Closure theory for high-collisionality multi-ion plasmas*

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NIMROD Team Meeting

August 2 2023, Madison WI

*PPCF **65**, 075014 (2023). Research supported by the U.S. DOE under Grant Nos. DE-SC0022048 and DE-FG02-04ER54746.

Outline

- A fluid model (5 moment, multiple species) and closures
- Moment expansion and collisional moments
- Moment equations for closures
- How to solve the moment equations: geometric method
- Electron closures for multiple ion species
- Ion closures for multiple ion species
- Deuterium-Carbon example calculations
- Convergence study by increasing the number of moments
- Two-temperature dependence of closure coefficients
- Conclusions and future work

A fluid model and closures

- Kinetic equation for a distribution function of species a, $D_a f_a = C f_a$,

$$\frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \nabla f_a + \frac{q_a}{m_a}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial}{\partial \mathbf{v}} f_a = \sum_b C(f_a, f_b) \qquad \text{(KE)}$$

- Five moment (density n_a, flow velocity $\mathbf{V}_a$, and temperature T_a) fluid model

$$\int \mathrm{d}^3 v \text{(KE)}: \boxed{\mathrm{d}_a n_a + n_a \nabla \cdot \mathbf{V}_a = 0}$$

$$\int \mathrm{d}^3 v m_a \mathbf{v} \text{(KE)}: \boxed{m_a n_a \mathrm{d}_a \mathbf{V}_a - n_a q_a (\mathbf{E} + \mathbf{V}_a \times \mathbf{B}) + \nabla p_a + \nabla \cdot \boldsymbol{\pi}_a = \mathbf{R}_a}$$

$$\int \mathrm{d}^3 v \frac{1}{2} m_a w_a^2 \text{(KE)}: \boxed{\frac{3}{2} n_a \mathrm{d}_a T_a + n_a T_a \nabla \cdot \mathbf{V}_a + \nabla \cdot \mathbf{h}_a + \nabla \mathbf{V}_a : \boldsymbol{\pi}_a = Q_a}$$

where $\mathrm{d}_a = \frac{\partial}{\partial t} + \mathbf{V}_a \cdot \nabla$, $p_a = n_a T_a$, and $\mathbf{w}_a = \mathbf{v} - \mathbf{V}_a$

- Closure relations express closure variables $\{\mathbf{h}_a, \boldsymbol{\pi}_a, \mathbf{R}_a, Q_a\}$ in terms of fluid variables $\{n_a, \mathbf{V}_a, T_a\}$

Moment expansion and collisional moments

- Moment expansion in terms of irreducible tensorial Hermite polynomials $\hat{\mathsf{p}}_a^{lk}$

$$f_a = f_a^{\mathrm{M}} \left(1 + \sum_{lk \neq \mathrm{M}} \hat{\mathsf{p}}_a^{lk} \cdot \mathsf{m}_a^{lk} \right) = f_a^{\mathrm{M}}[n_a, \mathbf{V}_a, T_a] + f_a^{\mathrm{N}}[\mathsf{m}_a^{lk}]$$

- Closures: heat flux $\mathbf{h}_a$, viscosity $\boldsymbol{\pi}_a$, collisional heating Q_a, and friction $\mathbf{R}_a$

$$\mathbf{h}_a = -\frac{\sqrt{5}}{2} n_a T_a v_{Ta} \mathsf{m}_a^{11}, \quad \boldsymbol{\pi}_a = \sqrt{2} n_a T_a \mathsf{m}_a^{20}$$

$$Q_a = \sum_b \left[\frac{n_a}{\tau_{ab}} \frac{3X_{ab}^{3/2}}{\mu_{ab}} (T_b - T_a) - \sqrt{\frac{3}{2}} T_a \sum_{k=2}^{\infty} \frac{n_a}{\tau_{ab}} (a_{ab}^{01p} \mathsf{m}_a^{0k} + b_{ab}^{01p} \mathsf{m}_b^{0k}) \right]$$

$$\mathbf{R}_a = \sum_b \left[\frac{m_a n_a}{\tau_{ab}} \left(X_{ab}^{3/2} (1 + \mu_{ba}) \mathbf{V}_{ba} + \frac{v_{Ta}}{\sqrt{2}} \sum_{k=1}^{\infty} (a_{ab}^{10k} \mathsf{m}_a^{1k} + b_{ab}^{10k} \mathsf{m}_b^{1k}) \right) \right]$$

where $X_{ab}^{3/2} = (1 + \theta_{ab}/\mu_{ab})^{-1}$, $\mu_{ab} = m_b/m_a$, $\theta_{ab} = T_b/T_a$, $\mathbf{V}_{ba} = \mathbf{V}_b - \mathbf{V}_a$

- Linearized (L) collisional moments (τ_{ab} collision time)

$$\int \mathrm{d}^3 v \hat{\mathsf{p}}_a^{lp} C\left(f_a^{\mathrm{M}} \hat{\mathsf{p}}_a^{lk} \cdot \mathsf{m}_a^{lk}, f_b^{\mathrm{M}}\right) \stackrel{\mathrm{L}}{=} \frac{n_a}{\tau_{ab}} a_{ab}^{lpk} [\mu_{ab}, \theta_{ab}]\, \mathsf{m}_a^{lk}$$

$$\int \mathrm{d}^3 v \hat{\mathsf{p}}_a^{lp} C\left(f_a^{\mathrm{M}}, f_b^{\mathrm{M}} \hat{\mathsf{p}}_b^{lk} \cdot \mathsf{m}_b^{lk}\right) \stackrel{\mathrm{L}}{=} \frac{n_a}{\tau_{ab}} b_{ab}^{lpk} [\mu_{ab}, \theta_{ab}]\, \mathsf{m}_b^{lk}$$

Moment equations for closures

- Moment equations: take moments $\int \mathrm{d}^3 v \hat{\mathsf{p}}_a^{lp}[D_a f_a = \sum_b C(f_a, f_b)]$

$$D_a f_a^{\mathrm{M}} + D_a f_a^{\mathrm{N}} = \sum_b \left[C(f_a^{\mathrm{M}}, f_b^{\mathrm{M}}) + C(f_a^{\mathrm{N}}, f_b^{\mathrm{M}}) + C(f_a^{\mathrm{M}}, f_b^{\mathrm{N}})\right]$$

