Semi-implicit Treatment of the Hall Effect in NIMROD Simulations

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Theme: An anisotropic fourth-order operator in the magnetic advance for the Hall term in Ohm's law is developed as one part of the two-fluid advance in the NIMROD project. This leads to a semi-implicit method that is numerically stabile for large time steps. The condition for stabilizing whistler wave at arbitrary large time-step is derived. Analytical and numerical dispersion relation are compared with the simulation result in numerical tests.

I. Introduction

- The need for two-fluid effects in fusion simulations
- Numerically challenging
- The existing semi-implicit operator for Hall term in the NIMROD code

II. Improved Semi-implicit Model

- Semi-implicit 4th order Hall operator formation
- Numerical stability analysis
- Complete time-advance

- Implementation with a finite element spatial representation

III. Numerical Tests

- Whistler wave dispersion relation
- Comparison between 2^{nd} order and 4^{th} order Hall operator

IV. Summary

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- Current Code development Status

The need for two-fluid effects in fusion simulations

- Two-fluid plasma models represent independently evolving electron and ion species that are connected through quasineutrality. The model can generate drift effects that introduce real frequencies and change stability limits of MHD modes. This can affect the fusion plasma beta limit.

- Two-fluid effects are critical for a full understanding of the fusion physics problems including neoclassical MHD instabilities, dynamo effects, magnetic reconnection.

- Full exploitation of experimental data requires simulation and prediction for dynamics of the electrons and ions.

Numerical Challenge

- Considering resistive MHD behavior alone, characteristic times and length for different effects may differ by 4-10 orders of magnitude in experiments.
- Two-fluid effects introduce further spatial and temporal stiffness.
- The Semi-implicit method is a tool to address the stiffness introduced by the Hall term in a two-fluid Ohm's law as applied to fusion plasmas

The existing semi-implicit operator for Hall term in NIMROD

- The operator is second order and has the form : $\nabla\times(\Delta t)\dfrac{|B_0|}{\mu_0 ne}\nabla\times$, while the electron inertia appears as the operator $\nabla\times\dfrac{m_e}{\mu_0 ne^2}\nabla\times$. Therefore the 2nd order Hall operator acts like electron inertia with an effective electron mass of $eB_0\Delta t$.

- Electron inertia gives rise to magnetic reconnection if the electron skin depth, c/ω_{pe} , is comparable to the resistive tearing layer width. If this depth, computed with the effective electron mass, is large, we can expect an unphysical amount of reconnection and tearing mode growth. Hence, the implementation requires small time-steps. The convergence properties of this operator are not acceptable.

- An alternative semi-implicit operator described in Ref [Harned and Mikic] is introduced in this poster to improve the large time step accuracy and the two-fluid convergence properties of NIMROD.

Semi-implicit 4th order Hall operator formation

- Focusing on the Hall term alone, we have a simple evolution equation from Faraday's Law:

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$

$$\text{where : } \vec{E} = \frac{1}{ne} \vec{J} \times \vec{B}$$

$$\text{and : } \vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B}$$

after linearization :

$$\frac{\partial \vec{b}}{\partial t} = -\nabla \times \left(\frac{1}{\mu_0 ne} (\nabla \times \vec{b}) \times \vec{B}_0\right)$$

- A self-adjoint operator can be formed by making a second-order wave equation from the relevant magnetic advance.

$$\frac{\partial^2 \vec{b}}{\partial t^2} = -\nabla \times \left(\frac{1}{\mu_0 ne} (\nabla \times \partial \vec{b} / \partial t) \times \vec{B}_0\right)$$

$$= \nabla \times \left(\frac{1}{\mu_0 ne} \left(\nabla \times \nabla \times \left(\frac{1}{\mu_0 ne} (\nabla \times \vec{b}) \times \vec{B}_0\right)\right) \times \vec{B}_0\right)$$

Note: The operator is 4th order

- Using this operator in a semi-implicit advance

$$\Delta\vec{b} - \nabla\times\left(\frac{\Delta t}{\mu_0 ne}\left(\nabla\times\nabla\times\left(\frac{\Delta t}{\mu_0 ne}\left(\nabla\times\Delta\vec{b}\right)\times\vec{\mathbf{B}}_0\right)\right)\times\vec{\mathbf{B}}_0\right) = rhs$$

This is similar to the Hall semi-implicit operator recommended in Ref. [Harned and Mikic].

- NIMROD also uses time-splitting for Hall advance to reduce errors of large time step.

Predictor Step

$$(1-\sigma(\Delta t)^2 S)\Delta\vec{b}^* = -\Delta t\nabla\times[\frac{1}{\mu_0 ne}(\nabla\times\vec{b}^n)\times\vec{B}_0]$$

Corrector Step

$$(1-\sigma(\Delta t)^2 S)\Delta\vec{b} = -\Delta t\nabla\times[\frac{1}{\mu_0 ne}(\nabla\times(\vec{b}^n + f\Delta\vec{b}^*)\times\vec{B}_0]$$

Note: f is centering coefficient, σ is semi-implicit coefficient, S is the the 4th order Hall operator. We have implemented this operator in NIMROD.

