Computational Modeling of Fully Ionized Magnetized Plasmas Using the Fluid Approximation

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- Proponent of using implicit methods and primitive equations
- Founding Director of NERSC (CTRCC)
- Leader and mentor
The Problem

• Compute the *low frequency* dynamics of hot magnetized plasmas in *realistic geometry* in the presence of *high frequency* oscillations

• Incorporate the effects of *lowest order kinetic corrections* to the usual MHD equations

• Develop *accurate and efficient algorithms* that enable these goals
Modeling Magnetized Plasmas

- Plasma kinetic equation

\[
\frac{d}{dt} f_\alpha(x, v, t) = \frac{\partial f_\alpha}{\partial t} + v \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha} (E + v \times B) \cdot \nabla_v f_\alpha = \sum_\beta C_{\alpha, \beta}(f_\alpha, f_\beta)
\]

- Maxwell’s equations

\[
\frac{\partial B}{\partial t} = -\nabla \times E \quad \quad \nabla \cdot E = \rho_q
\]

\[
\rho_q = \sum_\alpha q_\alpha \int f_\alpha d^3v \quad \nabla \times B - \frac{\partial E}{\partial t} = \mu_0 J \quad J = \sum_\alpha q_\alpha \int f_\alpha v d^3v
\]

- Contains all information about plasma dynamics
- Impossible to solve analytically except in special cases
- Impractical for low frequencies, global geometry
Fluid equations defined by taking moments of distribution function

• Define moments of distribution function

\[ M_n(x, t) = \int_{-\infty}^{\infty} f(x, v, t)v^n dv \]

• Knowledge of N moments allows (in principle) reconstruction of \( f \) at N points in velocity space

• N moments of plasma kinetic equation

\[ \Rightarrow \] N fluid equations satisfied by \( M_{N+1} \)

  – Each additional moment equation yields more information about velocity distribution

• Must truncate moment equation hierarchy

  – Approximate solution of kinetic equation
Closure of Moment Equations

• Use low-order truncation and closures
• Need to express high-order moments in terms of low-order moments
  \[ q = q[n, T, ...], \quad \Pi = \Pi(p, V, ....) \]
• Must be obtained from approximate solution of kinetic equation
  – Analytical
  – Numerical
• There is no general agreement on the closure of the moment equations for hot, magnetized plasmas!
Two-fluid Equations \((m_e \sim 0, n_e \sim n_i)\)

Lowest order moments for ions and electrons:

\[
\frac{\partial n}{\partial t} = -\nabla \cdot n \mathbf{V}_e = -\nabla \cdot n \mathbf{V}_i
\]

\[
m_n \frac{d \mathbf{V}_i}{dt} = -\nabla p_i + n e (\mathbf{E} + \mathbf{V}_i \times \mathbf{B}) - \nabla \cdot \Pi_i + \mathbf{R}
\]

\[
0 = -\nabla p_e - n e (\mathbf{E} + \mathbf{V}_e \times \mathbf{B}) - \nabla \cdot \Pi_e - \mathbf{R}
\]

\[
\mathbf{J} = n e (\mathbf{V}_i - \mathbf{V}_e)
\]

+ Energy equation (+??)

+ Maxwell’s equations \((V^2/c^2 << 1)\)

+ Closure expressions
Two-Fluid Equations Present Challenges for Computation

• Extreme separation of time scales
  \[ \tau_A \ll \tau_S \ll \tau_{\text{evol}} \ll \tau_R \]
  - Alfven transit time
  - Sound transit time
  - MHD evolution time
  - Resistive diffusion time

• Extreme separation of spatial scales
  - Internal boundary layers, localized and extended along magnetic field lines
    \[ \delta/L \sim S^{-\alpha} \ll 1 \text{ for } S \gg 1 \]
    \[ S = \tau_R/\tau_A \]

• Extreme anisotropy
  - E.g., accurate treatment of \( B \cdot \nabla \), \( \chi_{||}/\chi_{\perp} \sim 10^{10} \), etc.

