MHD Simulations with the NIMROD Code

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The success of tokamak experiments in obtaining high plasma pressure discharges has largely been due to the design and operation of the experiments around the stability of ideal magnetohydrodynamic modes[1]. Understanding slower-growing electromagnetic perturbations gains in importance as modern tokamak experiments operate near the ideal MHD limit in steady-state conditions. In addition, nonideal relaxation is often an inherent aspect of sustained operation in alternative confinement devices. Because modern experiments are virtually collisionless, the analysis of long-wavelength instabilities requires extension of the usual MHD model to include additional effects such as the two-fluid terms, anisotropic thermal transport, different ion and electron temperatures, neoclassical contributions to the stress tensor, and pressure contributions from energetic species. To compare a numerical simulation with experiment, the numerical code must be able to model the complex geometry of the experiments as well as operate in the nonlinear regime.

The principal goal of the NIMROD project is to develop a modern computer code suitable for the study of long-wavelength, low-frequency, nonlinear phenomena in realistic toroidal geometry. To accomplish this, the NIMROD team has the specific objectives that the code

- model all important effects for nonlinear electromagnetic phenomena,
- be easily accessible for use by fusion scientists of different levels of expertise,
- operate on modern computers with different architectures, from personal computers to massively parallel supercomputers, and
- be well-structured, commented and documented in order to allow users modify parts of the code without mastering the whole.

This paper briefly discusses some of the development and results that the NIMROD team has achieved.
1 NIMROD Code Description

Formulated to study instabilities traditionally studied with the MHD equations, the equations which are solved by NIMROD are the momentum equation

$$\mathbf{\rho} \left( \frac{\partial \mathbf{V}}{\partial t} + (\nabla \cdot \mathbf{V}) \mathbf{V} \right) = \mathbf{J} \times \mathbf{B} - \nabla p - \nabla \cdot \Pi,$$

the generalized Ohm’s law

$$\tilde{E} = - \frac{1}{\text{Ideal MHD}} \nabla \times \tilde{B} + \frac{\eta \mathbf{J}}{\text{Resistive MHD}} + \frac{1}{\epsilon_0 \omega_\text{pe}^2 (1 + \nu)} \left[ \frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot (\mathbf{V} \mathbf{J} + \mathbf{J} \nabla) \right]$$

$$+ \frac{1}{n_c (1 + \nu)} \left[ \frac{(1 - \nu) \mathbf{J} \times \tilde{B} - \nabla (p_c - \nu p_i) - \nabla \cdot (\Pi_c - \nu \Pi_i) }{\text{Hall Term}} \right],$$

and the temperature equations

$$\frac{\partial T_a}{\partial t} + \nabla_a \cdot \nabla T_a + \gamma T_a \nabla \cdot \mathbf{V}_a = - (\gamma - 1) \nabla \cdot \tilde{q}_a + (\gamma - 1) Q_a,$$

where $a$ refers to the fluid species. These are solved in conjunction with Maxwell’s equations.

These equations, which are fully equivalent to the two-fluid equations, are often called the “extended MHD” equations because they reduce to the resistive and ideal MHD limits, but can be easily extended to include additional physics (NIMROD simulations may incorporate as much or as little of the additional physics as desired by the user at runtime). The only assumptions used in deriving the equations are two plasma species, quasineutrality, and no displacement current in Faraday’s Law.

The code is structured to be flexible in the type of closures used. Currently, isotropic viscosity and neoclassical-type closures are implemented for the stress tensor $\Pi$, and gyrokinetic closures for energetic ion species are under development. For the heat flux, $\tilde{q}$, NIMROD currently uses a Braginskii-type closure

$$\tilde{q} = -\kappa || \mathbf{b} \cdot \nabla T - (\kappa_\perp - \kappa_||) \nabla \cdot \tilde{B} T,$$

but more rigorous closures valid in the long-mean-free-path regime are being investigated.\[2\]

Even in the MHD limit, solving these equations numerically presents many challenges. As an illustration, for parameters typical of DIII-D (and using number density $n = 5 \times 10^{19} \text{ m}^{-3}$ and the Lundquist number $S = 10^8$, the relative time scales for the resistive MHD equations are:
• compressional Alfvén wave, $\tau_{\text{comp}} = 0.09 \ \mu s$