Complete Time-Advance

- Here, We present the full time-advance with split Hall advance in NIMROD

Velocity Advance

$$(\rho^n - \Delta t f_V \nabla \cdot \rho^n \nu \nabla - \sigma \Delta t^2 L)(\Delta \vec{V})_{pass} =$$

$$-\Delta t(\rho^n \vec{V}^* \cdot \nabla \vec{V}^* + \vec{J}^n \times \vec{B}^n - \nabla p^n + \nabla \cdot \rho^n \nu \nabla \vec{V}^n + \nabla \cdot \Pi)$$

where L is the self-adjoint linear ideal MHD force operator

$$L(\vec{V}) = \frac{1}{\mu_0}(\nabla \times (\nabla \times (\vec{V} \times \vec{B}_0))) \times \vec{B}_0 + \vec{J}_0 \times \nabla \times (\vec{V} \times \vec{B}_0)$$

$$+ \nabla(\vec{V} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \vec{V})$$

For pass = predict, $\vec{V}^* = \vec{V}^n$

For pass = correct, $\vec{V}^* = \vec{V}^n + f_V (\Delta \vec{V})_{predict}$

And , $\vec{V}^{n+1} = \vec{V}^n + (\Delta \vec{V})_{correct}$

Magnetic Field Advance

MHD Part

$$(1+\nabla\times S_{MHD}\nabla\times)(\Delta\vec{B}_{pass}) =$$

$$\Delta t\nabla\times(\vec{V}^{n+1}\times\vec{B}^{*} - \frac{\eta}{\mu_0}\nabla\times\vec{B}^{n} + \frac{1}{ne}\nabla\cdot\Pi)$$

where S_{MHD} is impedance tensor, which contains electron inertia and resistive effects. $S_{MHD} = (\frac{m_e}{ne^2} + \Delta t f_{\eta}\frac{\eta}{\mu_0})$

For pass = predict, $\vec{B}^{*} = \vec{B}^{n}$
For pass = correct, $\vec{B}^{*} = \vec{B}^{n} + f_b(\Delta\vec{B})_{predict}$
And , $\vec{B}^{MHD} = \vec{B}^{n} + (\Delta\vec{B})_{correct}$

Hall Part

$$(1-\sigma(\Delta t)^2 S_{Hall})(\Delta\vec{B}_{pass}) =$$

$$-\Delta t\nabla\times[\frac{1}{ne}(\vec{J}^*\times\vec{B}_0-\nabla p_e^n)+\frac{m_e}{ne^2}\nabla\cdot(\vec{J}^*\vec{V}^{n+1}+\vec{V}^{n+1}\vec{J}^*)]$$

where $\vec{J}^* = \frac{1}{\mu_0}(\nabla\times\vec{B}^*)$ and S_{Hall} is the 4th order Hall operator

$$S_{Hall}(\vec{B})=\nabla\times(\frac{1}{\mu_0 ne}(\nabla\times\nabla\times(\frac{1}{\mu_0 ne}(\nabla\times\vec{B})\times\vec{B}_0))\times\vec{B}_0)$$

For pass = predict, $\vec{B}^* = \vec{B}^n$

For pass = correct, $\vec{B}^* = \vec{B}^n + f_{bhall}(\Delta\vec{B})_{predict}$

And , $\vec{B}^{n+1} = \vec{B}^{MHD} + (\Delta\vec{B})_{correct}$

Numerical stability analysis

- A 1D ($\vec{b} = \vec{b}_k e^{ikz}$) numerical stability analysis was performed on the 4th order Hall including the semi-implicit operator. The numerical eigenvalue, ($\Delta \vec{b}_k = (\lambda - 1)\vec{b}_k^n$), is

$$\lambda = (1 - \frac{f\xi^2}{(1+\sigma\xi^2)^2}) \pm i \frac{\xi}{1+\sigma\xi^2}$$, where $$\xi = \frac{\Delta t k^2 B_{0z}}{\mu_0 n e}$$

