General Neoclassical Closure Theory: Diagonalizing the Drift Kinetic Operator

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Introduction

- Goal of a general neoclassical closure theory:

  Capture collisional, free-streaming, trapping and drift-orbit physics in closure relations for long time scale, electromagnetic fluid simulations of magnetized plasmas.

- Allow for
  1. complicated geometry,
  2. varying collisionality, and
  3. numerical tractability.

Solve drift kinetic equation (DKE)

- Invert DKE operator $L(F)$ in

$$L(F) = \left(\frac{\partial}{\partial t} + (\vec{v}_D + \vec{v}_{\|}) \cdot \vec{\nabla}\right) F + a\frac{\partial F}{\partial v} - C_L(F) = G(f_M).$$

- Closures of interest for 5-moment fluid model are

$$\vec{q} = \int d^3v\vec{v}v^2 F, \qquad \text{and}$$

$$\mathbf{\Pi} = \int d^3v(\vec{v}\vec{v} - v^2\mathbf{I}/3)F.$$

- Difficulties:
  1. 6-dimensional configuration space, $(v, \xi, \vec{x}, t)$.
  2. disparate time scales, $10^7 \sim \vec{v}_{\|} \cdot \vec{\nabla}_l \sim \omega_b >> \partial(\ln B)/\partial t \sim 1$.

DKE operator is complicated

- Separating out parallel speed dependence,

$$
\begin{aligned}
L(F) = \quad & P_0(v_{||}/v) \quad \left[\frac{\partial}{\partial t} + \frac{\hat{\mathbf{b}}}{\Omega} \times \left(\frac{q\vec{E}}{m} + \frac{2}{3}v^2\vec{\nabla}\ln B + \frac{\mu_0 v^2}{3B}\vec{J}_\perp\right)\cdot\vec{\nabla}\right] F + \\
& P_1(v_{||}/v) \quad v(\hat{\mathbf{b}}\cdot\vec{\nabla})F + \\
& P_2(v_{||}/v) \quad \frac{\hat{\mathbf{b}}}{\Omega} \times \left(\frac{1}{3}v^2\vec{\nabla}\ln B + \frac{2\mu_0 v^2}{3B}\vec{J}_\perp\right)\cdot\vec{\nabla}F - \\
& C(F),
\end{aligned}
$$

where $v_{||}^2/v^2 = 1 - (B/B_0)(1-\xi^2)$.

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& P_1(v_\|/v) \quad v(\hat{\mathbf{b}}\cdot(\vec{\nabla}_l + \vec{\nabla}_L))F + \\
& P_2(v_\|/v) \quad \frac{\hat{\mathbf{b}}}{\Omega} \times \left(\frac{1}{3}v^2\vec{\nabla}\ln B + \frac{2\mu_0 v^2}{3B}\vec{J}_\perp\right)\cdot\vec{\nabla}F - \\
& \qquad \nu \qquad \frac{v_\|}{v\xi}\frac{\partial}{\partial\xi}\frac{1-\xi^2}{\xi}\left(\frac{v_\| B_0}{vB}\right)\frac{\partial F}{\partial\xi},
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where $v_\parallel^2/v^2 = 1 - (B/B_0)(1-\xi^2)$.

- Low-beta, bounce-averaged form is

$$
\begin{aligned}
L(F) \approx \left(\partial_t + \left\langle\partial_{\vec{E}\times\vec{B}}\right\rangle\right)F + v^2\left\langle P(v_\parallel/v)\partial_{\vec{\nabla}B}\right\rangle F \\
\sigma v\xi\left\langle\frac{B}{B_0}\partial_L\right\rangle F - \nu\partial_\xi\frac{1-\xi^2}{\xi}\left\langle\frac{v_\parallel}{v}\right\rangle\partial_\xi F
\end{aligned}
$$

Basis function expansion aides diagonalization

- Use expansion in spherical harmonics as guide

$$f = \sum_{lmn} a_{lmn} Y_{lm}(\xi, \phi) L_n^{l+\frac{1}{2}}(\frac{v^2}{v_{th}^2})(\frac{m}{2\pi T})^{\frac{3}{2}} e^{-\frac{mv^2}{2T}}.$$

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- Expand in solutions of separated bounce-averaged eigenvalue equation:

$$\frac{\partial}{\partial \xi} \frac{1-\xi^2}{\xi} \left\langle \frac{v_\|}{v} \right\rangle \frac{\partial C_l}{\partial \xi} + \lambda_l \langle J \rangle C_l = 0.$$

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- Laguerre polynomials serve as energy eigenfunctions:

$$F = \sum_{l,n} F_{l,n}(\vec{x}, t) C_l(\xi) L_n^{l+\frac{1}{2}}(\frac{v^2}{v_{th}^2}).$$