- For high collisionality

$$\xcancel{\left(\frac{\partial}{\partial t}, v_{Ta}\nabla, \frac{2q_a\mathbf{E}}{m_a v_{Ta}}\right)} \mathsf{m}_a + \Omega_a \mathbf{b}\breve{\times}\mathsf{m}_a^{lp} = \sum_b \boxed{\frac{1}{\tau_{ab}}} \left(a_{ab}^{lpk}\mathsf{m}_a^{lk} + b_{ab}^{lpk}\mathsf{m}_b^{lk}\right) + \mathcal{G}_a^{lp}$$

Multiply τ_{aa}

$$r_a \mathbf{b}\breve{\times}\mathsf{m}_a^{lp} = \sum_b z_{ab}\left(a_{ab}^{lpk}\mathsf{m}_a^{lk} + b_{ab}^{lpk}\mathsf{m}_b^{lk}\right) + \mathsf{g}_a^{lp}$$

where $r_a = \Omega_a \tau_{aa}$ (Hall parameter), $z_{ab} = \tau_{aa}/\tau_{ab}$, and nonvanishing drives

$$\mathsf{g}_a^{1p} = \delta_{p1}\frac{\sqrt{5}}{2}\frac{n_a v_{Ta}\tau_{aa}}{T_a}\nabla T_a + \sqrt{2} n_a \sum_{b\neq a} z_{ab} b_{ab}^{1p0}\frac{\mathbf{V}_{ba}}{v_{Tb}}$$

$$\mathsf{g}_a^{20} = -\frac{1}{\sqrt{2}} n_a \tau_{aa}\mathsf{W}_a,\ \mathsf{W}_a = \nabla\mathbf{V}_a + (\nabla\mathbf{V}_a)^{\mathrm{T}} - \frac{2}{3}\mathsf{I}\nabla\cdot\mathbf{V}_a$$

How to obtain closures: moment approach

- Moment equations for S ion species in matrix form

$$\begin{pmatrix} r_\mathrm{e}\mathbf{b}\breve{\times}\mathsf{m}_\mathrm{e}^l \\ r_1\mathbf{b}\breve{\times}\mathsf{m}_1^l \\ r_2\mathbf{b}\breve{\times}\mathsf{m}_2^l \\ r_S\mathbf{b}\breve{\times}\mathsf{m}_S^l \end{pmatrix} = \begin{pmatrix} \mathsf{c}_\mathrm{e}^l & 0 & 0 & 0 \\ 0 & \mathsf{c}_1^l & \mathsf{c}_{12}^l & \mathsf{c}_{1S}^l \\ 0 & \mathsf{c}_{21}^l & \mathsf{c}_2^l & \mathsf{c}_{2S}^l \\ 0 & \mathsf{c}_{S1}^l & \mathsf{c}_{S2}^l & \mathsf{c}_S^l \end{pmatrix} \begin{pmatrix} \mathsf{m}_\mathrm{e}^l \\ \mathsf{m}_1^l \\ \mathsf{m}_2^l \\ \mathsf{m}_S^l \end{pmatrix} + \begin{pmatrix} \mathsf{g}_\mathrm{e}^l \\ \mathsf{g}_1^l \\ \mathsf{g}_2^l \\ \mathsf{g}_S^l \end{pmatrix}$$

$$\Rightarrow\ \mathsf{R}\mathbf{b}\breve{\times}\mathsf{M}^l = \mathsf{C}^l\mathsf{M}^l + \mathsf{G}^l$$

where $c_a^{lpq} = \sum_{b=\mathrm{e}}^{S} a_{ab}^{lpq} + b_{aa}^{lpq}$ and $\mathsf{c}_{a\neq b}^{lpq} = z_{ab} b_{ab}^{lpq}$

- K-moment calculations: convergence checked by increasing K

$$\mathsf{m}_a^0 = \begin{pmatrix} \mathsf{m}_a^{02} \\ \mathsf{m}_a^{03} \\ \vdots \\ \mathsf{m}_a^{0,K+1} \end{pmatrix},\ \mathsf{m}_a^1 = \begin{pmatrix} \mathsf{m}_a^{11} \\ \mathsf{m}_a^{12} \\ \vdots \\ \mathsf{m}_a^{1K} \end{pmatrix},\ \mathsf{M}_a^{l\geq 2} = \begin{pmatrix} \mathsf{m}_a^{l0} \\ \mathsf{m}_a^{l1} \\ \vdots \\ \mathsf{m}_a^{l,K-1} \end{pmatrix}$$

- Solve the linear system for closures M^l

$$\mathsf{C}^l\mathsf{M}^l - \mathsf{R}\mathbf{b}\breve{\times}\mathsf{M}^l = -\mathsf{G}^l\ \Rightarrow\ \mathsf{M}^l = -\left(\mathsf{C}^l - \mathsf{R}\mathbf{b}\breve{\times}\right)^{-1}\mathsf{G}^l$$

How to solve the linear system $l = 1$: geometric method

- Vector decomposition $\mathbf{V} = \mathbf{V}_{\parallel} + \mathbf{V}_{\perp}$

$$\begin{aligned}
\mathbf{V}_{\parallel} &= \mathsf{b}_{\parallel}\mathbf{V} = \mathbf{b}\mathbf{b}\cdot\mathbf{V} \\
\mathbf{V}_{\perp} &= \mathsf{b}_{\perp}\mathbf{V} = -\mathbf{b}\times(\mathbf{b}\times\mathbf{V}) = (\mathsf{I}-\mathbf{b}\mathbf{b})\cdot\mathbf{V} \\
\mathbf{V}_{\times} &= \mathsf{b}_{\times}\mathbf{V} = \mathbf{b}\times\mathbf{V} = -\mathbf{V}\times\mathbf{b}
\end{aligned}$$

- Act $\mathsf{b}_{\parallel}$ on $\mathsf{C}^1\mathsf{M}^1 - \mathsf{R}\mathbf{b}\times\mathsf{M}^1 = -\mathsf{G}^1$

$$\mathsf{C}^1\mathsf{M}^1_{\parallel} = -\mathsf{G}^1_{\parallel} \;\Rightarrow\; \mathsf{M}^1_{\parallel} = -\left(\mathsf{C}^1\right)^{-1}\mathsf{G}^1_{\parallel}$$