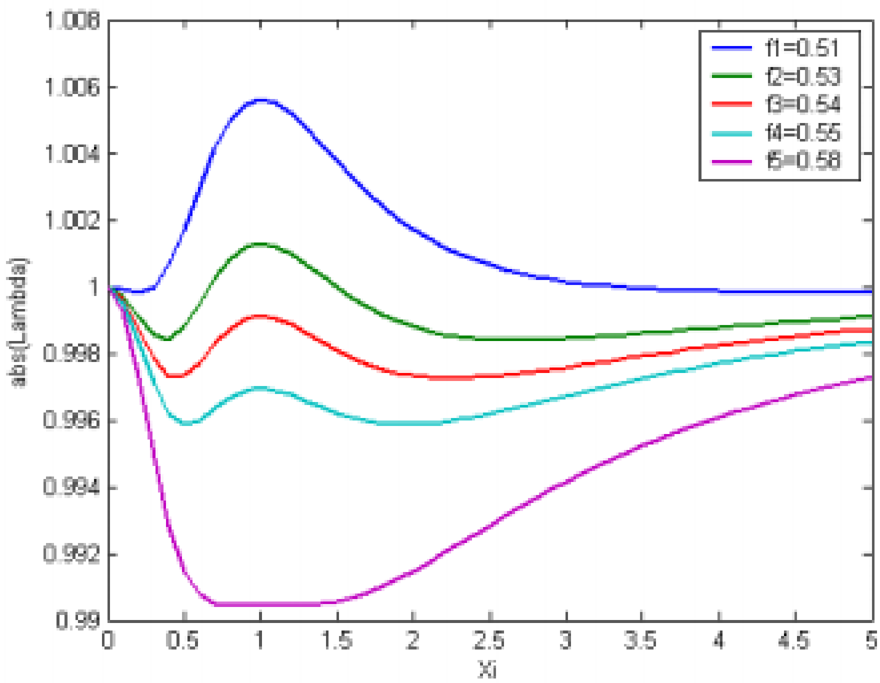


Fig. 1 Numerical Eigenvalue

From Fig.1, We found:

1. Dissipation error is slight for smaller f
2. If $f < 0.54$, the scheme is unstable.

Hence, We choose $f = 0.54$ in our numerical computation

Implementation with a finite element spatial representation

- A weak form with integration by parts is required to reduce the order of continuity required from the solution space. Hence, we need to satisfy an equation of the form

$$\int d\vec{x}\vec{A}\cdot\Delta\vec{b} +$$

$$\int d\vec{x}\nabla\times(\frac{\Delta t}{\mu_0 ne}(\nabla\times\Delta\vec{b})\times B_0)\cdot\nabla\times(\frac{\Delta t}{\mu_0 ne}(\nabla\times\vec{A})\times\vec{B}_0) = rhs$$

for all appropriate test functions $\vec{A}$.

- The self-adjoint fourth-order operator must be recast through a system of second-order equations, so that the existing spatial representation remains conforming. Hence, Introducing auxiliary fields $\vec{f}$, where $\vec{f} = \nabla\times(\frac{\Delta t}{\mu_0 ne}(\nabla\times\Delta\vec{b})\times B_0)$

$$\int d\vec{x}\vec{A}\cdot\Delta\vec{b} + \int d\vec{x}\nabla\times\vec{f}\cdot(\frac{\Delta t}{\mu_0 ne}(\nabla\times\vec{A})\times\vec{B}_0) = rhs$$

$$\int d\vec{x}\vec{g}\cdot\vec{f} - \int d\vec{x}\nabla\times\vec{g}\cdot(\frac{\Delta t}{\mu_0 ne}(\nabla\times\Delta\vec{b})\times\vec{B}_0) = 0$$

for all appropriate test functions $\vec{A}$ and $\vec{g}$.

- Implementation will require use of the non-symmetric solver library. Here, The AZTEC parallel linear solver library [http://www.cs.sandia.gov/CRF/aztec1.html] is used to solve the resulting non-symmetric system.

Whistler wave dispersion relation

- In order to examine the effects of the 4th order operator for the Hall term, we first consider simple whistler waves in a periodic domain. Here, Faraday's and Ohm's law gives the analytical whistler wave dispersion relation for $\vec{k} \,//\, \vec{B}_0$

$$\omega_{th} = \frac{k^2 \upsilon_A^2}{\Omega_i} \frac{1}{(1 + \frac{c^2 k^2}{\omega_{pe}^2})} \approx \frac{k^2 \upsilon_A^2}{\Omega_i} = \frac{k^2 B_0}{\mu_0 n e}$$

where υ_A is the Alfvén velocity, Ω_i is the ion cyclotron frequency

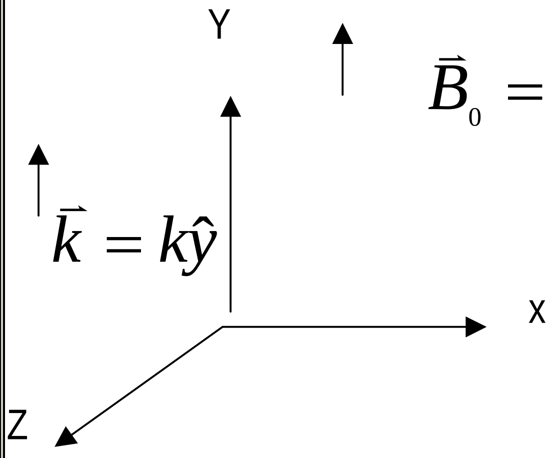
- The numerical dispersion relation is obtained from the numerical eigenvalue computation. Here, we write λ with real and imaginary part respectively.

$$\lambda_R = 1 - \frac{f\xi^2}{(1+\sigma\xi^2)^2} \quad \text{and} \quad \lambda_I = \frac{\xi}{1+\sigma\xi^2},$$

which give the following numerical dispersion relation:

$$\omega_R = \frac{1}{\Delta t} \tan^{-1}\left(\frac{\lambda_I}{\lambda_R}\right)$$

- Geometry of test problem



$\vec{B}_0 = B_0 \hat{y}$

$\vec{k} = k\hat{y}$

$k = 2\pi$

$n = 10^{20}$

$B_0 = 1$

- Analytical dispersion gives the theoretical value of ω_{th} :
  $\omega_{th} = 1.964 \times 10^6$ (MKS units)