• “Parasitic” modes
  - High frequency modes inherent in the formulation that may affect the low frequency dynamics
Dealing with Parasitic Modes

• The fundamental problem of computational MHD: *Compute low frequency dynamics in presence of high frequency parasitic modes*
  – “Reduction” of mathematical model
    • Eliminate parasitic modes analytically
      – Example: \( \nabla \cdot \mathbf{V} = 0 \) eliminates sound waves
      – Strong toroidal field allows elimination of fast waves from MHD model
  – “Primitive” equations and “strongly” implicit methods
    • No analytic reduction of equations
    • Use algorithms that allow very large time steps (CFL $\sim 10^{4-5}$)

• Will concentrate on the second approach
There are Different Fluid Models

- Within fluid formulation, different terms are important in different parameter regimes
- Leads to different fluid models of plasmas
  - MHD
  - Hall MHD
  - Drift MHD
  - Transport
- Models distinguished by degree of force balance
- Obtained by “non-dimensionalizing” equations and systematically ordering small parameters:

\[
\delta = \frac{\rho_i}{L} \ll 1, \quad \varepsilon = \frac{\omega}{\Omega_i}, \quad \xi = \frac{V}{V_{thi}}
\]
Non-dimensional Equations

Continuity:
\[ \varepsilon \frac{\partial n}{\partial t} = -\xi \delta \nabla \cdot n \mathbf{V}_i = -\xi \delta \nabla \cdot n \mathbf{V}_e \]

Ion momentum:
\[ \varepsilon \xi \frac{\partial \mathbf{V}_i}{\partial t} + \xi^2 \delta \mathbf{V}_i \cdot \nabla \mathbf{V}_i = -\frac{1}{n} \delta \left( \nabla p_i + \frac{\Pi_i}{p_0} \nabla \cdot \Pi_i \right) + \xi \left( \mathbf{E} + \mathbf{V}_i \times \mathbf{B} \right) , \]

Electron momentum:
\[ \xi \mathbf{E} = -\xi \mathbf{V}_e \times \mathbf{B} - \frac{1}{n} \delta \left( \nabla p_e + \frac{\Pi_e}{p_0} \nabla \cdot \Pi_e \right) \]

Pre-Maxwell:
\[ \varepsilon \frac{\partial \mathbf{B}}{\partial t} = -\xi \delta \nabla \times \mathbf{E} , \quad \mathbf{J} = \xi \nabla \times \mathbf{B} , \quad \mathbf{J} = n \left( \mathbf{V}_i - \mathbf{V}_e \right) \]

Orderings:
\[ \text{time} \quad \varepsilon = \frac{\omega}{\Omega} , \quad \text{flow} \quad \xi = \frac{V_0}{V_{thi}} , \quad \text{length} \quad \delta = \frac{\rho_i}{L} \ll 1 \]

Normalizations:
\[ E_0 = V_0 B_0 , \quad J_0 = n_0 e V_0 , \quad p_0 = mn_0 V_{thi}^2 \]
Equation of Motion and Generalized Ohm’s Law

- Adding and subtracting ion and electron equations:

\[
\xi J \times B - \frac{1}{n} \delta \nabla p = n \left( \varepsilon \xi \frac{\partial V_i}{\partial t} + \xi^2 \delta V_i \cdot \nabla V_i \right) - \frac{1}{n} \delta \frac{\Pi i_0}{p_0} \nabla \cdot \Pi_i
\]

"Equilibrium" forces

\[
\xi (E + V_i \times B) = \xi \frac{1}{n} J \times B - \delta \frac{1}{n} \left( \nabla p_e + \frac{\Pi e_0}{p_0} \nabla \cdot \Pi_e \right)
\]

Ideal MHD

2-fluid and FLR effects

\[ V \times B \text{ and } J \times B \text{ enter at same order in } \xi \]
## Stress Tensor Scaling

\[ \Pi = \Pi_{||} + \Pi^\wedge + \Pi_\perp \]

\[ \Pi_{||} = \mathbf{b b} \cdot \Pi \quad \Pi^\wedge = (\mathbf{I} \times \mathbf{b}) \cdot \Pi \quad \Pi_\perp = (\mathbf{I} - \mathbf{b b}) \cdot \Pi \]

<table>
<thead>
<tr>
<th>Component</th>
<th>Scaling</th>
<th>Remarks</th>
</tr>
</thead>
</table>
| \( \Pi_{||} / p_0 \) (Braginskii) | \( \xi \delta (v / \Omega) \) | • Diverges for low collisionality  
  \((v / \Omega \sim \delta^2)\) |
| \( \Pi_{||} / p_0 \) (Neo-classical) | \( (\xi / \delta) (v / \Omega) \) | • \( O(\xi \delta) \) at low collisionality  
  • Remains “in scale” |
| \( \Pi_\perp / p_0 \) | \( \xi \delta (v / \Omega) \) | • Vanishingly small at low collisionality  
  • Ignore |
| \( \Pi^\wedge / p_0 \) (Gyro-viscosity) | \( \xi \delta \) | • Independent of collisionality  
  • Not dissipative  
  • Important FLR effects |
Different Orderings Yield Different Fluid Models

\[ \xi \mathbf{J} \times \mathbf{B} - \frac{1}{n} \delta \nabla p = n \left( \xi \frac{\partial \mathbf{V}_i}{\partial t} + \xi^2 \nabla \mathbf{V}_i \cdot \nabla \mathbf{V}_i \right) - \frac{1}{n} \delta \frac{\Pi_i dV}{p_0} \nabla \cdot \Pi_i \]