- Act $\mathsf{b}_{\perp}$ and $\mathsf{b}_{\times}$ on $\mathsf{C}\mathsf{M}^1 - \mathsf{R}\mathbf{b}\times\mathsf{M}^1 = -\mathsf{G}^1$

$$\begin{array}{l} \mathsf{C}^1\mathsf{M}^1_{\perp} - \mathsf{R}\mathsf{M}^1_{\times} = -\mathsf{G}^1_{\perp} \\ \mathsf{C}^1\mathsf{M}^1_{\times} + \mathsf{R}\mathsf{M}^1_{\perp} = -\mathsf{G}^1_{\times} \end{array} \;\Rightarrow\; \begin{bmatrix} \mathsf{C}^1 & -\mathsf{R} \\ \mathsf{R} & \mathsf{C}^1 \end{bmatrix}\begin{bmatrix} \mathsf{M}^1_{\perp} \\ \mathsf{M}^1_{\times} \end{bmatrix} = -\begin{bmatrix} \mathsf{G}^1_{\perp} \\ \mathsf{G}^1_{\times} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \mathsf{M}^1_{\perp} \\ \mathsf{M}^1_{\times} \end{bmatrix} = -\frac{1}{\mathsf{R}\mathsf{C}^1\mathsf{R}^{-1}\mathsf{C}^1 + \mathsf{R}^2}\begin{bmatrix} \mathsf{R}\mathsf{C}^1\mathsf{R}^{-1} & \mathsf{R} \\ -\mathsf{R} & \mathsf{R}\mathsf{C}^1\mathsf{R}^{-1} \end{bmatrix}\begin{bmatrix} \mathsf{G}^1_{\perp} \\ \mathsf{G}^1_{\times} \end{bmatrix}$$

- Solution $\mathsf{M}^1 = \mathsf{M}^1_{\parallel} + \mathsf{M}^1_{\perp}$

$$\mathsf{M}^1 = -\left(\mathsf{C}^1\right)^{-1}\mathsf{G}^1_{\parallel} - \frac{1}{\mathsf{R}\mathsf{C}^1\mathsf{R}^{-1}\mathsf{C}^1 + \mathsf{R}^2}\left(\mathsf{R}\mathsf{C}^1\mathsf{R}^{-1}\mathsf{G}^1_{\perp} + \mathsf{R}\mathsf{G}^1_{\times}\right)$$

How to solve the linear system $l = 2$: geometric method

- Rank-2 tensor decomposition $\mathsf{W} = \mathsf{W}_{\parallel\parallel} + 2\mathsf{W}_{\parallel\perp} + \mathsf{W}_{\perp\perp}$

$$\mathsf{W}_{AB} = \tfrac{1}{2}(\mathsf{b}_A \otimes \mathsf{b}_B + \mathsf{b}_B \otimes \mathsf{b}_A)\mathsf{W}, \text{ where } A, B = \parallel, \perp, \times$$

$$\text{e.g. } \mathsf{W}_{\parallel\perp} = \tfrac{1}{2}\left[\mathbf{bb}\cdot\mathsf{W}\cdot(\mathsf{I}-\mathbf{bb}) + (\mathsf{I}-\mathbf{bb})\cdot\mathsf{W}\cdot\mathbf{bb}\right]$$

$$\mathsf{W}_{\times\times} = -\mathbf{b}\times\mathsf{W}\times\mathbf{b}$$

- Defining $\mathsf{W}_{(0)} = \mathsf{W}_{\parallel\parallel} + \frac{1}{2}(\mathsf{W}_{\times\times} + \mathsf{W}_{\perp\perp})$, $\mathsf{W}_{(1)} = \frac{1}{2}(\mathsf{W}_{\perp\perp} - \mathsf{W}_{\times\times})$, $\mathsf{W}_{(2)} = 2\mathsf{W}_{\parallel\perp}$, $\mathsf{W}_{(3)} = \mathsf{W}_{\times\perp}$, $\mathsf{W}_{(4)} = 2\mathsf{W}_{\parallel\times}$, decompose the $l = 2$ equation $\mathsf{C}^2\mathsf{M}^2 - \mathsf{R}(\mathbf{b}\times\mathsf{M}^2 - \mathsf{M}^2\times\mathbf{b}) = -\mathsf{G}^2$ into

$$\mathsf{C}^2\mathsf{M}^2_{(0)} = -\mathsf{G}^2_{(0)}$$

$$\begin{bmatrix} \mathsf{C}^2 & -\mathsf{R} \\ \mathsf{R} & \mathsf{C}^2 \end{bmatrix}\begin{bmatrix} \mathsf{M}^2_{(2)} \\ \mathsf{M}^2_{(4)} \end{bmatrix} = -\begin{bmatrix} \mathsf{G}^2_{(2)} \\ \mathsf{G}^2_{(4)} \end{bmatrix}$$

$$\begin{bmatrix} \mathsf{C}^2 & -2\mathsf{R} \\ 2\mathsf{R} & \mathsf{C}^2 \end{bmatrix}\begin{bmatrix} \mathsf{M}^2_{(1)} \\ \mathsf{M}^2_{(3)} \end{bmatrix} = -\begin{bmatrix} \mathsf{G}^2_{(1)} \\ \mathsf{G}^2_{(3)} \end{bmatrix} \text{ : same as the vector equations}$$

- Solution

$$\mathsf{M}^2 = \mathsf{M}^2_{\parallel\parallel} + 2\mathsf{M}^2_{\parallel\perp} + \mathsf{M}^2_{\perp\perp} = \mathsf{M}^2_{(0)} + \mathsf{M}^2_{(1)} + \mathsf{M}^2_{(2)}$$

Electron parallel closures for multiple ion species

- Equivalent to the theory of a single ion species with effective variables defined

$$Z = \sum_{j=\text{ions}} \frac{\tau_{\text{ee}}}{\tau_{\text{e}j}}$$

$$\frac{1}{\tau_{\text{ei}}} = \sum_{j=\text{ions}} \frac{1}{\tau_{\text{e}j}} = \frac{Z}{\tau_{\text{ee}}}$$

$$\mathbf{V}_{\text{i}} = \sum_{j=\text{ions}} \frac{z_{\text{e}j}\mathbf{V}_j}{Z} = \tau_{\text{ei}} \sum_{j=\text{ions}} \frac{\mathbf{V}_j}{\tau_{\text{e}j}}$$