- Numerical frequencies and NIMROD simulation results are plotted in Fig.2

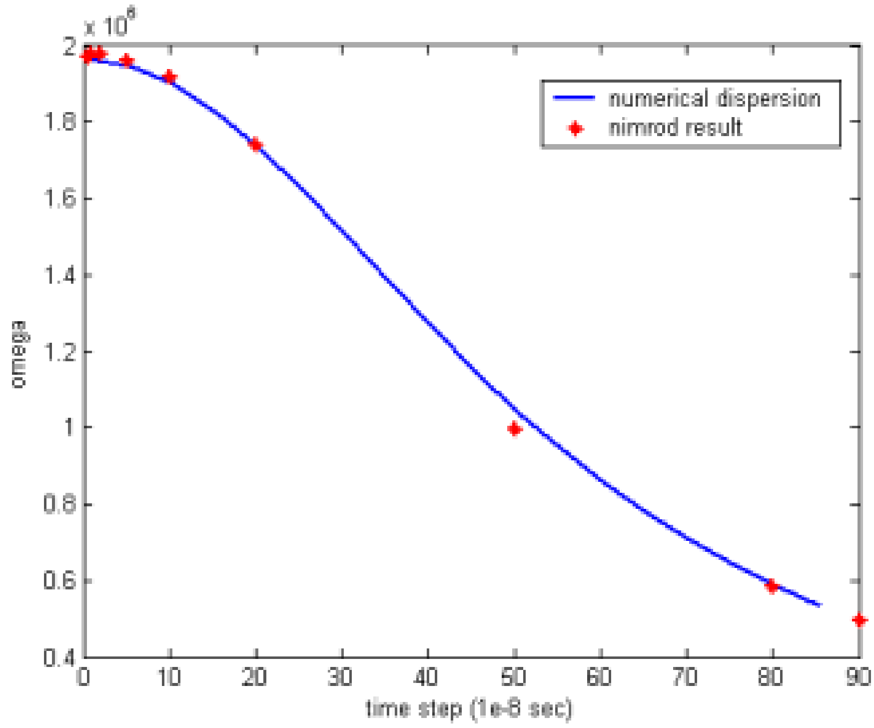


Fig. 2 Numerical whistler wave frequencies at different time steps(k=2π)

- From Fig.2, We found:

  1. The convergence of numerical whistler wave frequencies to the theoretical prediction is observed for time step equal to zero.

  2. NIMROD simulation result agrees with the numerical values very well.

- Numerical dispersion and NIMROD simulation results are plotted in Fig.3 and Fig.4