• shear Alfvén wave, $\tau_{\text{shear}} = 0.49 \ \mu s$

• tearing mode growth time, $\tau_{\text{tear}} \sim 1 \ m s$

• resistive diffusion time, $\tau_R = 49 \ s$

The numerical challenge is to calculate on the time scale of a tearing mode’s growth rate which is four orders of magnitude larger than the fastest time scale in the system. If the simplest explicit time-differencing schemes [3] are used, timesteps are limited by the Courant-Friedrichs-Levy (CFL) condition:

$$\Delta t \lesssim \frac{\Delta x}{V_A}. \quad (5)$$

The CFL limit would make numerical simulations computationally intractable in resistive MHD calculations where the time scale of interest is long compared to the fastest wave. The semi-implicit method [4] is a very advantageous scheme for overcoming this challenge. For the ideal MHD waves, NIMROD uses a semi-implicit operator similar to that of Lerbinger and Luciani: [5]

$$\left\{ I - \sigma_{\text{mhd}}(\Delta t)^2 L_{\text{ideal}} + \sigma_{\text{nl}} \frac{B_{n=0}^2}{\mu_0 \rho} (\Delta t)^2 \nabla^2 I \right\} \cdot \Delta \vec{V} = \frac{1}{\rho} \left( \vec{J}^n \times \vec{B}^n - \nabla p^n \right), \quad (6)$$

where $L_{\text{ideal}}$ is the linearized ideal MHD operator. The difference between NIMROD’s implementation of the semi-implicit operator and Ref. [5] is that NIMROD includes the $n = 0$ components of the solution fields in addition to the equilibrium fields. NIMROD also uses semi-implicit advances for the anisotropic heat conduction and Hall terms.

The challenges for spatial discretization are due to the boundary layers formed at rational surfaces for high Lundquist numbers, and the extreme anisotropy of the equations. These boundary layers, which have large gradients, occur on a scale length of approximately a millimeter in DIII-D plasmas – three orders of magnitude smaller than the size of the system. The anisotropy can be seen by considerations of the Braginskii closure of the heat flux given by Eq. (4). The experimental values of $\kappa_\parallel / \kappa_\perp = 10^{10}$ imply that the distinction between parallel and perpendicular must be determined very accurately by the code. For flexibility in matching to experimental configurations yet taking advantage of the axisymmetry in the experiments of interest, NIMROD uses finite elements in the poloidal plane (both structured quadrilateral elements and unstructured triangular elements) and Fourier decomposition in the toroidal direction. The NIMROD quadrilateral grid is aligned
with the magnetic field inside the separatrix and packed around the rational surfaces to overcome the spatial stiffness and anisotropy. The structured grids can be extended outside the separatrix to maximize computational efficiency, while the unstructured mesh can be used to match to the walls of the experiment. An example grid is shown in Figure 1. For efficiency in parallelization, different processors may be assigned either different finite element blocks, or toroidal mode numbers for efficiency in parallelization.

![Figure 1: Example of a finite element grid in DIII-D geometry.](image)

A recent advance in NIMROD’s capabilities (not yet in the production code) is the ability to use arbitrary degree polynomial basis functions in our Lagrange-type finite elements. This allows spatial convergence at rates faster than second-order on nonuniform meshes, thereby improving computational performance. An example of convergence changing with the basis function polynomial is shown in Fig. 2. Another benefit of this development is a vastly improved treatment of magnetic divergence. Representing the magnetic field directly with Lagrange-type elements does not enforce Gauss’s Law ($\nabla \cdot \vec{B} = 0$). To prevent growth of an unphysical longitudinal part of the magnetic field [6], we use an error diffusion technique [7], where the divergence of magnetic field is diffused:

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} + \kappa_B \nabla \nabla \cdot \vec{B}. \quad (7)$$

This technique is related to the “penalty method” used in incompressible Navier-Stokes and satisfies the divergence-stability condition [8] in the limit of large $\kappa_B$ only for polynomial
bases of degree two and larger. Spatial convergence then limits to a divergence-free space, and in practice, simulations are much less sensitive to the value of $\kappa_B$.