Perturbative expansion necessary for diagonalization

- Multiply through by $C_{l'}$ and integrate over ξ:

$$\int_{-1}^{1} d\xi \, \langle J \rangle \, C_{l'} L(F) \approx \underbrace{(-\mathbf{C} + \mathbf{I}\partial_t + \mathbf{I}\partial_{\vec{E}\times\vec{B}})\mathbf{F}}_{\text{diagonal}} \quad + v^2 \mathbf{D}_{\tilde{\nabla}\mathbf{B}} + v\mathbf{D}_{\mathbf{L}})\mathbf{F}.$$

electrons	$\omega_{be} >>$	$\vec{v}_{\parallel} \cdot \vec{\nabla}_L \sim \nu_e \geq$	$\vec{v}_D \cdot \vec{\nabla} \sim \partial/\partial t \sim a\frac{\partial}{\partial v}$	
frequency	10^7	10^5	10^4	10^3
ions		$\omega_{bi} >>$	$\vec{v}_D \cdot \vec{\nabla} \sim \partial/\partial t \geq$	$\vec{v}_{\parallel} \cdot \vec{\nabla}_L \sim \nu_i \sim a\frac{\partial}{\partial v}$

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- Diagonalizable systems include

$$L(F) \approx (-\mathbf{C} + \mathbf{I}\partial_t + \mathbf{I}\partial_{\vec{E}\times\vec{B}} + v^2\mathbf{D}_{\tilde{\nabla}\mathbf{B}} + v\mathbf{D}_{\mathbf{L}})\mathbf{F}$$

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Balance electron free-streaming with e-e and e-i collisions

- Electrons have

$$\nu \mathbf{I} \mathbf{F}^0 + v \bar{\mathbf{D}}_{\mathbf{L}} \mathbf{F}^0 = \mathbf{G}^0(\vec{v}_{||} \cdot \vec{\nabla} f_M, C(f_M))$$

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- Expand in eigenvectors of $\bar{\mathbf{D}}_{\mathbf{L}}$, $\mathbf{F}^0 = \mathbf{W}\mathbf{f}^0$, and multiply through by $\mathbf{W}^{-1}$:

$$\nu(v) f_m^0 + v\lambda_m \partial_L f_m^0 = g_m^0.$$

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- Solution to all orders has form

$$F^n = \sum_l \int^L dL' \mathbf{W}(l,:)\mathbf{f}^n(L', L, v) C_l(\xi),$$

Balance ion drifting with time dependence.

- Ions have

$$\mathbf{I}\left(\partial_t + \left\langle \partial_{\vec{E}\times\vec{B}} \right\rangle\right)\mathbf{F}^0 + v^2 \bar{\mathbf{D}}_{\tilde{\vec{\nabla}}\mathbf{B}} \mathbf{F}^0 = \mathbf{G}^0(\vec{v}_D \cdot \vec{\nabla} f_M, C(f_M), \partial_t f_M)$$

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- Expansion in Laguerre polynomials and substitution using $\mathbf{f_m}^0 = \mathbf{X}\mathbf{h_m}^0$ diagonalizes system:

$$\left(\partial_t + \left\langle \partial_{\vec{E}\times\vec{B}} + \lambda_{m,i} \partial_{\vec{\tilde{\nabla}}B} \right\rangle\right) h_{m,i}^0 = g_{m,i}^0.$$

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- Solution to all orders has form

$$F^n = \sum_{l,m} \int^{\tau_{m,i}} d\tau'_{m,i} \mathbf{W}(l,:)\mathbf{X}(m,:)\mathbf{h}^n(\tau_{m,i}, \tau'_{m,i}) C_l(\xi) L_m^{l+\frac{1}{2}}(\frac{v^2}{v_{th}^2}),$$

where $\tau_{m,i} = t - (\langle \partial_{\vec{E}\times\vec{B}} + \lambda_{m,i}\partial_{\vec{\nabla} B} \rangle)^{-1} = t - L_\perp.$

Conclusions

- From F^n calculate closures of interest

$$\vec{q} = \int d^3v \vec{v} v^2 F, \qquad \text{and}$$

$$\mathbf{\Pi} = \int d^3v (\vec{v}\vec{v} - v^2 \mathbf{I}/3) F.$$

- Diagonalizing via expansion in basis functions permits
  1. integration along essential characteristics of F (drift or free-streaming orbits) in complicated magnetic fields,
  2. truncation of integration helped by collisional effects,
  3. reduction of configuration space variables hence numerical tractability, and,
  4. simple prescription for higher-order corrections.
- Closures include collisional, free-streaming, trapping and drift-orbit physics for ions and electrons.