"Equilibrium" forces

\[ \xi (\mathbf{E} + \mathbf{V}_i \times \mathbf{B}) = \xi \left( \frac{1}{n} \mathbf{J} \times \mathbf{B} - \frac{1}{n} \nabla p_e + \frac{\Pi e_0}{p_0} \nabla \cdot \Pi_e \right) \]

Dynamical response

2-fluid and FLR effects

<table>
<thead>
<tr>
<th>Model</th>
<th>( V )</th>
<th>( \omega )</th>
<th>Force Balance</th>
<th>Ohm’s Law</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hall MHD</td>
<td>( V_{th}/\delta )</td>
<td>( \Omega_{ci} )</td>
<td>( \mathbf{J} \times \mathbf{B} = n \frac{d\mathbf{V}_i}{dt} )</td>
<td>( \mathbf{E} + \mathbf{V}_i \times \mathbf{B} = \frac{1}{n} \mathbf{J} \times \mathbf{B} )</td>
</tr>
<tr>
<td>MHD</td>
<td>( V_{th} )</td>
<td>( \delta \Omega_{ci} )</td>
<td>( \mathbf{J} \times \mathbf{B} = \delta \left( n \frac{d\mathbf{V}_i}{dt} + \nabla p \right) + \delta^2 \nabla \cdot \Pi_i^{gy} )</td>
<td>( \mathbf{E} + \mathbf{V}_i \times \mathbf{B} = \frac{1}{n} \mathbf{J} \times \mathbf{B} - \delta \frac{1}{n} \nabla p_e )</td>
</tr>
<tr>
<td>Drift MHD</td>
<td>( \delta V_{th} )</td>
<td>( \delta^2 \Omega_{ci} )</td>
<td>( -\nabla p + \mathbf{J} \times \mathbf{B} = \delta^2 \left( n \frac{d\mathbf{V}_i}{dt} + \nabla \cdot \Pi_i^{gy} \right) )</td>
<td>( \mathbf{E} + \mathbf{V}_i \times \mathbf{B} = \frac{1}{n} \left( \mathbf{J} \times \mathbf{B} - \nabla p_e \right) )</td>
</tr>
</tbody>
</table>
The “Standard” Drift Model

- In MHD, \( V_{\perp i} = E \times B / B^2 = V_E \)
- In drift ordering, \( V_{\perp i} = V_E + V_\ast + O(\delta^2) \)
- Write drift equations in terms of \( V_E \):

\[
\begin{align*}
\text{Ohm’s Law} & \quad E = -\left( V_E + V_\ast \right) \times B - \frac{1}{n} \nabla P_e + O(\delta^2), \\
 & = -V_E \times B - \frac{1}{n} \nabla ||P_e + \frac{1}{n} \left( -\nabla \perp p + J \times B \right) + O(\delta^2), \\
 & = -V_E \times B - \frac{1}{n} \nabla ||P_e \\
\end{align*}
\]

\( \ast \) Valid only for slight deviations from equilibrium

\[
\begin{align*}
\text{Equation of Motion} & \quad \delta^2 \left( n \frac{d}{dt} (V_{\| i} + V_E) + n \frac{dV_\ast}{dt} + \nabla \cdot \Gamma_i^{\text{GVC}} (V_i) \right) = \\
 & = -\nabla p + J \times B + O(\delta^4)
\end{align*}
\]

\( \ast \) Gyro-viscous cancellation gives simplified equations
- Exact form uncertain
- Only applicable to slight deviations from equilibrium
- We ignore for general application
Extended MHD Model

\[ Mn \frac{dV}{dt} = -\nabla p + \mathbf{J} \times \mathbf{B} - \nabla \cdot \Pi_{\parallel i} - \nabla \cdot \Pi_{gvi} \]

\[ \mathbf{E} = -\nabla \times \mathbf{B} + \frac{1}{n_e} \left( \mathbf{J} \times \mathbf{B} - \nabla p_e - \nabla \cdot \Pi_{\parallel e} \right) + \eta \mathbf{J} \]

\[ \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \quad , \quad \mu_0 \mathbf{J} = \nabla \times \mathbf{B} \]

- + Continuity and Energy equations
- + Closure expressions
- Encompasses Hall, MHD, and Drift models
- Terms can be selected by the “user”
  - GV cancellation not explicitly implemented
Extended MHD Properties

- **Dispersion**
  - Contains all MHD modes ($\omega^2 \sim k^2$)
  - Introduces dispersive modes ($\omega^2 \sim k^4$)
    - Electrons (whistlers)
    - Ions + electrons (kinetic Alfvén wave)
    - Ions only (“gyro-viscous” waves)
  - If extended MHD just produced more troublesome parasitic modes, who cares? However…..