- Electron closures for a single ion species [Braginskii 1965, Ji & Held 2013]

$$\begin{aligned}\mathbf{R}_{\text{e}} =& \frac{m_{\text{e}} n_{\text{e}}}{\tau_{\text{ei}}}(-\hat{\alpha}_{\|}\mathbf{V}_{\text{ei}\|} - \hat{\alpha}_{\perp}\mathbf{V}_{\text{ei}\perp} + \hat{\alpha}_{\times}\mathbf{V}_{\text{ei}\times}) \\ &+ n_{\text{e}}(-\hat{\beta}_{\|}\nabla_{\|}T_{\text{e}} - \hat{\beta}_{\perp}\nabla_{\perp}T_{\text{e}} - \hat{\beta}_{\times}\nabla_{\times}T_{\text{e}}) = \sum_j \mathbf{R}_{\text{e}j} = \sum_j \frac{z_{\text{e}j}}{Z}\mathbf{R}_{\text{e}}\end{aligned}$$

$$\begin{aligned}\mathbf{h}_{\text{e}} =& n_{\text{e}} T_{\text{e}}(\hat{\beta}_{\|}\mathbf{V}_{\text{ei}\|} + \hat{\beta}_{\perp}\mathbf{V}_{\text{ei}\perp} + \hat{\beta}_{\times}\mathbf{V}_{\text{ei}\times}) \\ &+ \frac{n_{\text{e}} T_{\text{e}} \tau_{\text{ee}}}{m_{\text{e}}}(-\hat{\kappa}^{\text{e}}_{\|}\nabla_{\|}T_{\text{e}} - \hat{\kappa}^{\text{e}}_{\perp}\nabla_{\perp}T_{\text{e}} - \hat{\kappa}^{\text{e}}_{\times}\nabla_{\times}T_{\text{e}})\end{aligned}$$

$$\boldsymbol{\pi}_{\text{e}} = -\eta_{\text{e}(0)}\mathsf{W}_{\text{e}(0)} - \eta_{\text{e}(1)}\mathsf{W}_{\text{e}(1)} - \eta_{\text{e}(2)}\mathsf{W}_{\text{e}(2)} - \eta_{\text{e}(3)}\mathsf{W}_{\text{e}(3)} - \eta_{\text{e}(4)}\mathsf{W}_{\text{e}(4)}$$

Ion closures for multiple ion species

- Closure relations for multiple ion species $(i, j = 1, 2, \cdots, S)$

$$
\begin{aligned}
\mathbf{h}_i =& \sum_j \frac{n_i T_i \tau_{jj}}{m_j} \left(-\hat{\kappa}_{\parallel,ij} \nabla_\parallel T_j - \hat{\kappa}_{\perp,ij} \nabla_\perp T_j + \hat{\kappa}_{\times,ij} \nabla_\times T_j\right) \\
&+ n_i T_i \sum_{j,k} \left(\hat{\beta}^{TV}_{\parallel,ijk} \mathbf{V}_{\parallel,jk} + \hat{\beta}^{TV}_{\perp,ij} \mathbf{V}_{\perp,jk} - \hat{\beta}^{TV}_{\times,ij} \mathbf{V}_{\times,jk}\right) \\
\mathbf{R}_i =& \mathbf{R}_{ie} + \sum_j n_j \left(-\hat{\beta}^{VT}_{\parallel,ij} \nabla_\parallel T_j - \hat{\beta}^{VT}_{\perp,ij} \nabla_\perp T_j + \hat{\beta}^{VT}_{\times,ij} \nabla_\times T_j\right) \\
&- \frac{m_i n_i}{\tau_{ii}} \sum_{j,k} \left(\hat{\alpha}_{\parallel,ijk} \mathbf{V}_{\parallel,jk} + \hat{\alpha}_{\perp,ijk} \mathbf{V}_{\perp,jk} + \hat{\alpha}_{\times,ijk} \mathbf{V}_{\times,jk}\right) \\
\boldsymbol{\pi}_i =& p_i \sum_j \tau_{jj} \Big[-\hat{\eta}_{(0)ij} \mathsf{W}_{(0)j} - \hat{\eta}_{(1)ij} \mathsf{W}_{(1)j} - \hat{\eta}_{(2)ij} \mathsf{W}_{(2)j} \\
&+ \hat{\eta}_{(3)ij} \mathsf{W}_{(3)j} + \hat{\eta}_{(4)ij} \mathsf{W}_{(4)j}\Big]
\end{aligned}
$$

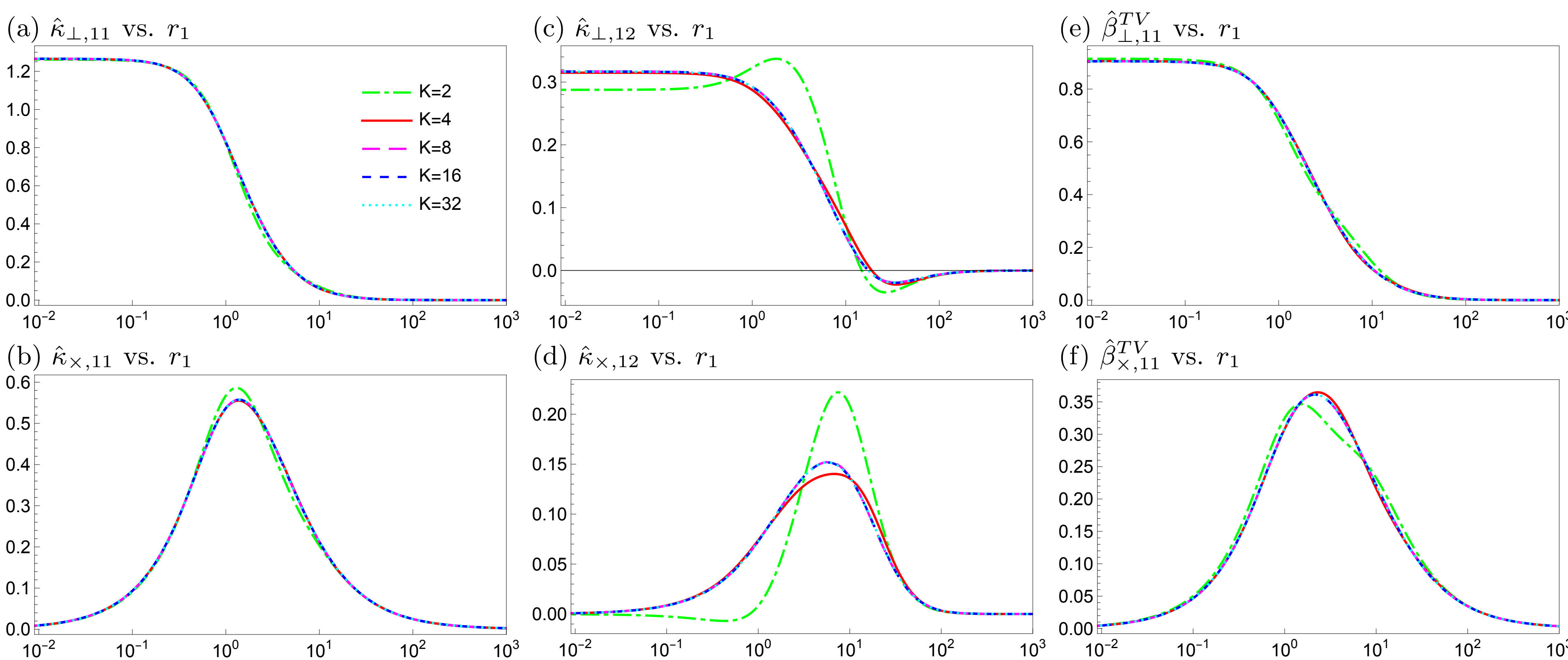
  where the summation indices j and k run over ion species only