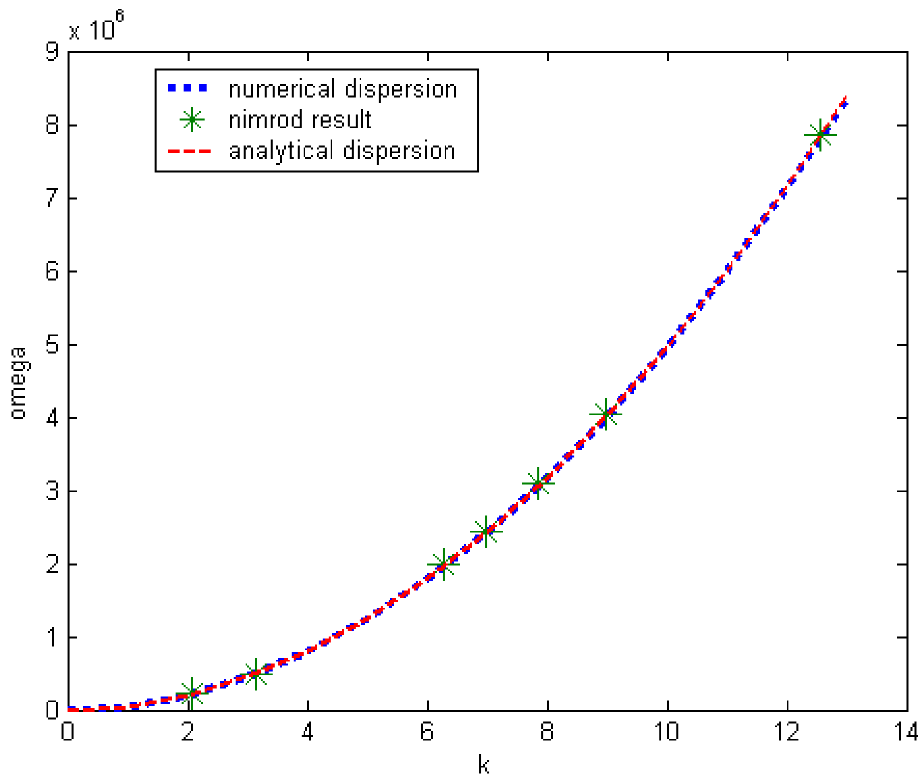


Fig 3. Numerical dispersion for Whistler wave for time step Δt =1e-8sec

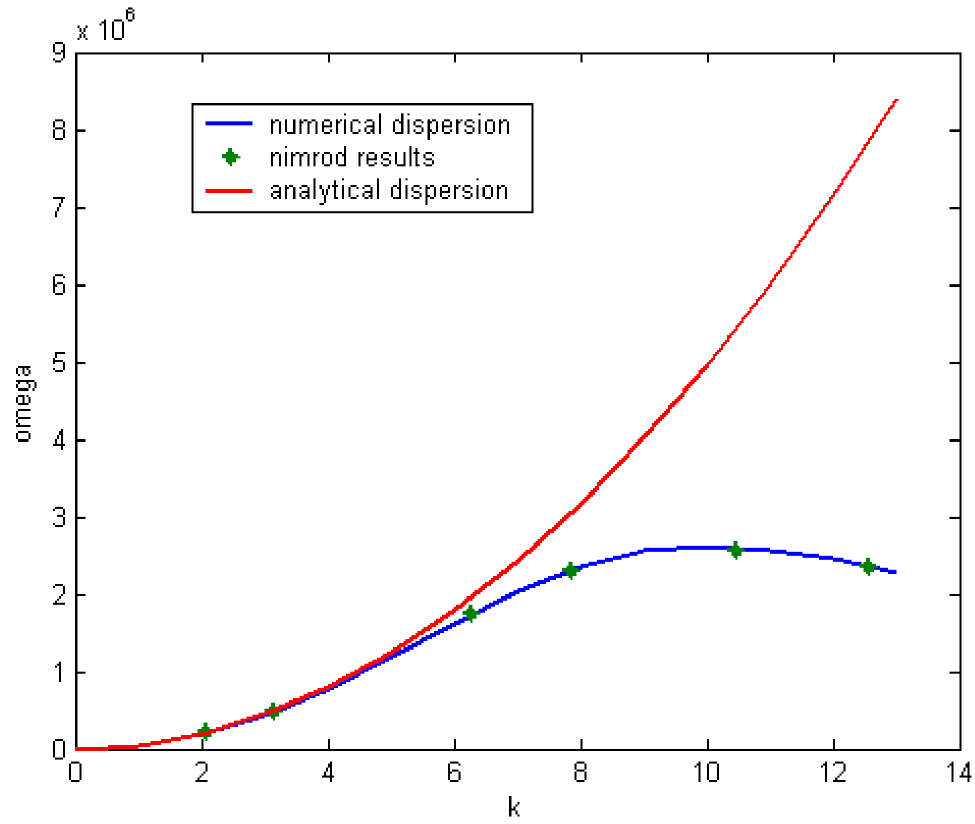


Fig 4. Numerical dispersion for Whistler wave for time step Δt =2e-7sec

- At small time step, both the numerical dispersion and simulation result agree with the analytical dispersion relation very well. (See Fig.3)

- At large time step, they also agree with theoretical curve at low k, while the discrepancy is obvious at high k. Semi-implicit operator will cause a negative group velocity for large time step at high k. (See Fig.4)

Comparison between 2nd order and 4th order Hall operator

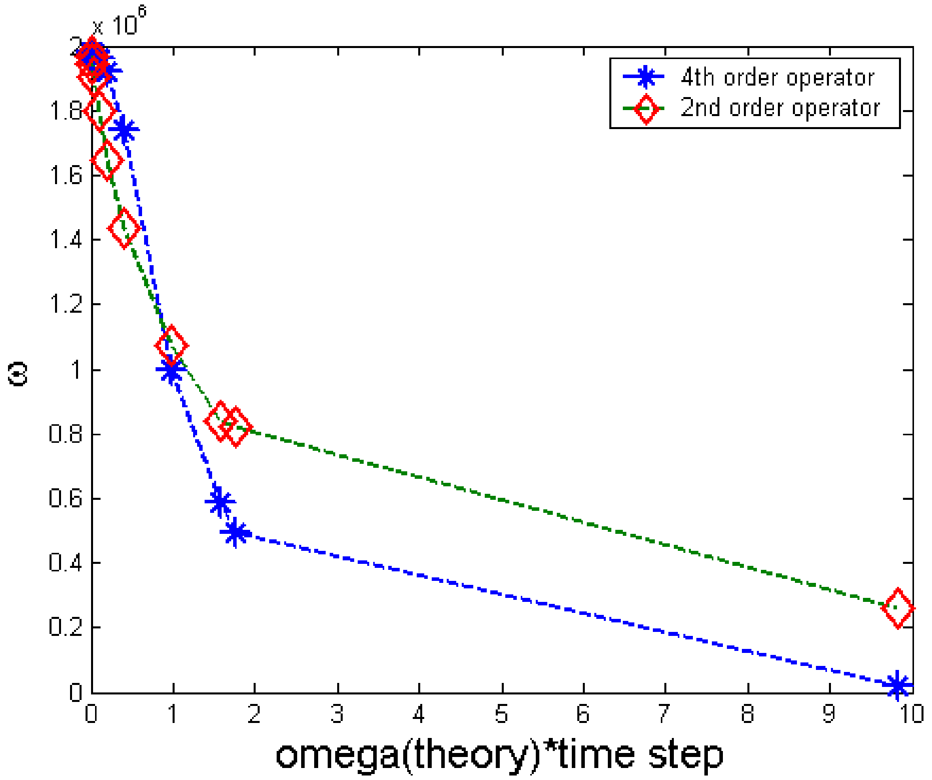


Fig. 5 (a)