- **Stability**
  - Drift stabilization at moderate to high $k$
  - Neo-classical de-stabilization of magnetic islands
  - ++++
  - ++++?
# Dispersion in Extended MHD

## Mode | Origin | Wave Equation | Dispersion | Comments
--- | --- | --- | --- | ---
Whistler | $\mathbf{J} \times \mathbf{B}$ in Ohm | $\frac{\partial^2 \mathbf{B}}{\partial t^2} = -\left(\frac{V_\perp^2}{\Omega}\right) (\mathbf{b} \cdot \nabla)^2 \nabla^2 \mathbf{B}$ | $\omega^2 = V_A^2 k^2 \left[ 1 + \frac{1}{\beta} \left( \rho_{\parallel} k_{\parallel} \right)^2 \right]$ | finite $k_{\parallel}$, electron response
KAW | $V_{||} p_e$ in Ohm | $\frac{\partial^2 \mathbf{B}}{\partial t^2} = \left( \frac{V_A V_{th}^*}{\Omega} \right) (\mathbf{b} \cdot \nabla)^2 \nabla \times [\mathbf{b} \cdot \nabla \times \mathbf{B}]$ | $\omega^2 = V_A^2 k_{\parallel}^2 \left[ 1 + \left( \rho_{s} k_{\perp} \right)^2 \right]$ | finite $k_{||}$, $k_{\perp}$, ion and electron response
Parallel ion GV | $\eta_4$ term in $\nabla \cdot \Pi^{GV}$ | $\rho \frac{\partial^2 \mathbf{V}_{\perp}}{\partial t^2} = -\eta_4 \nabla^2 \mathbf{V}_{\perp}$ | $\omega \pm = V_A k_{\parallel} \left[ \pm 1 + \frac{1 + \beta}{2\sqrt{\beta}} \left( \rho_{\parallel} k_{\parallel} \right) \right]$ | finite $k_{||}$, ion response
Perp. ion GV | $\eta_3$ term in $\nabla \cdot \Pi^{GV}$ | $\rho \frac{\partial \mathbf{V}_{\perp}}{\partial t} = -\eta_3 \nabla^4 \mathbf{V}_{\perp}$ | $\omega^2 = V_A^2 k_{\perp}^2 \left[ 1 + \frac{\gamma \beta}{2} + \frac{\beta}{16} \left( \rho_{k_{\perp}} \right)^2 \right]$ | finite $k_{\perp}$, ion response

Notation: $\rho_i = V_{th_i} / \Omega$ is the ion gyro-radius; $V_{th}^* = \sqrt{T_e / m_i}$; $\rho_s = V_{th}^* / \Omega$; $\eta_4 = nT_i / 2\Omega$; $\eta_3 = 2\eta_4$

$\omega^2 \sim k^4 \rightarrow \Delta t \sim \Delta x^2$ Requires implicit methods
Stability: Gravitational Interchange

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \]

\[ \rho \frac{d\mathbf{V}}{dt} = -\nabla \left( p + \frac{B^2}{2\mu_0} \right) + \rho \mathbf{g} - \nabla \cdot \Pi \]

\[ \mathbf{E} = -\mathbf{V} \times \mathbf{B} + \frac{M}{\rho e} \left[ \rho \frac{d\mathbf{V}}{dt} + \nabla p_i - \rho \mathbf{g} + \nabla \cdot \Pi \right] \]

Assume electrostatic:

\[ \nabla \times \mathbf{E} = 0 \Rightarrow \]

\[ \nabla \cdot \mathbf{V} + \frac{1}{\Omega} \nabla \times \frac{d\mathbf{V}}{dt} - \frac{1}{\Omega \rho^2} \nabla \rho \times \nabla \cdot \Pi = 0 \]

Extended MHD

\[
\begin{align*}
(\nabla \cdot \Pi)_x &= -(\rho_0 v_0)''ikV_x + \rho_0 v_0 k^2 V_y \\
(\nabla \cdot \Pi)_y &= -(\rho_0 v_0)''ikV_y - \rho_0 v_0 k^2 V_x
\end{align*}
\]