- Example study for deuterium (D) and carbon (C): $Z_\mathrm{D} = +1$, $Z_\mathrm{C} = +6$, $m_\mathrm{D} = 2.014\mathrm{u}$, $m_\mathrm{C} = 12.00\mathrm{u}$, $n_\mathrm{D} = 0.64 n_\mathrm{e}$, $n_\mathrm{C} = 0.06 n_\mathrm{e}$, $T_\mathrm{D} = 1.5 T_\mathrm{e}$, $T_\mathrm{C} = 1.8 T_\mathrm{e}$, and $\ln \Lambda_\mathrm{ee} = 17$

Heat flux density for two ion species: convergence

Heat-flux closure coefficients of ion species 1: sum over $A = \parallel, \perp \, [\overset{A}{\pm} = -], \times[+]$ is implied, $\hat{\kappa}_{\parallel} = \hat{\kappa}_{\perp}(r_1 = 0)$

$$\mathbf{h}_1 = \overset{A}{\pm} \frac{n_1 T_1 \tau_{11}}{m_1} \hat{\kappa}_{A,11} \nabla_A T_1 \overset{A}{\pm} \frac{n_1 T_1 \tau_{22}}{m_2} \hat{\kappa}_{A,12} \nabla_A T_2 + n_1 T_1 \hat{\beta}^{TV}_{A,11} \left(\mathbf{V}_1 - \mathbf{V}_2\right)_A$$

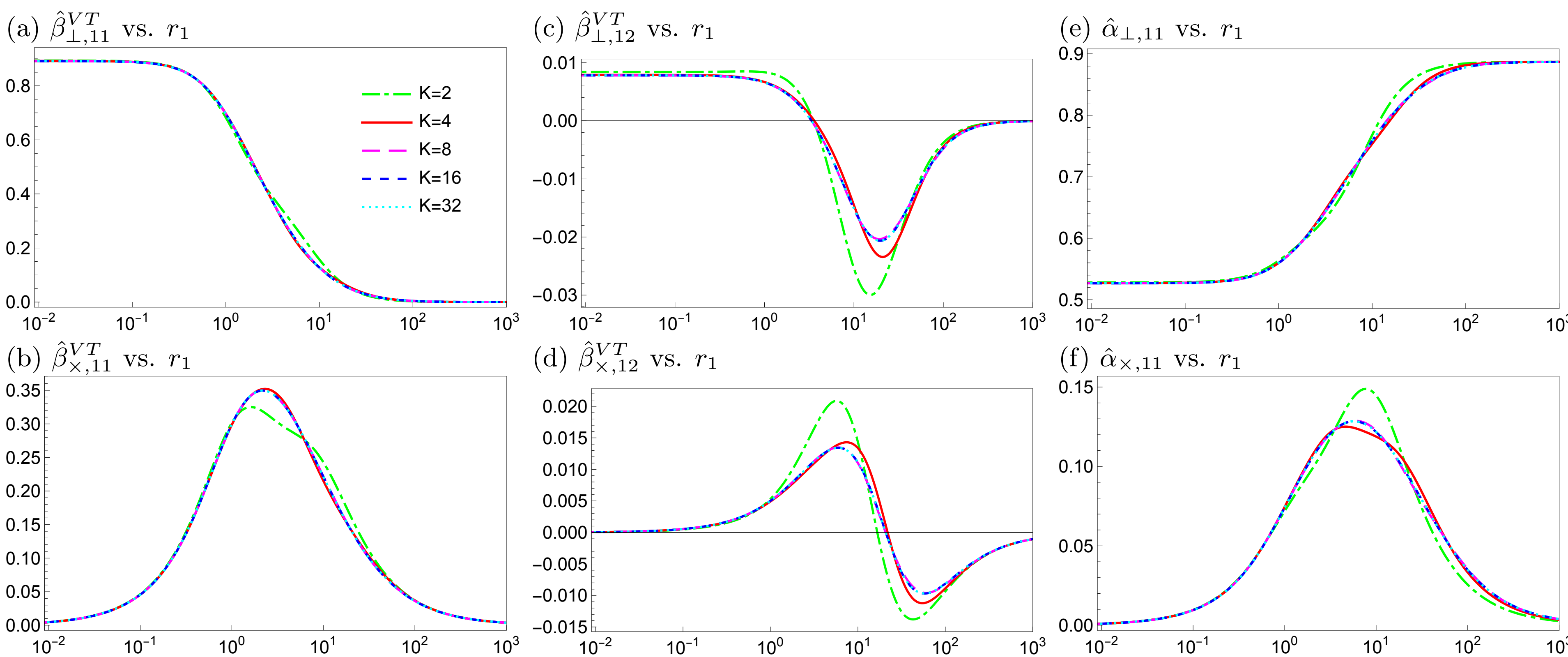


Convergent closure coefficients are obtained by increasing the number of moments from $K = 2$ to $K = 32$

