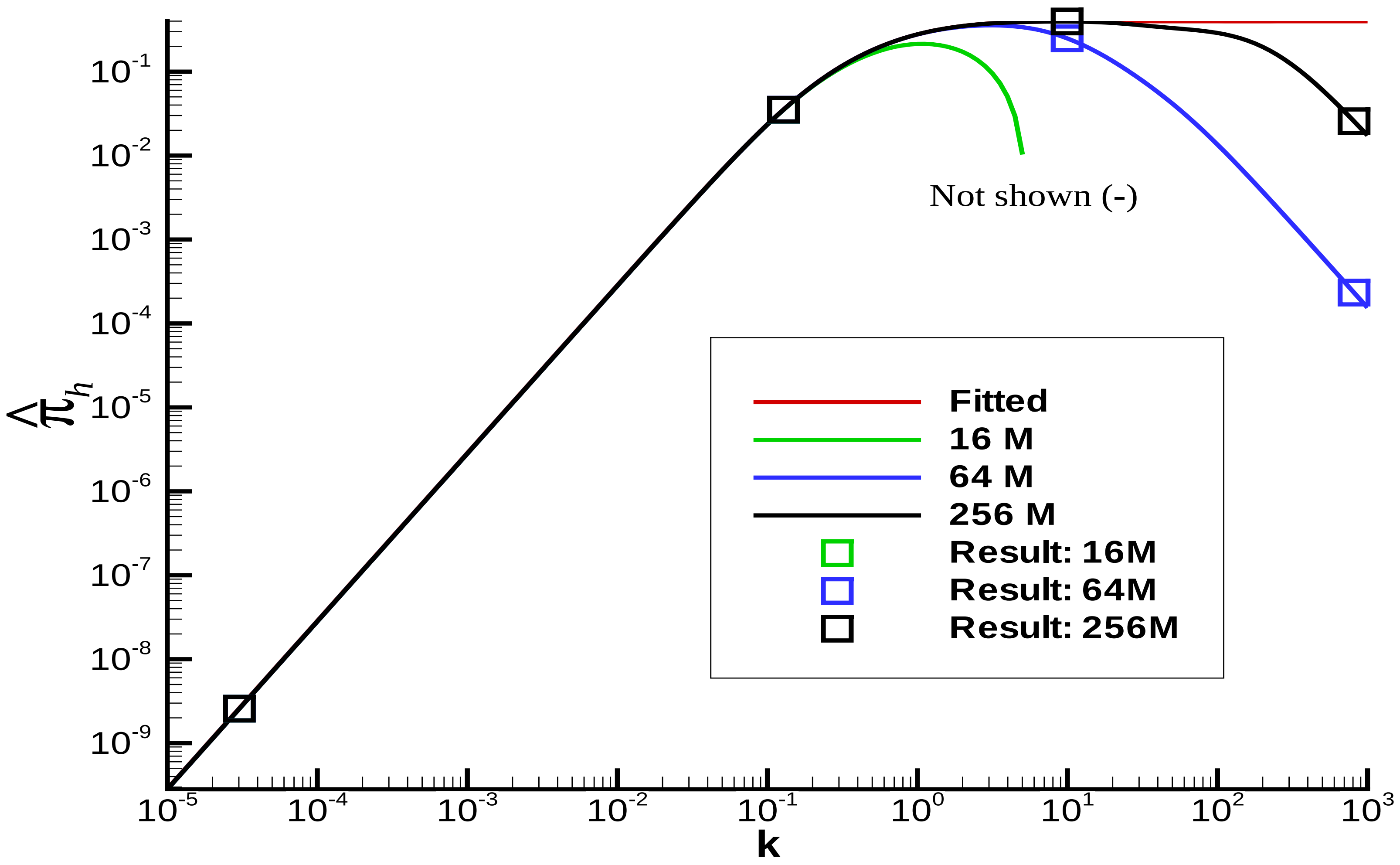
## Ion $\hat{h}_h$ closure ($T_\mathrm{i} = T_\mathrm{e}$)