Gyro-viscosity:

# G-mode stabilization

<table>
<thead>
<tr>
<th></th>
<th>Dispersion Relation</th>
<th>Solution</th>
<th>Stabilizing Wave Number</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>MHD</strong> ((\xi = 0, \ \zeta = 0))</td>
<td>(\omega^2 + g\eta = 0)</td>
<td>(\omega = i\sqrt{g\eta})</td>
<td>None</td>
</tr>
<tr>
<td><strong>2-Fluid</strong> ((\xi = 0, \ \zeta = 1))</td>
<td>(\omega^2 - \frac{g k}{\Omega_0} \omega + g\eta = 0)</td>
<td>(2\omega = \frac{g k}{\Omega_0} \pm \sqrt{\left(\frac{g k}{\Omega_0}\right)^2 - 4g\eta})</td>
<td>(k^2 &gt; \frac{4\eta\Omega_0^2}{g})</td>
</tr>
<tr>
<td><strong>Gyro-Viscosity</strong> ((\xi = 1, \ \zeta = 0))</td>
<td>(\omega^2 - \nu_0 \eta k \omega + g\eta = 0)</td>
<td>(2\omega = \nu_0 \eta k \pm \sqrt{(\nu_0 \eta k)^2 - 4g\eta})</td>
<td>(k^2 &gt; \frac{4g}{\nu_0^2 \eta})</td>
</tr>
<tr>
<td><strong>Full Extended MHD</strong> ((\xi = 1, \ \zeta = 1))</td>
<td>(\omega^2 - \left(\frac{g k}{\Omega_0} + \nu_0 \eta k\right) \omega + g\eta = 0)</td>
<td>(2\omega = \frac{g k}{\Omega_0} + \nu_0 \eta k \pm \sqrt{\left(\frac{g k}{\Omega_0} + \nu_0 \eta k\right)^2 - 4g\eta})</td>
<td>(k^2 &gt; \frac{4g\eta}{\left(\frac{g}{\Omega_0} + \nu_0 \eta\right)^2})</td>
</tr>
</tbody>
</table>
Form of the Gyro-viscous Stress (Hooke’s Law for a Magnetized Plasma)

- Braginskii: 
  \[ \Pi^g = \Pi^g = \frac{p}{4\Omega} \left[ (b \times W) \cdot (I + 3bb) + \text{transpose} \right] \]

  \[ W = \nabla V + \nabla V^T - \frac{2}{3} I \nabla \cdot V \]

- Suggested modifications for consistency (Mikhailovskii and Tsypin, Hazeltine and Meiss, Simakov and Catto, Ramos) involve adding term proportional to the ion heat rate of strain:

  \[ \Pi^q = \frac{2}{5\Omega} \left[ b \times W_q \cdot (I + 3bb) + \text{transpose} \right] \]

  \[ W_q = \nabla q_i + \nabla q_i^T - \frac{2}{3} I \nabla \cdot q_i \]

- Implicit numerical treatment difficult

- What is the effect of this term on dispersion and stability?
  - Does it introduce new normal modes?
  - Does it alter stability properties?
Ion Heat Stress Has Little Effect on Important Dynamics

\[ \Pi^q = \frac{2}{5\Omega} \left[ b \times W_q \cdot (I + 3bb) + \text{transpose} \right] \]

\[ W_q = \nabla q_i + \nabla q_i^T - \frac{2}{3} I \nabla \cdot q_i \]

\[ q = -\kappa \| \nabla \| T - \kappa \perp \nabla T - \kappa \wedge b \times \nabla T \]

\[ \rho_0 \frac{\partial \nabla}{\partial t} = -\nabla p - \nabla \cdot \Pi^q \]

\[ \frac{\partial p}{\partial t} = -\nu p_0 \nabla \cdot \nabla \]

\[ \omega^2 = C_s^2 k^2 \left[ 1 + f(\theta)(\rho_i k)^2 \right] \quad f(0) = 0 \quad f(\pi/2) = 1 \]

- Dispersive effect on compressional waves, but……
- Negligible effect on g-mode stability
- Simplification: ignore these terms (for now!)
Careful Computational Approach is Required

- **Spatial approximation**
  - Must capture anisotropy and global geometry
    - Flux aligned grids
    - High order finite elements

- **Temporal approximation**
  - Must compute for long times
    - Require implicit methods
    - Semi-implicit methods have proven useful
Solenoidal Constraint

• Faraday: \( \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \quad \Rightarrow \quad \frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0 \)