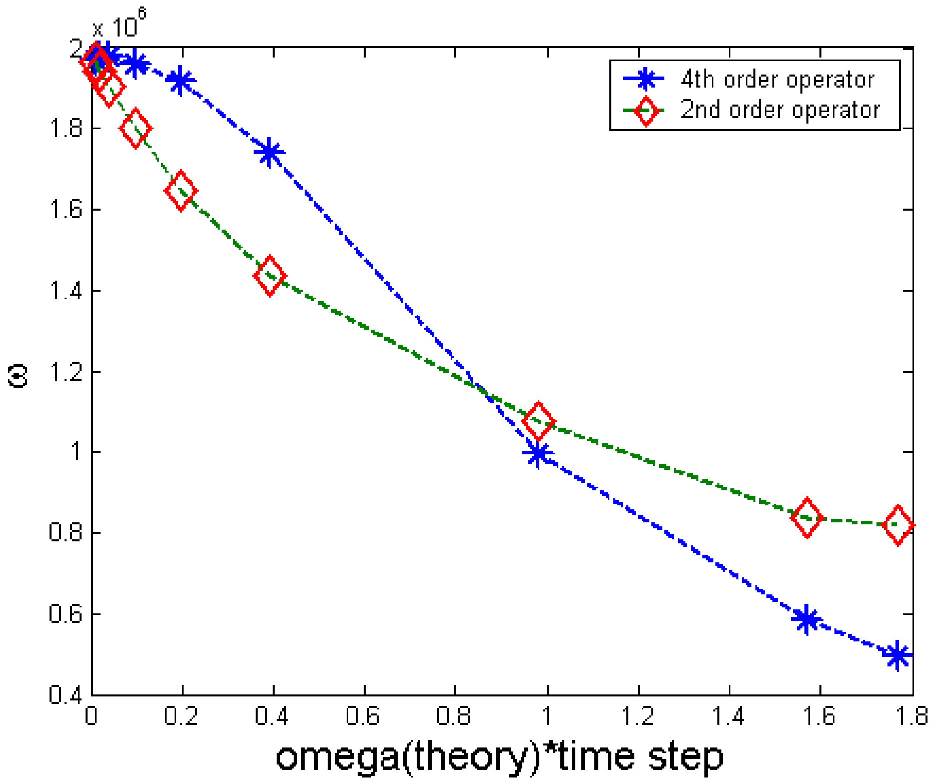


Fig 5 (b). Comparison between Hall operators on Numerical whistler wave frequencies at different time steps (k=2π)

- For $\omega_{th}\Delta t \leq 0.8$, The 4th order Hall operator gives more accurate result than 2nd order operator does. (See Fig.5)

- The 4th order operator has less numerical dissipation than the 2nd order operator for large time steps. The magnetic field decays more quickly as time grows using 2nd order operator. Decay coefficients vs different time steps is plotted in Fig.6

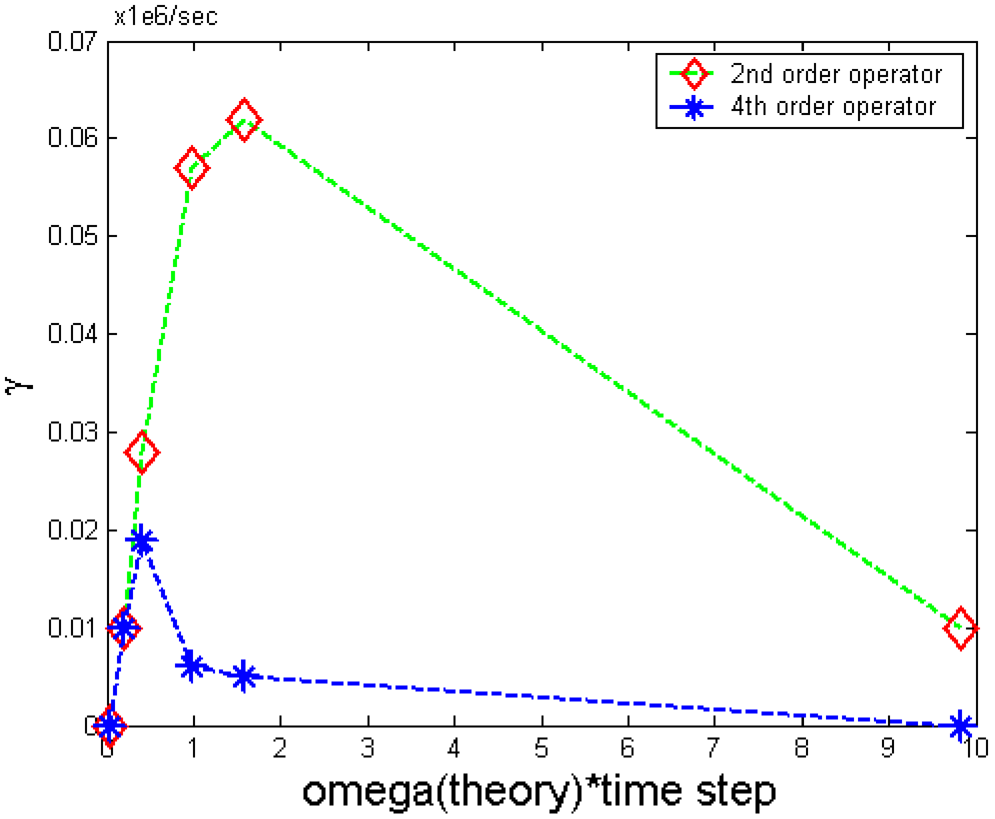


Fig.6 (a)