Friction density for two ion species: convergence

Friction closure coefficients of ion species 1: sum over $A = \|, \perp\ [\overset{A}{-} = -], \times [+]$ is implied, $\hat{\alpha}_{\|} = \hat{\alpha}_{\perp}(r_1 = 0)$

$$\mathbf{R}_{12} = \overset{A}{\pm}\, n_1 \hat{\beta}^{VT}_{A,11} \nabla_A T_1 \overset{A}{\pm}\, n_2 \hat{\beta}^{VT}_{A,12} \nabla_A T_2 - \frac{m_1 n_1}{\tau_{11}} \hat{\alpha}_{A,11} \left(\mathbf{V}_1 - \mathbf{V}_2\right)_A$$



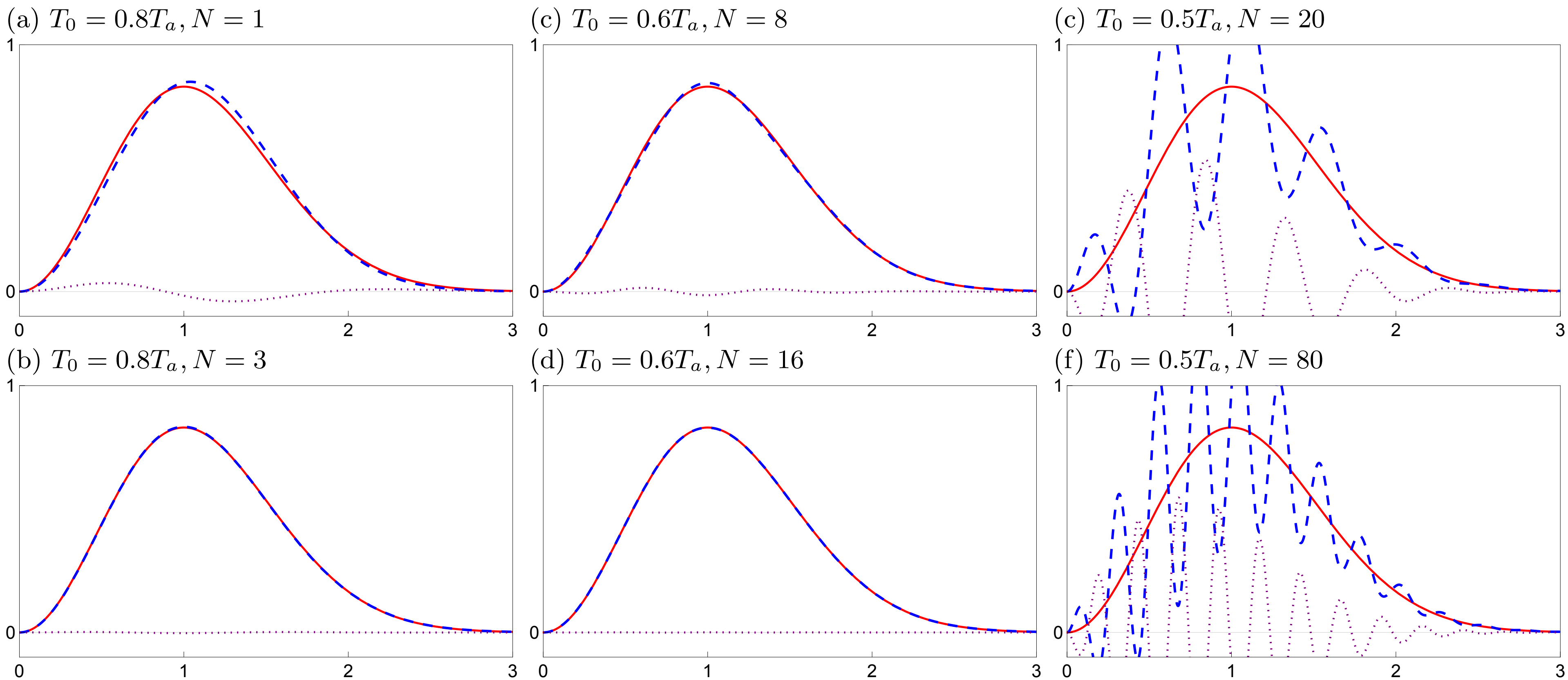
For the example, $K = 8$ calculations seem to yield convergent results

Problems of one temperature formulation $T_0 = \dfrac{n_a T_a + n_b T_b}{n_a + n_b}$

Moment expansion fails for $T_0 \le T_a/2$ in the moment expansion

$$\mathbf{s}_a = \frac{\mathbf{v}}{v_{Ta}} = \frac{\mathbf{v}}{\sqrt{2T_a/m_a}}, \quad \mathbf{s}_{a0} = \frac{\mathbf{v}}{v_{Ta0}} = \frac{\mathbf{v}}{\sqrt{2T_0/m_a}}, \quad a_k = \left(1 - \frac{v_{Ta}^2}{v_{Ta0}^2}\right)^k$$

$$\frac{n_a}{\pi^{3/2} v_{Ta}^3} e^{-s_a^2} = \frac{n_a}{\pi^{3/2} v_{T0}^3} e^{-s_{a0}^2} \left[1 + a_1 L_1^{(1/2)}(s_{a0}^2) + \cdots + a_N L_N^{(1/2)}(s_{a0}^2)\right]$$