• Depends on \( \nabla \cdot \nabla \times = 0 \)

• Different discrete approximations
  – Modified wave system \( \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \mathbf{F} = \mathbf{R} \nabla \cdot \mathbf{B} \)
  – Projection \( \mathbf{B}' = \mathbf{B} + \nabla \phi \quad \nabla^2 \phi = -\nabla \cdot \mathbf{B} \)
  – Diffusion \( \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} + \kappa \nabla \nabla \cdot \mathbf{B} \quad , \quad \frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = \nabla \cdot \kappa \nabla \nabla \cdot \mathbf{B} \)
  – Grid properties \( \nabla_a \cdot \nabla_b \times = 0 \quad \text{E.g., staggered grid, “dual mesh”} \)
Galerkin Methods

- Finite differences and finite volumes minimize error locally
  - Based on Taylor series expansion
- Galerkin methods minimize weighted error
  - Based on expansion in basis functions
- Solve “weak form” of problem

\[
\frac{\partial u(x,t)}{\partial t} = Lu(x,t) \quad \rightarrow \quad \int v \left( \frac{\partial u}{\partial t} - Lu \right) dx = 0
\]

Minimize error by expansion in basis functions and determining coefficients
Galerkin Discrete Approximation

\[ M_{ij} \frac{du_j}{dt} = L_{ij} u_j \]

- Solution generally requires inverting the mass matrix, even for “explicit” methods
- Different basis functions give different methods
  - Usually: \( \beta_i = \alpha_i \)
  - \( \alpha_i = \exp(ikx) \) => Fourier spectral methods
  - \( \alpha_i = \text{localized polynomial} \) => finite element methods
Finite Elements

- Project onto basis of locally defined polynomials of degree $p$
- Polynomials of degree $p$ can converge as fast as $h^{p+1}$
- Integrate by parts:

$$
\int \alpha_i \alpha_j \frac{d\mathbf{V}_j}{dt} dV = -\int \alpha_i \nabla \cdot \Pi(\alpha_j \mathbf{V}_j) dV = \int \nabla \alpha_i \cdot \Pi(\alpha_j \mathbf{V}_j) dV - \int \alpha_j \mathbf{n} \cdot \Pi(\alpha_j \mathbf{V}_j) dS
$$

  - Simplifies implementation of complex closure relations
  - Natural implementation of boundary conditions

- Automatically preserves self-adjointness
- Works well with arbitrary grid shapes
Three Examples of Favorable Properties of High Order Elements

Grid used for ELM studies
Non-uniform meshes retain high-order convergence rate

Magnetic divergence constraint
Scalings show expected convergence rates

Critical island width for temperature flattening
Dealing with extreme anisotropy
Agreement on scaling
Multiple Time Scales
(Parasitic Waves)

- MHD contains widely separated time scales (eigenvalues)
  \[
  \frac{\partial u}{\partial t} = \Omega u = Fu + Su \quad \text{Full MHD operator}
  \]
  - Fast time scales: Alfvén waves, soundwaves, etc (parasitic waves)
  - Slow time scales: Resistive instabilities, island evolution, (interesting physics)

- “Parasitic” waves are properties of the physics problem but are not the dynamics of interest
- Treat only “fast” part of operator implicitly to avoid time step restriction
  \[
  \frac{u^{n+1} - u^n}{\Delta t} = Fu^{n+1} + Su^n
  \]
- Precise decomposition of \( \Omega \) for complex nonlinear system is often difficult or impractical to achieve algebraically
Dealing with Parasitic Waves

• Original idea from André Robert (1971)
  – Gravity waves in climate modeling

• $F$ and $\Omega$ are often known, but an expression for $S$ is difficult to achieve
  – $\Omega$: full MHD operator
  – $F$: linearized MHD operator

• Use operator splitting: $\Omega = F + S \Rightarrow S = \Omega - F$

$$\frac{u^{n+1} - u^n}{\Delta t} = Fu^{n+1} + (\Omega - F)u^n = \Omega u^n + \Delta t F \left( \frac{u^{n+1} - u^n}{\Delta t} \right)$$

• Expression for $S$ not needed
Semi-Implicit Method

\[
\frac{u^{n+1} - u^n}{\Delta t} = Fu^{n+1} + (\Omega - F)u^n = \Omega u^n + \Delta t F \left( \frac{u^{n+1} - u^n}{\Delta t} \right)
\]

- Recognize that the operator \( F \) is completely arbitrary!!

\[
(I - \Delta t G) u^{n+1} = (I - \Delta t \Omega) u^n - \Delta t G u^n
\]

- \( G \) can be chosen for accuracy and ease of inversion
  - \( G \) should be easier to invert than \( F \) (or \( \Omega \), e.g., toroidal coupling)
  - \( G \) should approximate \( F \) for “modes of interest”
  - Some choices are better than others!