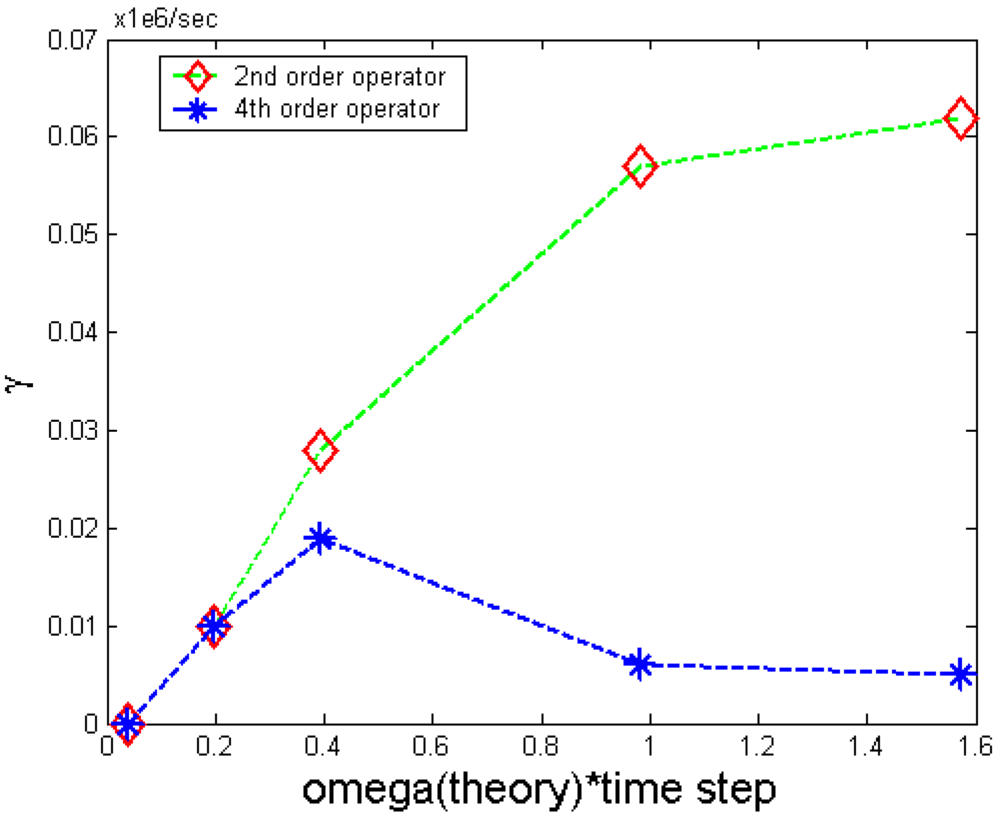


Fig. 6 (b) Comparison between Hall operators on numerical dissipation at different time steps (k=2π)