![Figure 2: Increasing the polynomial degree of the finite element bases results in improved convergence as demonstrated in this plot of growth rate of a tearing mode in cylindrical geometry computed by NIMROD.](image)

**2 Spheromak Simulations**

Among all of the devices being considered for magnetically confined fusion plasmas, none rely more heavily on plasma currents to form the magnetic configuration than the electrostatically driven spheromak. Establishing the magnetic fields requires little more than a potential difference, $V_0$, applied between electrode surfaces that are linked by magnetic fields, as shown in Fig. 3. The applied potential then provides magnetic helicity injection so that fields linking the electrodes are twisted by plasma currents, and reconnection arising from nonideal electromagnetic activity leads to a configuration with little resemblance to the initial state. Using simple experimental configurations is attractive from a reactor design standpoint; however, the electromagnetic activity from the relaxation impacts the confinement. Therefore developing a detailed understanding of spheromak formation is a critical component of spheromak research.
NIMROD is well-qualified to study the formation of the spheromak plasma since its flexibility allows it to simulate plasmas in simply connected domains. The geometry presented in this paper is shown in Figure 3 and the results are representative from more comprehensive studies done with the NIMROD code [9, 10]. The model used is zero-beta resistive MHD with the Lundquist number, $S$, of the order $10^3 - 10^4$ and the Prandtl number, the ratio of viscosity to resistivity ($P_m = \mu_0 \nu / \eta$), set to unity.

Above the symmetric pinch instability threshold, nonlinear NIMROD results show that MHD activity results in the amplification of the poloidal flux as seen in the experiments. The MHD dynamo converts the toroidal flux into the poloidal flux as seen in Figure 4, where contours of the $n = 0$ poloidal flux are shown. The MHD activity also relaxes the mean parallel current profile toward the uniform-$J \cdot \bar{B} / B^2$ Taylor state[11], producing $n = 0$ magnetic fields that match the first $\nabla \times \bar{B} = \mu \bar{B}$ eigenfunctions rather well, except near boundaries (Fig. 5).

Although simulations with applied potential above the pinch instability threshold show closed contours of $n = 0$ poloidal flux, as in Fig. 4, large closed magnetic flux surfaces are not sustained in typical conditions. The amplitudes of the $n > 0$ components (shown in Fig. 6) are sufficient to produce chaotic scattering of the magnetic field throughout the volume. The largest component has $n = 1$ and is the saturated state of the axisymmetric pinch instability. It produces the helical deformation of the current illustrated in Fig. 7 via the dynamo electric field field it induces. The dynamo action deflects power from the central
Figure 4: Contours of n=0 poloidal flux. Flux amplification is defined as $\Psi_a/\Psi_e$ where $\Psi_a$ is the amplified flux in excess of $\Psi_e$.

region, and coupling to $n > 0$ helps spread the fluctuation induced mean current across the domain. The large variation $J \cdot B/B^2$ evident in Fig. 7 imply that while the Taylor state provides a rough description of the toroidally averaged magnetic field, a uniform $J \cdot B/B^2$ state is not achieved. Furthermore, the sustained state relies on large-scale MHD activity, not ubiquitous and benign small-scale fluctuations.

If the voltage is turned off and the spheromak is allowed to decay, then flux surfaces seem to form readily as the spheromak is allowed to relax as seen in Fig. 8. The formation of these flux surfaces would result in increased confinement which, when considered with the coincident Ohmic heating, provides a possible explanation of the increased temperatures observed during decay in spheromak experiments[12, 13].
Figure 5: Comparison of axial component of the $n = 0$ magnetic field of the saturated nonlinear state with the Taylor eigenfunction. The comparison is better at the midplane than near the lower electrode due to line-tying effects.