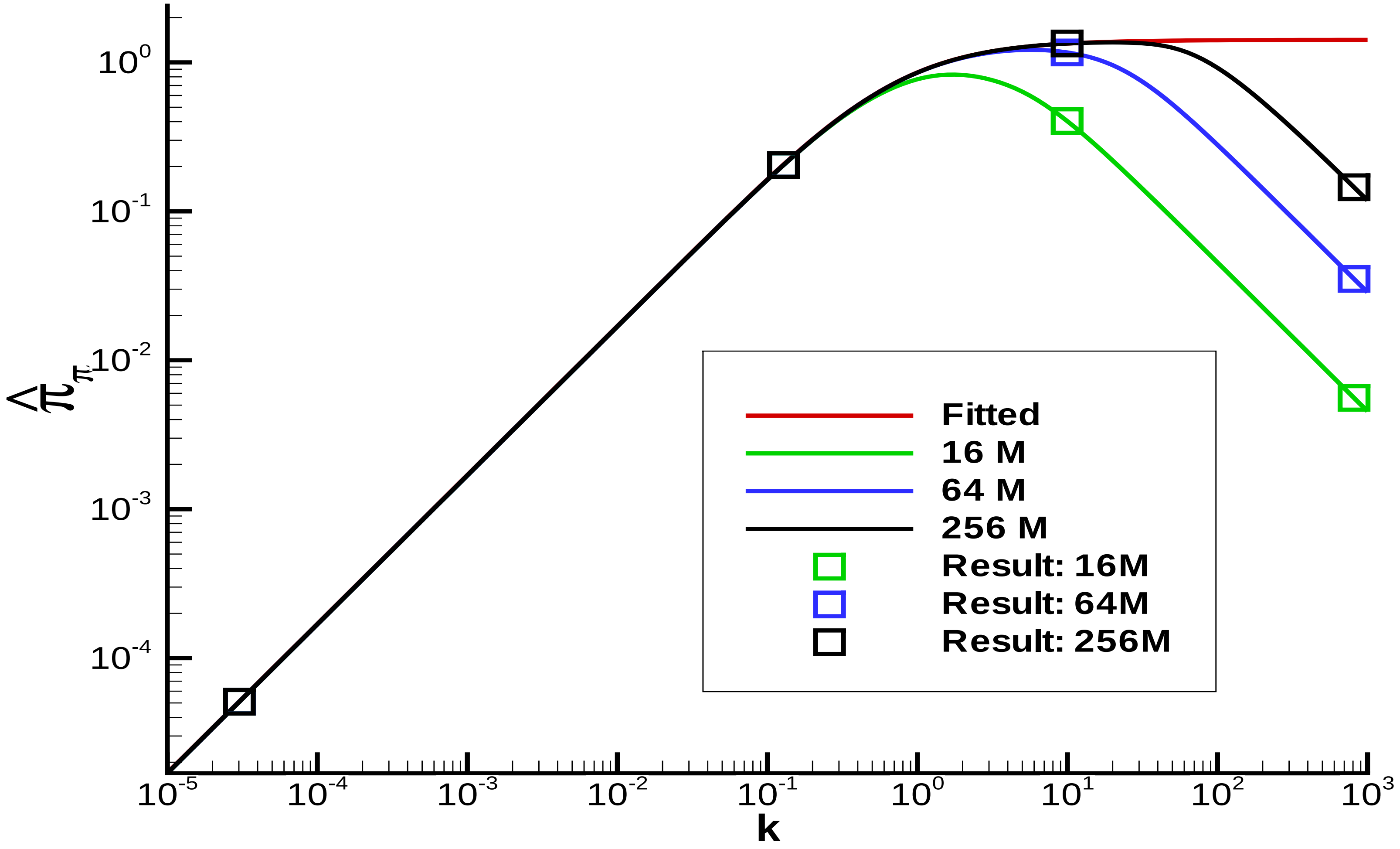
# Ion $\hat{h}_\pi$ closure $(T_\mathrm{i} = T_\mathrm{e})$



# Ion $\hat{\pi}_h$ closure ($T_\mathrm{i} = T_\mathrm{e}$)



# Ion $\hat{\pi}_\pi$ closure ($T_\mathrm{i} = T_\mathrm{e}$)



# Future plan

$$\partial_t \bar{N}_{\parallel}^{lp} + \sum_k \Bigg[ \bar{\Xi}_{pk}^{l} (\partial_t \ln T) \bar{N}_{\parallel}^{lk} + v_T (\bar{\Psi}_{pk}^{l+} \partial_{\parallel}^{l+} + \bar{\Phi}_{pk}^{l+} \partial_{\parallel} \ln T + \frac{q}{2T} \bar{\Theta}_{pk}^{l+} E_{\parallel}) \bar{N}_{\parallel}^{l+1,k}$$

$$+ v_T (\bar{\Psi}_{pk}^{l-} \partial_{\parallel}^{l-} + \bar{\Phi}_{pk}^{l-} \partial_{\parallel} \ln T + \frac{q}{2T} \bar{\Theta}_{pk}^{l-} E_{\parallel}) \bar{N}_{\parallel}^{l-1,k} \Bigg] = \sum C^{lpk} \bar{N}_{\parallel}^{lk} + C^{(2)} + G^{lp}$$

$$\partial_{\parallel}^{l+} = \partial_{\parallel} - \frac{l+2}{2} (\partial_{\parallel} \ln B)$$

$$\partial_{\parallel}^{l-} = \partial_{\parallel} + \frac{l-1}{2} (\partial_{\parallel} \ln B)$$

- Adding temperature gradient coupling.
- Adding magnetic field gradient coupling (green terms).