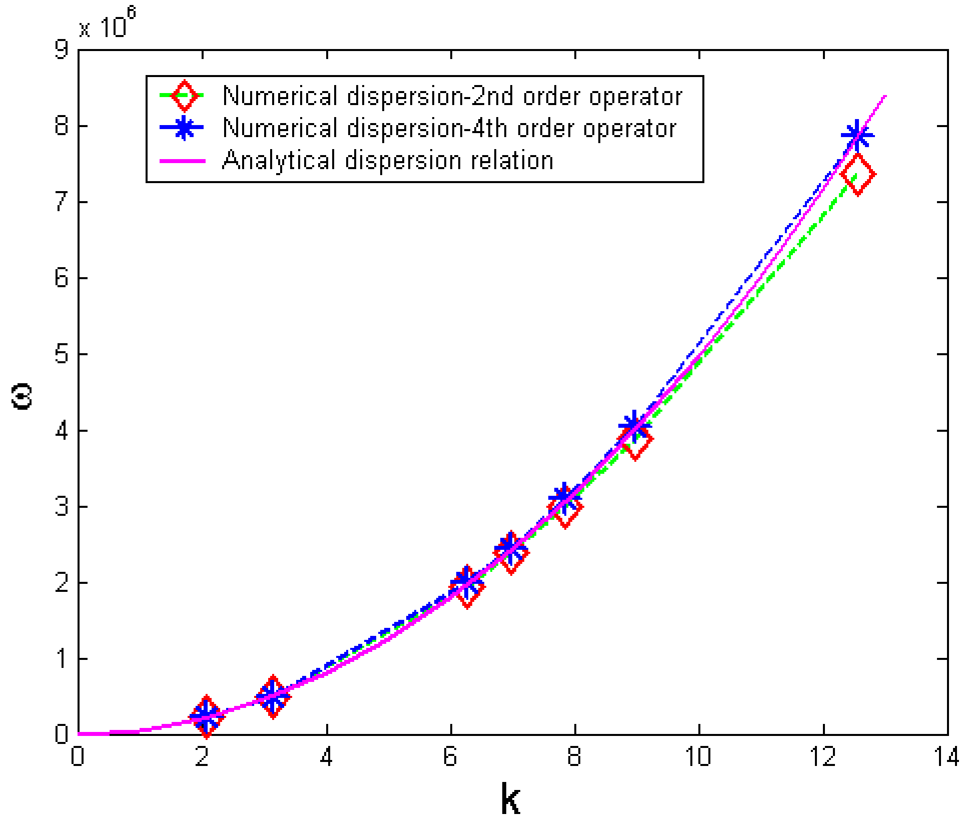


Fig. 7 Comparison between Hall operators on numerical dispersion for Whistler wave for time step Δt =1e-8sec

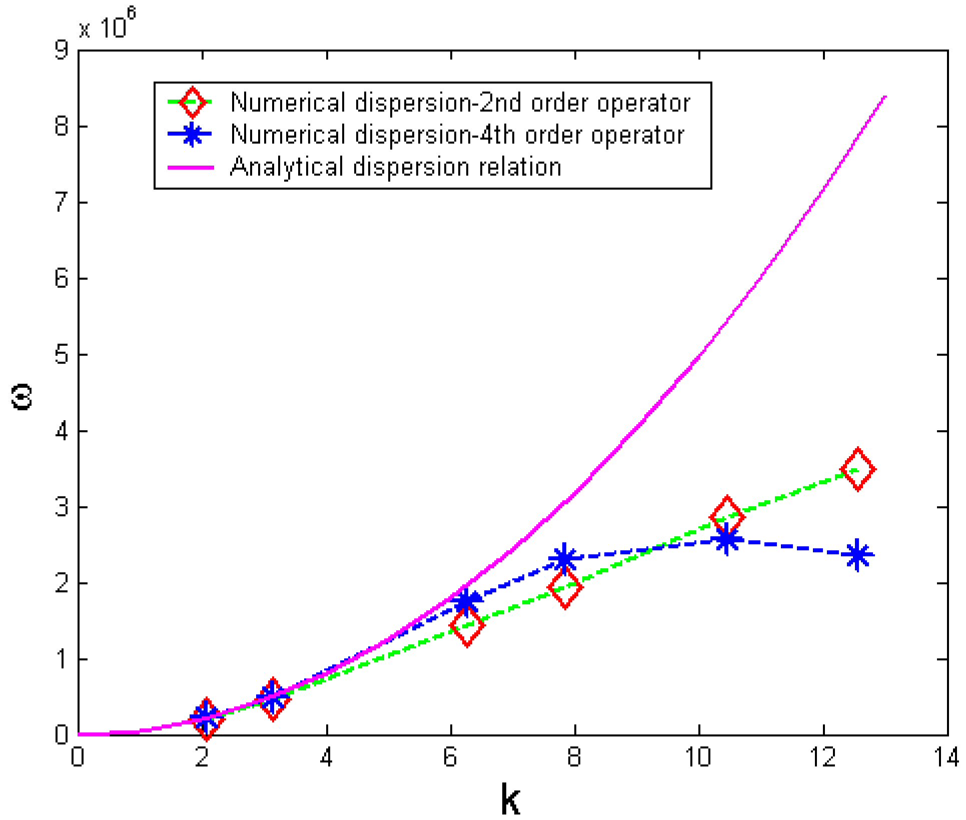


Fig. 8 Comparison between Hall operators on numerical dispersion for Whistler wave for time step Δt =2e-7sec

- Fig.7 and Fig.8 give the comparison between the two Hall operators on numerical dispersion relation, along with the analytical dispersion relation. For small time step, $\Delta t = 1\times 10^{-8}$ sec or $\omega_{th}\Delta t = 0.02$, the 4th order Hall operator gives more accurate result than 2nd order operator does. (Fig.6) For large time step, $\Delta t = 2\times 10^{-7}$ sec or $\omega_{th}\Delta t = 0.4$, the 4th order Hall operator still behaves better than 2nd order operator at low k. (Fig. 8)

Summary and Conclusion

- A semi-implicit technique for treating the Hall term has been developed and implemented in NIMROD. The operator is 4^{th} order and eliminates the stability limit on the time step due to the Hall term.

- The split algorithm is applied for the 4^{th} order operator. Numerical and analytical dispersion relation are compared. Good agreement is achieved for low k or small time step.

- The 4^{th} order Hall operator generates less numerical dissipation compare to the 2^{nd} order one especially for large time steps.

- The 4^{th} order operator is also superior to the 2^{nd} order operator in terms of accuracy and it's up to 10% more accurate out to $\omega_{th}\Delta t = 0.8$

Current Code Development Status

- Dirichlet boundary condition for the 6-component vector is being developed and applied. We are currently testing the operator on different geometries in finite domain.

- A direct solver, SuperLU [http://crd.lbl.gov/~xiaoye/SuperLU/] is applied to solve the non-symmetric system besides AZTEC solver.