- Has proven to be very useful for resistive and extended MHD
  - Used for spheromak, RFP, tokamak, and solar corona modeling
SI Operator for MHD

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\nabla \times \mathbf{B} - \eta \mathbf{J})
\]

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{V}
\]

\[
\frac{\partial p}{\partial t} = -\nabla \cdot \rho \mathbf{V} - (\gamma - 1) \rho \nabla \cdot \mathbf{V}
\]

\[
\rho \frac{\partial \mathbf{V}}{\partial t} = -\rho \mathbf{V} \cdot \nabla \mathbf{V} - \nabla p + \mathbf{J} \times \mathbf{B} + \alpha \Delta t^2 \left[ \nabla \times \nabla \times \left( \frac{\partial \mathbf{V}}{\partial t} \times \mathbf{B}_0 \right) \times \mathbf{B}_0 + \nabla \gamma p_0 \nabla \cdot \frac{\partial \mathbf{V}}{\partial t} \right]
\]

- Ideal MHD operator (Lerbinger and Luciani)
- Anisotropic, self-adjoint
- Avoids implicit toroidal coupling (great simplification)
- Accurate linear results for CFL \( \sim 10^{4-5} \) (\( \Rightarrow \) Condition number \( \sim 10^{10}!! \))

\( k \Delta t < 1 \)
Semi-Implicit Leap Frog

- Variables staggered at different time levels
- SI operator on velocity

\[ \Delta V = V^j - V^{j-1} \]
\[ \Delta p = p^{j+1/2} - p^{j-1/2} \]

\[ \frac{\Delta V}{\Delta t} = -\nabla p^{j-1/2} + \alpha \Delta t \left( \frac{\Delta V}{\Delta t} \right) \]

\[ V^j = V + \Delta V \]

\[ \frac{\Delta p}{\Delta t} = -\gamma P_0 \nabla \cdot V^j \]

\[ p^{j+1/2} = p^{j-1/2} + \Delta p \]
Extended MHD Time Advance

- “Implicit leap-frog” (also used in MHD)
  - Maintains numerical stability without introducing numerical dissipation
- MHD advance unchanged (semi-implicit self-adjoint operators)
- Need to invert non-self-adjoint operators at each step for dispersive modes
- Requires high performance parallel linear algebra software
Implicit Leap Frog for Extended MHD

\[
m_i n^{j+1/2} \left( \frac{\Delta V}{\Delta t} + \frac{1}{2} V^j \cdot \nabla \Delta V + \frac{1}{2} \Delta V \cdot \nabla V^j \right) = \frac{\Delta t L^{j+1/2}(\Delta V)}{\text{SI MHD}} + \nabla \cdot \Pi_i (\Delta V) = \frac{\Delta t L^{j+1/2}(\Delta V)}{\text{SI MHD}} + \nabla \cdot \Pi_i (\Delta V)
\]

\[
\text{Momentum}
\]

\[
\frac{\Delta n}{\Delta t} + \frac{1}{2} V^j \cdot \nabla \Delta n = -\nabla \cdot \left( V^j \cdot n^{j+1/2} \right)
\]

\[
\text{Continuity}
\]

\[
\frac{3n}{2} \left( \frac{\Delta T_\alpha}{\Delta t} + \frac{1}{2} V^\alpha \cdot \nabla \Delta T_\alpha \right) + \frac{1}{2} \nabla \cdot q_\alpha (\Delta T_\alpha)
\]

\[
\text{Anisotropic thermal conduction}
\]

\[
- \frac{3n}{2} V^j \cdot \nabla T_\alpha^{j+1/2} - n T_\alpha^{j+1/2} \nabla \cdot V^\alpha - \nabla \cdot q_\alpha \left( T_\alpha^{j+1/2} \right) + Q_\alpha^{j+1/2}
\]

\[
\text{Energy}
\]

\[
\frac{\Delta B}{\Delta t} + \frac{1}{2} V^j \cdot \nabla \Delta B + \frac{1}{2} \nabla \times \left[ \frac{1}{ne} \left( J^{j+1/2} \times \Delta B + \Delta J \times B^{j+1/2} \right) \right] + \frac{1}{2} \nabla \times \eta \Delta J
\]

\[
\text{Implicit HALL term}
\]

\[
- \nabla \times \left[ \frac{1}{ne} \left( J^{j+1/2} \times B^{j+1/2} - \nabla p_e \right) - V^j \times B^{j+1/2} + \eta J^{j+1/2} \right]
\]

\[
\text{Implicit resistive term}
\]

\[
\text{Maxwell/Ohm}
\]
Nonlinear ELM Evolution

- Anisotropic thermal conduction
- ELM interaction with wall
- 70 kJ lost in 60 µsec
- 2-fluid and gyro-viscosity have little effect on linear properties
Two-fluid Reconnection
GEM Problem