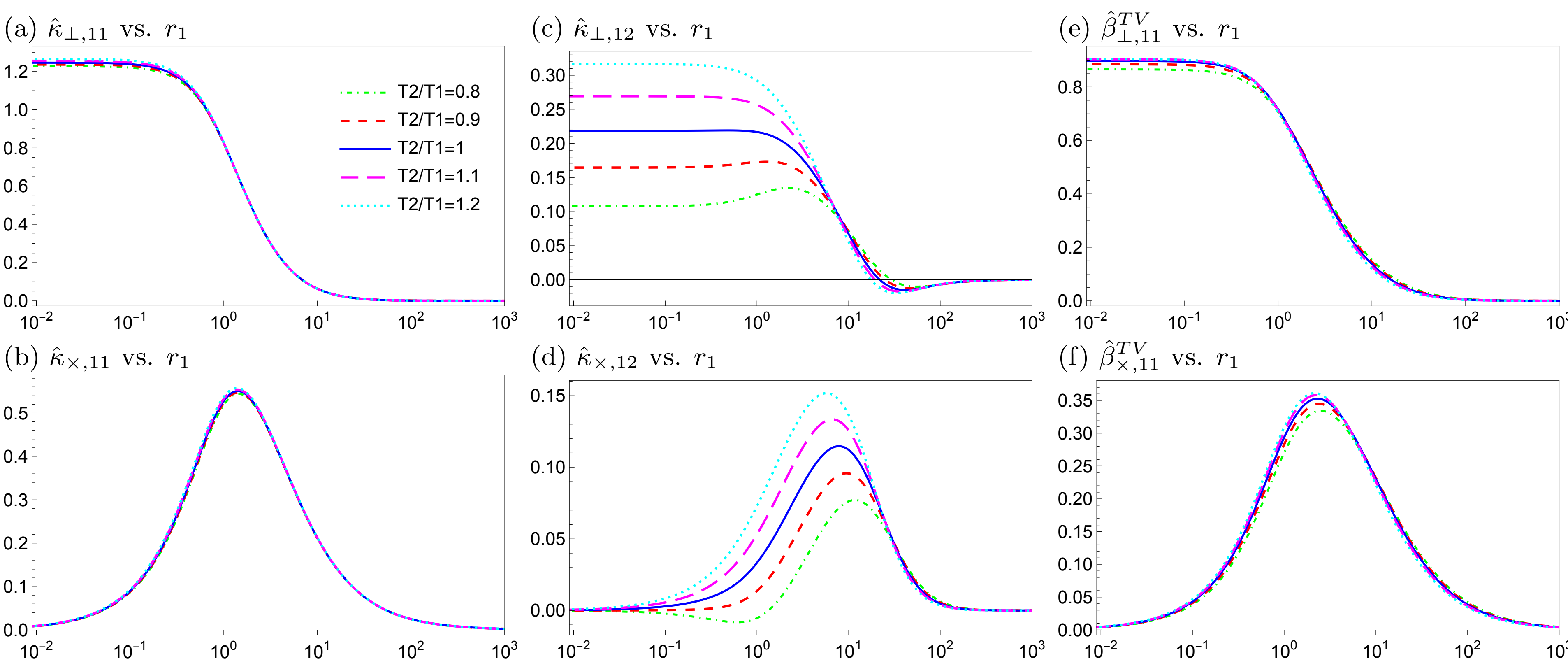


Furthermore, calculating collision coefficients with $T_a = T_b = T_0$ introduces considerable errors

Two-temperature dependence of heat flux closure coefficients

Heat-flux closure coefficients of ion species 1: sum over $A = \|, \perp\ [\overset{A}{\pm} = -], \times [+]$ is implied, $\hat{\kappa}_{\|} = \hat{\kappa}_{\perp}(r_1 = 0)$

$$\mathbf{h}_1 = \overset{A}{\pm} \frac{n_1 T_1 \tau_{11}}{m_1} \hat{\kappa}_{A,11} \nabla_A T_1 \overset{A}{\pm} \frac{n_1 T_1 \tau_{22}}{m_2} \hat{\kappa}_{A,12} \nabla_A T_2 + n_1 T_1 \hat{\beta}^{TV}_{A,11} \left(\mathbf{V}_1 - \mathbf{V}_2\right)_A$$

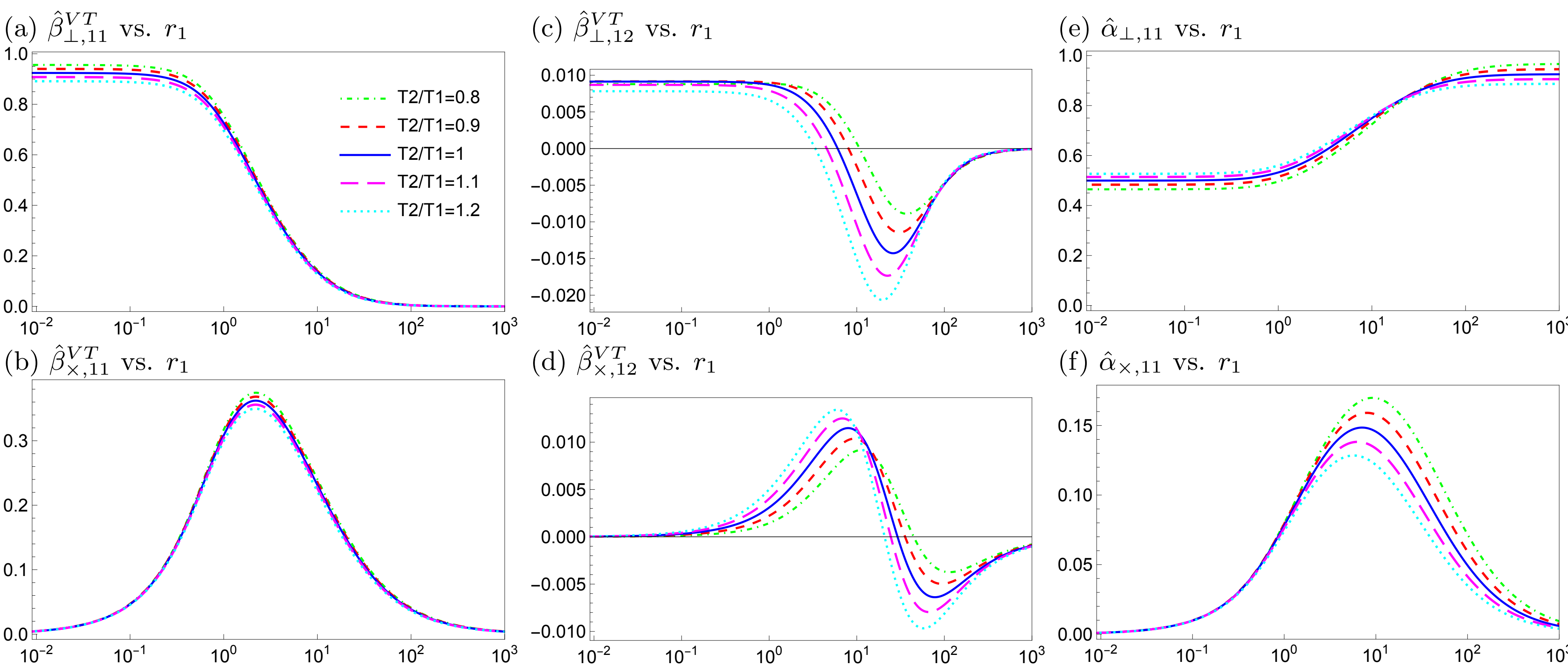


One-temperature calculations can be significantly erroneous

Two-temperature dependence of friction closure coefficients

Friction closure coefficients of ion species 1: sum over $A = \parallel, \perp\ [\overset{A}{-} = -], \times[+]$ is implied, $\hat{\alpha}_{\parallel} = \hat{\alpha}_{\perp}(r_1 = 0)$