Figure 6: Evolution of magnetic energy for each Fourier component from an unstable axisymmetric pinch.
Figure 7: Isosurfaces of large (central) and small (surrounding) $J \cdot B/B^2$ values, showing order unity variation in the three dimensional parallel current profile for the sustained state of Fig. 4.
Figure 8: Contours of $n=0$ poloidal flux and Poincare surfaces of section at the times indicated during a decay of a saturated state. The applied voltage is set to 0 at $t=0.187$. 
3 Implementation of Vacuum Region into NIMROD

MHD stability limits in experiments depend on the presence of a vacuum region and on the location of a (geometrically-complicated) conducting wall. Unlike linear codes where one can use Green’s functions to predict the linear response of a fixed plasma-vacuum boundary, the implementation of a vacuum region in a nonlinear initial-value code is more challenging. A vacuum-region model in NIMROD should give the same results linearly as a linear MHD code, be capable of nonlinear simulations with a moving plasma-vacuum interface, and be computationally tractable.

NIMROD models the vacuum as a very resistive plasma. By making the resistivity very large, any currents which are generated in the vacuum are quickly damped. The numerical challenge in this approach is maintaining a sharp plasma/vacuum interface. To do so, we use an interface tracking technique where NIMROD advects a “shape variable,” $\sigma$, which is used to indirectly determine the resistivity. Initially, the shape variable is

$$\sigma = \begin{cases} 
0 & \text{in plasma} \\
1 & \text{in vacuum} 
\end{cases} \quad (8)$$

while after the plasma has evolved, the shape variable is used to determine the vacuum-plasma boundary:

$$\sigma < \frac{1}{2} \quad \Rightarrow \quad \text{in plasma}$$

$$\sigma > \frac{1}{2} \quad \Rightarrow \quad \text{in vacuum.} \quad (9)$$

A cylindrical case which is unstable to an external kink [14] is used for linear benchmarking. The current profile is constant inside the plasma and zero outside. The resultant safety factor profile is 0.5 in the core, and then rises linearly to above 2 at the conducting wall, which is at twice the plasma minor radius. Analytic solutions for the eigenfunctions and growth rates may be found in Ref. [14]. The resistive vacuum model gives good linear convergence as we increase the ratio of the resistivity in the vacuum to the resistivity at the plasma center as seen in Fig. 9. Examination of the eigenfunctions in Fig. 10 for the magnetic field reveals that the currents in the vacuum region decrease as desired.

As the resistivity in the vacuum region is increased, the nature of Ohm’s law changes from a predominantly hyperbolic equation in the core, to a predominantly elliptic equation in the vacuum region. This is a classic ill-conditioned problem and manifests itself as an increase
Figure 9: The growth rate calculated by NIMROD converges to the analytic growth rate as the resistivity in the vacuum region is increased.

Figure 10: The radial magnetic eigenfunctions at $\eta_{\text{vac}}/\eta_{\text{core}} = 10^5$ (left) and $\eta_{\text{vac}}/\eta_{\text{core}} = 10^9$ (right).

in the number of iterations required to invert the matrix for the magnetic field advance as seen in Fig. 11. Note that the x axis is a log scale showing number of iterations increases logarithmically with increasing vacuum resistivity.

Work on the vacuum region is continuing. Currently, a toroidal benchmark based on DIII-D Shot #97441 is underway, as well as the nonlinear extension of the cylindrical benchmark.
Figure 11: Scaling of the number of iterations of the matrix inversion for the magnetic field shows favorable logarithmic scaling.

4 Summary

Nonlinear simulations of long-wavelength, low-frequency electromagnetic perturbations in modern experiments present many challenges. While developing a nonlinear, initial-value code solving the extended MHD equations using finite elements has presented many new research issues, much progress towards a code capable of solving these challenges has been made, and the code is already being used to solve unique problems.

Resistive MHD calculations with NIMROD reproduce the most prominent features of laboratory spheromaks. These features include flux amplification, the dominance of the $n = 1$ perturbation during sustainment, and contributions to the dynamo electric field. In addition, the formation of closed flux surfaces during decay in the simulations provides a possible explanation for the high temperatures seen during spheromak decay in the laboratory.

A model for the vacuum region has been developed for NIMROD is able to reproduce the linear stability criterion. Development is actively underway to test the model in nonlinear regimes and toroidal geometry.
References