- 2-D slab
- $\eta = 0.005$
- Good agreement with many other calculations
- Computed with same code used for tokamaks, spheromaks, RFPs
“Heuristic Closure” Captures Essential Neoclassical Physics

Neo-classical theory gives flux surface average

Local form for stress tensor forces:

$$\nabla \cdot \Pi_\alpha = \rho_\alpha \mu_\alpha \left\langle B^2 \right\rangle \frac{\mathbf{V}_\alpha \cdot \mathbf{e}_\theta}{(B_\alpha \cdot \mathbf{e}_\theta)^2} \mathbf{e}_\theta$$

• Valid for both ion and electrons
• Energy conserving and entropy producing
• Gives:
  • bootstrap current
  • neoclassical resistivity
  • polarization current enhancement

(Gianakon et al., Phys. Plasmas 9, 536 (2002))
Beyond Extended MHD: Parallel Kinetic closures

• Parallel closures for $q_\parallel$ and $\Pi_\parallel$ derived using Chapman-Enskog-like approach.

• Non-local; requires integration along perturbed field lines.

• General closures map continuously from collisional to nearly collisionless regime.

• General $q_\parallel$ closure predicts collisional response for heat flow inside magnetic island. As plasma becomes moderately collisional ($T > 50$ eV), general closure predicts correct flux limited response.

• Incorporated into global extended MHD algorithms.

Thermal diffusivity as function of $T$ showing $T^{5/2}$ response of Braginskii and general closure.
Beyond Extended MHD: Kinetic Minority Species

- Minority ions species affects bulk evolution:
  \[ n_h << n_0, \quad \beta_h \sim \beta_0 \]

  \[
  Mn \frac{dV}{dt} = J \times B - \nabla \cdot \Pi \]

  Bulk Plasma

  \[-\nabla \cdot \Pi_h \]

  Hot Minority Ion Species

  \[
  \delta \Pi_h = \int M (v - V_h)(v - V_h) \delta f(x, v) d^3v
  \]

- \( \delta f \) determined by kinetic particle simulation in evolving fields
- Demonstrated transition from internal kink to fishbone
- Benchmark of three codes
Balance of algorithm performance and problem requirements with available cycles

\[
\frac{N^\alpha Q}{\Delta t} = 3 \times 10^7 \frac{\varepsilon P}{CT}
\]

**Algorithms:**
- \( N \) - # of meshpoints for each dimension
- \( \alpha \) - # of dimensions
  - 1.5 - transport
  - 3 (spatial) fluid
  - 5-6 kinetic (spatial + velocity)
- \( Q \) - code-algorithm requirements (Tflop / meshpoint / timestep)
- \( \Delta t \) - time step (seconds)

**Constraints:**
- \( P \) - peak hardware performance (Tflop/sec)
- \( \varepsilon \) - hardware efficiency
  - \( \varepsilon P \) - delivered sustained performance
- \( T \) - problem time duration (seconds)
- \( C \) - # of cases / year
  - 1 case / week ==> \( C \sim 50 \)
“The Future”

Assumptions:
- Performance is delivered
- Implicit algorithm
- \( Q \) ind. of \( \Delta t \) (!)

Requirements:
- At least 3-D physics required
- Required problem time: 1 msec - 1 sec

Conclusions:
- 3-D (i.e., fluid) calculations for times of ~ 10 msec within reach
- Longer times require next generation computers (or better algorithms)
- **Higher dimensional (kinetic)** long time calculations unrealistic
- **Integrated kinetic effects must come through low dimensionality fluid closures**
Summary

- Fluid models are an approximation to the plasma kinetic equation, but are *required* for modeling low frequency response of hot, magnetized plasmas with global geometry
  - Direct kinetic calculations are impractical
- Primitive equations and implicit methods have proven successful in modeling a variety of plasmas
- Implicit methods are required for handling the dispersive terms of MHD. An understanding of the dispersive characteristics of discretized equations needed.
- “Kinetic” effects must be captured through fluid closures
  - “Best” form of fluid equations still unknown
  - Often problem dependent
- Next step is direct coupling of kinetic/fluid/transport models