$$\mathbf{R}_{12} = \overset{A}{\pm}\, n_1 \hat{\beta}^{VT}_{A,11} \nabla_A T_1 \overset{A}{\pm}\, n_2 \hat{\beta}^{VT}_{A,12} \nabla_A T_2 - \frac{m_1 n_1}{\tau_{11}} \hat{\alpha}_{A,11} \left(\mathbf{V}_1 - \mathbf{V}_2\right)_A$$



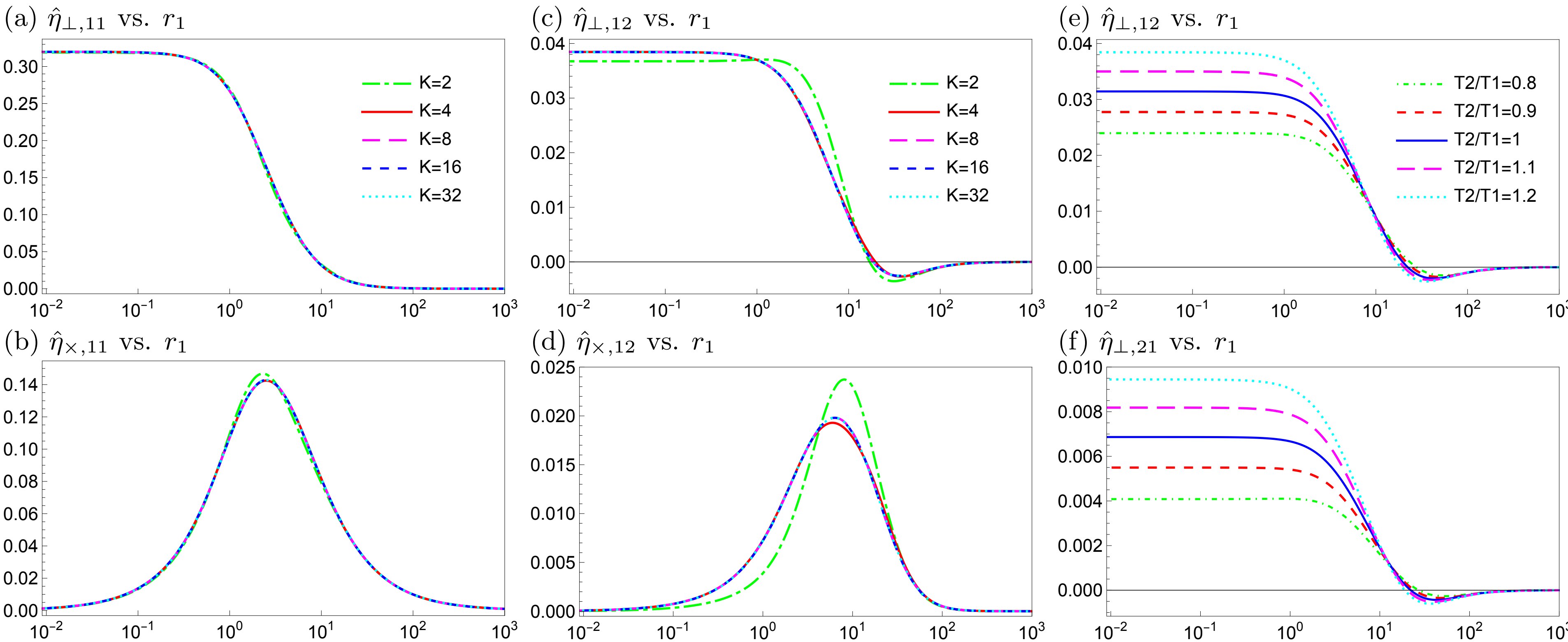
One-temperature calculations can be significantly erroneous

Viscosity closure for two ion species: convergence and two-temperature dependence

Viscosity coefficients of ion species 1: sum over $A = 0, 1, 2, [\overset{A}{\pm} = -], 3, 4[+]$ is implied

$$\hat{\eta}_{(0)} = \hat{\eta}_{\perp}(r_1 = 0),\ \hat{\eta}_{(1)} = \hat{\eta}_{\perp}(2r_1),\ \hat{\eta}_{(2)} = \hat{\eta}_{\perp}(r_1),\ \hat{\eta}_{(3)} = \hat{\eta}_{\times}(2r_1),\ \hat{\eta}_{(4)} = \hat{\eta}_{\times}(r_1)$$

$$\boldsymbol{\pi}_1 = \overset{A}{\pm}\, p_1 \hat{\eta}_{(A),11} \tau_{11} \mathsf{W}_{(A),1} \overset{A}{\pm}\, p_1 \hat{\eta}_{(A),12} \tau_{22} \mathsf{W}_{(A),2}$$



Conclusions and future work

- Closure relations for multiple ion species
  - ▷ General formalism is developed for arbitrary temperatures, masses, charges, and densities
- One temperature formalism yields inaccurate results
  - ▷ Closure coefficients are sensitive to the temperature ratio
  - ▷ Moment expansion fails for $T_0 = \dfrac{n_a T_a + n_b T_b}{n_a + n_b} < \dfrac{T_a}{2}$ or $\dfrac{T_b}{2}$
- Machine learning can develop simple fitted coefficients for two ion species
  - ▷ Multiple ion species: find effective parameters equivalent to two ion species
- Integral parallel closures for low (arbitrary) collisionality